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SOME LYAPUNOV TYPE POSITIVE OPERATORS ON ORDERED BANACH SPACES*

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Abstract

In this paper we investigate several properties of Lyapunov type operators occurring in connection with the characterization of exponential stability in mean square of systems of linear Itô differential equations perturbed by a Markov process with an infinite countable number of states. A criterion for exponential stability of linear differential equations defined by a Lyapunov operator is derived under the assumption of property of detectability adequately defined for this type of operator valued functions.

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1 Introduction

The Lyapunov type linear operators play a central role in the characterization of exponential stability of linear systems of differential equations both in deterministic and stochastic framework, continuous time and discrete-time cases. For the readers convenience, we refer to [13, 14] for the deterministic framework and [1, 2, 9, 10, 15] for the stochastic context. In the stochastic case, the differential equations defined by the Lyapunov type operator valued functions offer a deterministic framework for the characterization of exponential stability in mean square for stochastic linear differential equations with multiplicative white noise perturbations and/ or Markovian jumping. The systematic investigation of linear differential equations defined by Lyapunov type operator valued functions regarded as mathematical object with interest in itself, started with the time invariant case, namely, the differential equation is defined by a Lyapunov operator not depending upon the time. In this case, criteria for exponential stability of differential equations defined by a Lyapunov operator were derived based on properties of resolvent positive operators [4, 5, 6]. In the time varying case, criteria for exponential stability of differential equations defined by a Lyapunov type operator valued function were obtained in [8, 7] for the finite dimensional case and in [11] for infinite dimensional case. The derived criteria are expressed in terms of existence of some global bounded and uniform positive solutions of some suitable affine differential equations. It is known that in the case when the operator valued function defining the Lyapunov type differential equation is either periodic function or constant function, the unique global bounded solution of the affine differential equations involved in study of exponential stability criteria, is also periodic or constant function. Therefore, in this important special cases, the existence of a positive definite solution may be tested by numerical computations. Usually, to a system of stochastic linear differential equations one may associate two kinds of Lyapunov type operator valued functions which define two types of differential equations on some linear spaces of symmetric matrices: a forward differential equation and a backward differential equation. In the case of finite dimensional linear space of symmetric matrices one may introduce in a natural way an inner product which induces a Hilbert space structure. In this case the two types of Lyapunov type operators are interconnected, one of them being the adjoint operator of the other. A detailed study of linear differential equations defined by the Lyapunov type operator valued functions on a finite dimensional linear space of symmetric matrices may be found in Chapter 2 in [9]. The goal of the present paper is to study the Lyapunov type operator valued functions on an infinite dimensional linear space of bounded sequences of symmetric matrices. Such linear spaces can be organized as infinite dimensional ordered Banach spaces. We shall prove that under some additional assumptions we may identify some suitable subspaces which may be organized as infinite dimensional ordered Hilbert spaces where the two Lyapunov type linear operators are interconnected. More precisely, the restriction of the Lyapunov type operator defining a backward differential equation coincides with the adjoint of the Lyapunov operator defining a forward differential equation. Finally, we shall introduce a concept of detectability for Lyapunov type operators and we shall derive a new criterion for exponential stability for differential equations defined by the Lyaponov type operator valued functions.

2 Linear differential equations with positive evolution on an ordered Banach space

2.1 Linear evolution operators on ordered Banach spaces

Let $(\mathcal{X}, \|\cdot\|)$ be a real Banach space. Let $\mathcal{I} \subset \mathbf{R}$ be an interval of real numbers. Let $\mathcal{L} : \mathcal{I} \to \mathbf{B}(\mathcal{X})$ be a strongly continuous operator valued function. This means that for each $x \in \mathcal{X}$ the vector valued function $t \to \mathcal{L}(t)x$ is continuous on $\mathcal{I}$.

We consider the linear differential equation on $\mathcal{X}$:

$$\frac{d}{dt}x(t) = \mathcal{L}(t)x(t). \tag{1}$$

Based on the developments in Chapter 3 of [3] we deduce that for each $(t_0, x_0) \in \mathcal{I} \times \mathcal{X}$ there exists a unique C^1-function $x(\cdot; t_0, x_0) : \mathcal{I} \to \mathcal{X}$ satisfying (1) and the initial condition $x(t_0; t_0, x_0) = x_0$.

In Chapter 3 of [3] it is shown that there exists an operator valued function $T_{\mathcal{L}} : \mathcal{I} \times \mathcal{I} \to \mathbf{B}(\mathcal{X})$ with the property that $x(t; t_0, x_0) = T_{\mathcal{L}}(t, t_0)x_0$ for all $t, t_0 \in \mathcal{I}$ and $x_0 \in \mathcal{X}$. The operator valued function $(t, \tau) \to T_{\mathcal{L}}(t, \tau)$ or $T_{\mathcal{L}}(t, \tau)$ for shortness is named *the linear evolution operator on $\mathcal{X}$* defined by the operator valued function $\mathcal{L}(\cdot)$ or, equivalently, the linear evolution operator defined on $\mathcal{X}$ by the linear differential equation (1).

Often, we shall write $T(t,\tau)$ instead of $T_{\mathcal{L}}(t,\tau)$ if confusions are not possible.

Remark 1 *A linear evolution operator $T(t,\tau)$ on a Banach space $\mathcal{X}$ has the properties (see [3] Chapter 3 for details).*

(i) *$t \to T(t,\tau)$ is the unique solution of the problem with given initial value on $\mathbf{B}(\mathcal{X})$*

$$\frac{d}{dt}X(t) = \mathcal{L}(t)X(t), \quad X(\tau) = I_{\mathcal{X}}$$

where $I_{\mathcal{X}}$ is the identity operator on $\mathcal{X}$. More precisely,

$$\frac{d}{dt}T(t,\tau) = \mathcal{L}(t)T(t,\tau), t \in \mathcal{I} \qquad T(\tau,\tau) = I_{\mathcal{X}}. \tag{2}$$

(ii) *$\tau \to T(t,\tau) : \mathcal{I} \to \mathbf{B}(\mathcal{X})$ satisfies*

$$\frac{d}{d\tau}T(t,\tau) = -T(t,\tau)\mathcal{L}(\tau), \quad \forall \tau \in \mathcal{I}. \tag{3}$$

(iii)

$$T(t,\tau)T(\tau,s) = T(t,s), \quad (\forall) \ t,\tau,s \in \mathcal{I}. \tag{4}$$

(iv) *For each $(t,\tau) \in \mathcal{I} \times \mathcal{I}$, the operator $T(t,\tau)$ is invertible and $T^{-1}(t,\tau) \in \mathbf{B}(\mathcal{X})$. More precisely, we have $T^{-1}(t,\tau) = T(\tau,t)$.*

(v) $\|T(t,\tau)\| \leq e^{|\int_\tau^t \|\mathcal{L}(s)\| ds|}$, $\forall \ t,\tau \in \mathcal{I}$.

(vi) *If $\mathcal{L}(t) = \mathcal{L} \in \mathbf{B}(\mathcal{X})$ then $T(t,\tau) = e^{\mathcal{L}(t-\tau)}$, where $e^{\mathcal{L}t} = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathcal{L}^k$. This series is convergent in the topology induced by the operator norm uniformly on any compact subinterval of $\mathbf{R}$.*

(vii) *If $\mathcal{I} = \mathbf{R}$ and $\mathcal{L}(t+\theta) = \mathcal{L}(t)$ for all $t \in \mathbf{R}$ and some $\theta > 0$ then $T(t+k\theta, \tau+k\theta) = T(t,\tau)$ for any $t,\tau \in \mathbf{R}$, $k \in \mathbf{Z}$.*

The strongly continuous operator valued function $\mathcal{L}(\cdot)$ defines also the linear differential equation

$$\frac{d}{dt}y(t) + \mathcal{L}(t)y(t) = 0. \tag{5}$$

Applying the results from Chapter 3 of [3] to the operator valued function $t \to -\mathcal{L}(t)$ we deduce that for each $(t_0, y_0) \in \mathcal{I} \times \mathcal{X}$ the linear differential equation (5) has a unique solution $y(\cdot;t_0,y_0):\mathcal{I}\to\mathcal{X}$ which satisfy the initial condition $y(t_0;t_0,y_0)=y_0$. In what follows we denote by $T^a_{\mathcal{L}}(t,t_0)$ the causal evolution operator generated by $-\mathcal{L}(t)$ and we call it *the anticausal linear evolution operator on* $\mathcal{X}$ generated by the operator valued function $\mathcal{L}(\cdot)$. One proves also that $y(t;t_0,y_0)=T^a_{\mathcal{L}}(t,t_0)y_0$ for all $(t;t_0,y_0)\in\mathcal{I}\times\mathcal{I}\times\mathcal{X}$ and therefore the evolution operator $T^a_{\mathcal{L}}(t,t_0):\mathcal{I}\times\mathcal{I}\to\mathbf{B}(\mathcal{X})$ will be also called *the anticausal linear evolution operator on* $\mathcal{X}$ generated by the linear differential equation (5).

In the sequel, we shall write $T^a(t,t_0)$ instead of $T^a_{\mathcal{L}}(t,t_0)$, if confusions are not possible.

Remark 2 *Many of the assertions of Remark 1 remain valid if the causal linear evolution operator $T(t,\tau)$ is replaced by the anticausal linear evolution operator $T^a(t,\tau)$.*

In the case of anticausal linear evolution operator, the statements (i), (ii), (vi) from Remark 1 become:

$(i^{'})$ for each $\tau\in\mathcal{I}$, $t\to T^a(t,\tau)$ satisfies the linear differential equation on $\mathbf{B}(\mathcal{X})$:

$$\frac{d}{dt}T^a(t,\tau)=-\mathcal{L}(t)T^a(t,\tau) \tag{6}$$

and the initial condition $T^a(\tau,\tau)=I_{\mathcal{X}}$.

$(ii^{'})$ for each $t\in\mathcal{I}$, $\tau\to T^a(t,\tau)$ satisfies:

$$\frac{d}{d\tau}T^a(t,\tau)=T^a(t,\tau)\mathcal{L}(\tau). \tag{7}$$

$(vi^{'})$ If $\mathcal{L}(t)=\mathcal{L}\in\mathbf{B}(\mathcal{X})$, $t\in\mathbf{R}$, then, $T^a_{\mathcal{L}}(t,\tau)=e^{\mathcal{L}(\tau-t)}$, $\forall\ t,\tau\in\mathbf{R}$.

Beside the linear differential equations (1) and (5) respectively, we associate the following affine differential equations:

$$\frac{d}{dt}x(t)=\mathcal{L}(t)x(t)+f(t) \tag{8}$$

and

$$\frac{d}{dt}y(t)+\mathcal{L}(t)y(t)+g(t)=0 \tag{9}$$

where $\mathcal{L}(\cdot)$ is an operator valued function as before, and $f:\mathcal{I}\to\mathcal{X}$, $g:\mathcal{I}\to\mathcal{X}$ are continuous vector valued functions.

The solutions of (8) and (9) have the following representation formulae:

$$x(t;t_0,x_0) = T(t,t_0)x_0 + \int_{t_0}^{t} T(t,s)f(s)ds \tag{10}$$

for all $t \in [t_0,\infty) \cap \mathcal{I}$, $x_0 \in \mathcal{X}$ and

$$y(t;t_0,y_0) = T^a(t,t_0)y_0 + \int_{t}^{t_0} T^a(t,s)g(s)ds \tag{11}$$

for all $t \in (-\infty;t_0] \cap \mathcal{I}$, $y_0 \in \mathcal{X}$.

In the development from this paper, the affine differential equation of type (8) will be called *forward affine differential equation* while the affine differential equations of type (9) will be named *backward affine differential equations.*

Remark 3 *If $y(t)$ is a solution of the backward affine differential equation*

$$\frac{d}{dt}y(t) + \mathcal{L}(t)y(t) + g(t) = 0$$

then $\hat{x}(t)$ defined by $\hat{x}(t) = y(-t)$ is a solution of the forward affine equation $\frac{d}{dt}x(t) = \hat{\mathcal{L}}(t)x(t) + \hat{f}(t)$ where $\hat{\mathcal{L}}(t) = \mathcal{L}(-t)$ and $\hat{f}(t) = g(-t)$.

Moreover if $T^a(t,t_0)$ is the anticausal evolution operator defined by the operator valued function $\mathcal{L}(\cdot)$ then $\hat{T}(t,t_0)$ defined by

$$\hat{T}(t,t_0) = T^a(-t,-t_0), \ \ \forall t,t_0 \in \hat{\mathcal{I}} = \{t \in \mathbf{R}; -t \in \mathcal{I}\} \tag{12}$$

is the causal evolution operator defined by the operator valued function $\hat{\mathcal{L}} : \hat{\mathcal{I}} \to \mathbf{B}(\mathcal{X})$, $\hat{\mathcal{L}}(t) = \mathcal{L}(-t)$.

2.2 Linear differential equations with positive evolutions on ordered Banach spaces

Let $(\mathcal{X}, \|\cdot\|)$ be a real Banach space ordered by a solid, closed, normal, convex cone $\mathcal{X}_+$.

Let $\mathcal{L} : \mathcal{I} \to \mathbf{B}(\mathcal{X})$ be a strongly continuous operator valued function.

Definition 1 *We say that the operator valued function $\mathcal{L}(\cdot)$ generates:*

(i) a causal positive evolution on $\mathcal{X}$*, or a positive evolution (for shortness) if $T_{\mathcal{L}}(t,t_0)\mathcal{X}_+ \subset \mathcal{X}_+$, for all $t \geq t_0$, $t,t_0 \in \mathcal{I}$.*

(ii) an anticausal positive evolution on $\mathcal{X}$*, if $T^a_{\mathcal{L}}(t,t_0)\mathcal{X}_+ \subset \mathcal{X}_+$, for all $t \leq t_0$, $t,t_0 \in \mathcal{I}$.*

In other words, the operator valued function $\mathcal{L}(\cdot)$ generates a positive evolution on $\mathcal{X}$ if the solutions of the linear differential equation (1) have the property that $x(t;t_0,x_0) \in \mathcal{X}_+$ for all $t \geq t_0$, $t,t_0 \in \mathcal{I}$ if $x_0 \in \mathcal{X}_+$. In this case we shall say that the linear differential equation (1) defines a positive evolution on $\mathcal{X}$.

Similarly the operator valued function $\mathcal{L}(\cdot)$ generates an anticausal positive evolution on $\mathcal{X}$ if and only if the solutions of linear differential equation (5) have the property that $y(t;t_0,y_0) \in \mathcal{X}_+$, for all $t \leq t_0$, $t,t_0 \in \mathcal{I}$, if $y_0 \in \mathcal{X}_+$. In this case, we shall say that the linear differential equation (5) defines an anticausal positive evolution on $\mathcal{X}$.

Based on (12) we obtain:

Corollary 1 *Let $\mathcal{L} : \mathcal{I} \to \mathbf{B}(\mathcal{X})$ be a strongly continuous operator valued function and $\hat{\mathcal{L}}(t) = \mathcal{L}(-t)$, $t \in \hat{\mathcal{I}}$. Then the operator valued function $\mathcal{L}(\cdot)$ defines an anticausal positive evolution if and only if the operator valued function $\hat{\mathcal{L}}(\cdot)$ generates a causal positive evolution on $\mathcal{X}$.*

The next result was proved in [8] for the finite dimensional case. In infinite dimensions, the proof can be found in [11].

Corollary 2 *Let $\mathcal{L}(\cdot), \Pi(\cdot)$ be two strongly continuous operator valued functions defined on $\mathcal{I}$, taking values in $\mathbf{B}(\mathcal{X})$. Assume that $\Pi(t) \geq 0$ for all $t \in \mathcal{I}$. Then the following are true:*

(i) *If $\mathcal{L}(\cdot)$ generates a positive evolution on $\mathcal{X}$ then $\mathcal{L}(\cdot)+\Pi(\cdot)$ generates a positive evolution on $\mathcal{X}$.*

(ii) *If $\mathcal{L}(\cdot)$ generates an anticausal positive evolution on $\mathcal{X}$, then $\mathcal{L}(\cdot) + \Pi(\cdot)$ generates an anticausal positive evolution on $\mathcal{X}$.*

(iii) $\Pi(\cdot)$ *generates both a causal positive evolution and anticausal positive evolution on $\mathcal{X}$.*

Definition 2 (i) *We say that the zero state equilibrium of the linear differential equation (1) is exponentially stable, or equivalently, the operator* *valued function $\mathcal{L}(\cdot)$ generates an exponentially stable evolution if there exist the constants $\beta \geq 1$, $\alpha > 0$ such that*

$$\|T(t,t_0)\| \leq \beta e^{-\alpha(t-t_0)} \tag{13}$$

for all $t \geq t_0$, $t, t_0 \in \mathcal{I}$.

(ii) *We say that the zero state equilibrium of the linear differential equation (5) is anticausal exponentially stable, or equivalently, the operator valued function $\mathcal{L}(\cdot)$ generates an anticausal exponentially stable evolution on $\mathcal{X}$ if there exist the constants $\beta \geq 1$, $\alpha > 0$ such that*

$$\|T^a(t,t_0)\| \leq \beta e^{\alpha(t-t_0)} \tag{14}$$

for all $t \leq t_0$, $t, t_0 \in \mathcal{I}$.

Since both (1) and (5) are linear differential equations we will often say that the linear differential equation (1) is exponentially stable and the linear differential equation (5) is anticausal exponentially stable, respectively if (13) and (14), respectively, are fulfilled.

It is worth mentioning that under the considered assumptions for any $\xi \in Int\mathcal{X}_+$ the corresponding Minkovski operator norm $\|\cdot\|_\xi$ is equivalent with the norm $\|\cdot\|$ (see the Appendix for the definition of $\|\cdot\|_\xi$). Therefore, the above definition may be stated in terms of the operator norm $\|\cdot\|_\xi$ for some $\xi \in Int\mathcal{X}_+$.

Based on the identity (12) together with the Definition 2 we obtain:

Corollary 3 *Let $\mathcal{L} : \mathcal{I} \to \mathbf{B}(\mathcal{X})$ be a strongly continuous operator valued function and $\hat{\mathcal{L}}(t) = \mathcal{L}(-t)$, $t \in \hat{\mathcal{I}} = \{t \in \mathbf{R}; -t \in \mathcal{I}\}$. Then the operator valued function $\mathcal{L}(\cdot)$ defines an anticausal exponentially stable evolution if and only if the operator valued function $\hat{\mathcal{L}}(\cdot)$ generates a causal exponentially stable evolution.*

The above corollary allows us to derive criteria for anticausal exponential stability of a linear differential equation defined by an operator valued function $\mathcal{L}(\cdot)$ directly from the criteria for causal exponential stability for the linear differential equation defined by the operator valued function $\hat{\mathcal{L}}(\cdot)$.

Criteria for exponential stability of differential equations with positive evolution on an ordered Banach space are derived in [11].

2.3 The case of differential equations with positive evolution on ordered Hilbert spaces

Throughout this subsection $(\mathcal{X}; \langle \cdot, \cdot \rangle)$ is a real Hilbert space, ordered by the ordering "$\leq$" induced by the closed, solid, selfdual, convex cone. Based on Proposition 2.4 in [10] we deduce that the norm $\| \cdot \|$, induced by the inner product is monotone with respect to the cone $\mathcal{X}_+$. So, $\mathcal{X}_+$ is a normal cone with a constant $\tilde{b} = 1$ (see Definition 5 from the Appendix).

Let $\xi \in Int\mathcal{X}_+$ be fixed and $| \cdot |_\xi$ be the corresponding Minkovski norm. As we can see, applying Proposition 4 and Proposition 5 one deduces that $| \cdot |_\xi$ is equivalent with the norm $\| \cdot \|$ of the Hilbert space $\mathcal{X}$. It is known that, if $T \in \mathbf{B}(\mathcal{X})$ and T^* is its adjoint operator, then, $\|T^*\| = \|T\|$. The equality $\|T^*\|_\xi = \|T\|_\xi$ is not, in general, true. However, one can prove, via the equivalence of the operator norms $\| \cdot \|$ and $\| \cdot \|_\xi$, that there exist positive constants, $\tilde{c}_1, \tilde{c}_2$ such that

$$\tilde{c}_1 \|T\|_\xi \leq \|T^*\|_\xi \leq \tilde{c}_2 \|T\|_\xi, \ \ \forall \, T \in \mathbf{B}(\mathcal{X}). \tag{15}$$

Let $\mathcal{L} : \mathcal{I} \to \mathbf{B}(\mathcal{X})$ be a continuous operator valued function, $\mathcal{I} \subset \mathbf{R}$ being a right unbounded interval. In this case $t \to \mathcal{L}^*(t) : \mathcal{I} \to \mathbf{B}(\mathcal{X})$ is also a continuous operator valued function. It is known that if $T(t, \tau), \ \ t, \tau \in \mathcal{I}$, is the linear evolution operator defined by the linear differential equation

$$\frac{d}{dt} x(t) = \mathcal{L}(t) x(t), \ \ t \in \mathcal{I} \tag{16}$$

then, $\tau \to T^*(t, \tau)$ verifies

$$\begin{aligned} \frac{\partial}{\partial \tau} T^*(t, \tau) &= -\mathcal{L}^*(\tau) T^*(t, \tau) \\ T^*(t, t) &= I_{\mathcal{X}}. \end{aligned} \tag{17}$$

So we have:

$$T^*(t, \tau) = T^a_{\mathcal{L}^*}(\tau, t) \tag{18}$$

$\forall \ \ t, \tau \in \mathcal{I}$, where $T^a_{\mathcal{L}^*}(\tau, t)$ is the anticausal linear evolution operator defined by the operator valued function $\mathcal{L}^*(\cdot)$. This means that, $T^a_{\mathcal{L}^*}(\tau, t)$ is a linear evolution operator associated to the linear differential equation

$$\frac{d}{d\tau} y(\tau) = -\mathcal{L}^*(\tau) y(\tau). \tag{19}$$

Combining (15), (18) one obtains the following result:

Proposition 1 *If $\mathcal{L} : \mathcal{I} \to \mathbf{B}(\mathcal{X})$ is a continuous operator valued function, then the following statements are true:*

(i) *The operator valued function $\mathcal{L}(\cdot)$ defines a causal positive evolution on $\mathcal{X}$, iff the operator valued function $\mathcal{L}^*(\cdot)$ defines an anticausal positive evolution on $\mathcal{X}$.*

(ii) *The operator valued function $\mathcal{L}(\cdot)$ defines an exponentially stable evolution on $\mathcal{X}$ iff the operator valued function $\mathcal{L}^*(\cdot)$ defines an anticausal exponentially stable evolution on $\mathcal{X}$.*

Using criteria for the anticausal exponential stability of the linear differential equation defined by the operator valued function $\mathcal{L}^*(\cdot)$ and taking into account the equality (18) and Proposition 1 one obtains a set of criteria for the causal exponential stability of the linear differential equation (16). For details see [11]. Such criteria are specific to the linear differential equations with positive evolution on ordered Hilbert spaces.

3 Ordered vector spaces of sequences of symmetric matrices

This section collects several examples of real ordered Banach spaces. As usual $|x|$ stands for the euclidian norm of a vector $x \in \mathbf{R}^n$, that is, $|x| = (x^T x)^{1/2}$. For a matrix $A \in \mathbf{R}^{m\times n}$, $|A|$ stands for the matrix norm induced by the euclidian norm $|\cdot|$, that is

$$|A| = \sup_{|x|\leq 1} |Ax| \tag{20}$$

Also, we shall use the notation $|A|_2$ for the Frobenius norm of the matrix A, i.e.

$$|A|_2 = \left(Tr[A^T A]\right)^{1/2} \tag{21}$$

where $Tr[\cdot]$ stands for the trace operator. Beside the two norms introduced before, we shall use also the norm

$$|A|_1 = Tr\left[(A^T A)^{1/2}\right] \tag{22}$$

where $(A^T A)^{1/2}$ is the unique positive semidefinite matrix X such that $X^2 = A^T A$.

Let $\mathcal{S}_n \subset \mathbf{R}^{n\times n}$ be the linear subspace of symmetric matrices of size $n \times n$, that is $S \in \mathcal{S}_n$ iff $S = S^T$.

The restrictions of the norm (20)-(22) to the subspace $\mathcal{S}_n$ take the equivalent form:

$$|S| = max\{|\lambda|; \lambda \in \Lambda(S)\} = \sup_{|x|\leq 1}\{|x^T Sx|\} \tag{23}$$

$$|S|_2 = \left(\sum_{i=1}^{n} \lambda_i^2\right)^{1/2} \tag{24}$$

$$|S|_1 = \sum_{i=1}^{n} |\lambda_i| \tag{25}$$

where $\lambda_1, ..., \lambda_n \in \Lambda(S)$ with $\Lambda(S)$ is the spectrum of the matrix S.

For a matrix $S \in \mathcal{S}_n$ the following hold:

$$|S| \leq |S|_2 \leq |S|_1 \leq n|S|. \tag{26}$$

Throughout this paper $\mathcal{D}$ denotes either the set $\{1, 2, ..., d\}$ or the set $\mathbf{Z}_+$.

3.1 The space $\mathcal{S}_n^{\mathcal{D}}$

Let $\mathcal{X} = \mathcal{S}_n^{\mathcal{D}} = \ell^\infty\{\mathcal{D}, \mathcal{S}_n\}$ be the linear space of the bounded sequences of symmetric matrices, that is

$$\ell^\infty\{\mathcal{D}, \mathcal{S}_n\} = \left\{\mathbf{X} = \{X(i)\}_{i\in\mathcal{D}} | X(i) \in \mathcal{S}_n, i \in \mathcal{D},\ \sup_{i\in\mathcal{D}} |X(i)| < +\infty\right\}.$$

The space $\mathcal{S}_n^{\mathcal{D}}$ equipped with the norm

$$\|\mathbf{X}\|_\infty = \sup_{i\in\mathcal{D}} |X(i)| \tag{27}$$

is a real Banach space. On $\mathcal{S}_n^{\mathcal{D}}$ we consider the ordering induced by the cone $\mathcal{X}_+ = \mathcal{S}_{n+}^{\mathcal{D}} = \ell^\infty\{\mathcal{D}, \mathcal{S}_{n+}\}$ where

$$\ell^\infty\{\mathcal{D}, \mathcal{S}_{n+}\} = \{\mathbf{X} = \{X(i)\}_{i\in\mathcal{D}}; X(i) \geq 0, i \in \mathcal{D}\}. \tag{28}$$

Here $X(i) \geq 0$ means that $X(i)$ is positive semidefinite. One verifies that $\mathcal{X}_+$ is a closed, solid, convex cone. Its interior $Int\mathcal{X}_+$ consists of the subset of the sequences $\mathbf{X} = \{X(i), i \in \mathcal{D}; X(i) \geq \delta I_n, \forall i \in \mathcal{D}$ *for some* $\delta > 0$ *independent of* $i\}$.

Based on the monotonicity of the norm $|\cdot|$ on $\mathcal{S}_n$ one obtains that $\|\cdot\|_\infty$ introduced by *(27)* is monotone with respect to the cone $\mathcal{X}_+$. Hence $\mathcal{X}_+$ is a normal cone.

Further we shall use $\mathcal{S}_n^d$ instead of $\mathcal{S}_n^{\mathcal{D}}$ and $\mathcal{S}_{n+}^d$ for $\mathcal{S}_{n+}^{\mathcal{D}}$ when $\mathcal{D} = \{1, 2, .., d\}$, while $\mathcal{S}_n^\infty$ is used for $\mathcal{S}_n^{\mathcal{D}}$ when $\mathcal{D} = \mathbf{Z}_+$. In the last case, $\mathcal{S}_{n+}^\infty$ stands for the convex cone $\ell^\infty(\mathbf{Z}_+, \mathcal{S}_{n+})$. It is obvious that $(\mathcal{S}_n^d, \|\cdot\|_\infty)$ is a finite dimensional real ordered Banach space, while $(\mathcal{S}_n^\infty, \|\cdot\|_\infty)$ is an infinite dimensional real ordered Banach space.

Specializing the results from Theorem 4, Proposition 4, Proposition 5, to the ordered Banach space $\mathcal{S}_n^{\mathcal{D}}$ we obtain:

Corollary 4 *In the case of the Banach space* $(\mathcal{S}_n^{\mathcal{D}}, \|\cdot\|_\infty)$*, the following hold:*

(i) If $\mathcal{D} = \{1, 2, ..., d\}$ *and* $J^d = \underbrace{(I_n, I_n, ..., I_n)}_{d} \in Int\mathcal{S}_{n+}^d$ *then the Minkovski norm defined by (112) for* $\xi = J^d$ *is:*

$$|X|_{J^d} = \|X\|_\infty \tag{29}$$

for all $X = (X(1), X(2), ..., X(d)) \in \mathcal{S}_n^d$.

(ii) If $\mathcal{D} = \mathbf{Z}_+$ *and* $J^\infty = (I_n, I_n, ..., I_n, ...) \in Int\mathcal{S}_{n+}^\infty$ *then the Minkovski norm introduced by (112) for* $\xi = J^\infty$ *is given by*

$$|\mathbf{X}|_{J^\infty} = \|\mathbf{X}\|_\infty \tag{30}$$

for all $\mathbf{X} = \{X(i)\}_{i \in \mathbf{Z}_+}$.

Remark 4 *For the sake of brevity we shall use* $|X|$ *and* $|\mathbf{X}|$*, respectively, instead of* $|X|_{J^d}$ *and* $|\mathbf{X}|_{J^\infty}$*, respectively, for the Minkovski norms defined by* (29) *and* (30)*, respectively, if no confusions are possible.*

3.2 The space $\ell^1(\mathcal{D}, \mathcal{S}_n)$

Let $\mathcal{X} = \ell^1(\mathcal{D}, \mathcal{S}_n)$, where

$$\ell^1(\mathcal{D}, \mathcal{S}_n) = \{\mathbf{X} = \{X(i)\}_{i \in \mathcal{D}} \subset \mathcal{S}_n; \sum_{i \in \mathcal{D}} |X(i)|_1 < \infty\}.$$

Taking

$$\|\mathbf{X}\|_1 = \sum_{i\in\mathcal{D}} |X(i)|_1 \tag{31}$$

one obtains that $(\mathcal{X}, \|\cdot\|_1)$ is a real Banach space.

On the Banach space $(\mathcal{X}, \|\cdot\|_1)$ we consider the order relation induced by the convex cone $\mathcal{X}_+ = \ell^1(\mathcal{D}, \mathcal{S}_{n+}) = \{\mathbf{X} \in \ell^1(\mathcal{D}, \mathcal{S}_n); \mathbf{X} = \{X(i)\}_{i\in\mathcal{D}}, X(i) \geq 0\}$. It is a closed convex cone. In the case $\mathcal{D} = \{1, 2, ..., d\}$, $\ell^1(\mathcal{D}, \mathcal{S}_n)$ coincides with $\mathcal{S}_n^d$ and $\ell^1(\mathcal{D}, \mathcal{S}_{n+})$ coincides with $\mathcal{S}_{n+}^d$. In the case $\mathcal{D} = \mathbf{Z}_+$, $\mathcal{X} = \ell^1(\mathbf{Z}_+, \mathcal{S}_n) \subset \mathcal{S}_n^\infty$. The convex cone $\ell^1(\mathbf{Z}_+, \mathcal{S}_{n+})$ has an empty interior. Finally, let us remark that based on *(26)* we may introduce a new norm on $\mathcal{X}$, by

$$\tilde{\|}\mathbf{X}\tilde{\|}_1 = \sum_{i\in\mathcal{D}} |X(i)|. \tag{32}$$

Based on *(26)*, *(32)* we deduce that the norms $\|\cdot\|_1$ and $\tilde{\|}\cdot\tilde{\|}_1$ are equivalent, more precisely we have:

$$\tilde{\|}\mathbf{X}\tilde{\|}_1 \leq \|\mathbf{X}\|_1 \leq n\tilde{\|}\mathbf{X}\tilde{\|}_1 \tag{33}$$

for all $\mathbf{X} = \{X(i)\}_{i\in\mathcal{D}} \in \ell^1(\mathcal{D}, \mathcal{S}_n)$.

3.3 The space $\ell^2(\mathcal{D}, \mathcal{S}_n)$

Let $\mathcal{X} = \ell^2(\mathcal{D}, \mathcal{S}_n) = \{\mathbf{X} = \{X(i)\}_{i\in\mathcal{D}} \subset \mathcal{S}_n; \sum\limits_{i\in\mathcal{D}} (|X(i)|_2)^2 < \infty\}$. On $\ell^2(\mathcal{D}, \mathcal{S}_n)$ we introduce the inner product:

$$\langle \mathbf{X}, \mathbf{Y}\rangle_2 = \sum_{i\in\mathcal{D}} Tr[X(i)Y(i)] \tag{34}$$

for all $\mathbf{X} = \{X(i)\}_{i\in\mathcal{D}}$, $\mathbf{Y} = \{Y(i)\}_{i\in\mathcal{D}}$ from $\ell^2(\mathcal{D}, \mathcal{S}_n)$.

To show that the sum from the right hand side of *(34)* is convergent, let us remark that

$$\sum_{i\in\mathcal{D}} Tr[X(i)Y(i)] = \frac{1}{4}\sum_{i\in\mathcal{D}} \left\{|X(i) + Y(i)|_2^2 - |X(i) - Y(i)|_2^2\right\} =$$

$$\frac{1}{4}\left\{\sum_{i\in\mathcal{D}} |X(i) + Y(i)|_2^2 - \sum_{i\in\mathcal{D}} |X(i) - Y(i)|_2^2\right\} \in \mathbf{R}$$

because

$$\sum_{i\in\mathcal{D}}|X(i)+Y(i)|_2^2<+\infty$$

$$\sum_{i\in\mathcal{D}}|X(i)-Y(i)|_2^2<+\infty.$$

One may check that the inner product $\langle\cdot,\cdot\rangle_2$ induces a real Hilbert space structure on $\ell^2(\mathcal{D},\mathcal{S}_n)$. Set

$$\|\mathbf{X}\|_2=\langle\mathbf{X},\mathbf{X}\rangle_2^{1/2}. \tag{35}$$

On the space $\ell^2(\mathcal{D},\mathcal{S}_n)$ we consider the order relation induced by the convex cone $\mathcal{X}_+=\ell^2(\mathcal{D},\mathcal{S}_{n+})=\{\mathbf{X}=\{X(i)\}_{i\in\mathcal{D}}\in\ell^2(\mathcal{D},\mathcal{S}_n);X(i)\geq 0,i\in\mathcal{D}\}$. The cone $\ell^2(\mathcal{D},\mathcal{S}_{n+})$ is a closed cone. If $\mathcal{D}=\mathbf{Z}_+$ its interior is empty.

Remark 5 *In the case $\mathcal{D}=\{1,2,...,d\}$ the linear spaces $\ell^\infty(\mathcal{D},\mathcal{S}_n)$, $\ell^1(\mathcal{D},\mathcal{S}_n)$, $\ell^2(\mathcal{D},\mathcal{S}_n)$ coincide with $\mathcal{S}_n^d=\underbrace{\mathcal{S}_n\times\mathcal{S}_n\times...\times\mathcal{S}_n}_{d}$.*

On $\mathcal{S}_n^d$ we have three norms:
$\|\cdot\|_\infty$ introduced via (27),
$\|\cdot\|_1$ defined in (31) and
$\|\cdot\|_2$ introduced by (35) for $\mathcal{D}=\{1,2,...,d\}$.

We have $\|S\|_\infty\leq\|S\|_2\leq\|S\|_1\leq nd\|S\|_\infty$ for all $S\in\mathcal{S}_n^d$. The convex cone $\ell^2(\mathcal{D},\mathcal{S}_{n+})$ coincides with the convex cone $\mathcal{S}_{n+}^d=\underbrace{\mathcal{S}_{n+}\times\mathcal{S}_{n+}...\mathcal{S}_{n+}}_{d}$ if $\mathcal{D}=\{1,2,...,d\}$. The cone $\mathcal{S}_{n+}^d$ is a closed, solid, selfdual convex cone. It is selfdual with respect to the inner product

$$\langle X,Y\rangle=\sum_{i=1}^{d}Tr[X(i)Y(i)] \tag{36}$$

for all $X=(X(1),X(2),...,X(d)),Y=(Y(1),Y(2),...,Y(d))\in\mathcal{S}_n^d$ which is the special form of (34) for the case $\mathcal{D}=\{1,2,...,d\}$.

In the case $\mathcal{D}=\mathbf{Z}_+$ we have the following result.

Proposition 2 *If $\ell^1(\mathbf{Z}_+,\mathcal{S}_n)$ and $\ell^2(\mathbf{Z}_+,\mathcal{S}_{n+})$ are the linear spaces introduced in a previous subsections for $\mathcal{D}=\mathbf{Z}_+$ then*

$$\ell^1(\mathbf{Z}_+,\mathcal{S}_n)\subset\ell^2(\mathbf{Z}_+,\mathcal{S}_{n+}).$$

Proof is similar to the one given in [19] for the case of sequences of nuclear operators and Hilbert-Schmith operators.

4 Lyapunov type linear differential equations on the space $\mathcal{S}_n^{\mathcal{D}}$

In this section we emphases several properties of an important class of operator valued functions on the Banach spaces $\mathcal{S}_n^{\mathcal{D}}$ and $\ell^1(\mathbf{Z}_+, \mathcal{S}_n)$, respectively. These operators extend to this framework the well known Lyapunov operators and they will play an important role in the characterization of the exponential stability in mean square of stochastic linear differential equation.

4.1 Extended Lyapunov operators

Let $\mathcal{M}_{mn}^{\mathcal{D}} := \ell^\infty(\mathcal{D}, \mathbf{R}^{m\times n})$ be the space of the bounded sequences of matrices $A = \{A(i)\}_{i\in\mathcal{D}}$ where $A(i) \in \mathbf{R}^{m\times n}$. We introduce the norm $\|A\|_\infty = \sup\limits_{i\in\mathcal{D}} |A(i)|$ where $|A(i)|$ is defined by (20). One obtains that $(\mathcal{M}_{mn}^{\mathcal{D}}, \|\cdot\|_\infty)$ is a real Banach space. If $m = n$ we shall write $\mathcal{M}_n^{\mathcal{D}}$ instead of $\mathcal{M}_{nn}^{\mathcal{D}}$. If $\mathcal{D} = \mathbf{Z}_+$, $\mathcal{M}_{mn}^\infty$ and $\mathcal{M}_n^\infty$, respectively stand for $\mathcal{M}_{mn}^{\mathcal{D}}$ and $\mathcal{M}_n^{\mathcal{D}}$. It is obvious that $\mathcal{S}_n^{\mathcal{D}} \subset \mathcal{M}_n^{\mathcal{D}}$.

We make the following convention of notation:

(a) If $A = \{A(i)\}_{i\in\mathcal{D}} \in \mathcal{M}_{mn}^{\mathcal{D}}$, $X = \{X(i)\}_{i\in\mathcal{D}} \in \mathcal{M}_{np}^{\mathcal{D}}$, by $Y = AX$ we understand the sequence $Y = \{Y(i)\}_{i\in\mathcal{D}} \in \mathcal{M}_{mp}^{\mathcal{D}}$, $Y(i) = A(i)X(i)$, $i \in \mathcal{D}$.

(b) If $A = \{A(i)\}_{i\in\mathcal{D}} \in \mathcal{M}_{mn}^{\mathcal{D}}$ then $A^T = \{A^T(i)\}_{i\in\mathcal{D}} \in \mathcal{M}_{nm}^{\mathcal{D}}$.

Let $A : \mathcal{I} \to \mathcal{M}_n^{\mathcal{D}}$ be a continuous function. This means that $A(t) = \{A(t,i)\}_{i\in\mathcal{D}}$, where $t \to A(t,i)$ are matrix valued functions which are continuous on $\mathcal{I}$ uniformly with respect to $i \in \mathcal{D}$.

The extended Lyapunov operators associated to $A(t)$:

$$\mathcal{L}_A(t) : \mathcal{S}_n^{\mathcal{D}} \to \mathcal{S}_n^{\mathcal{D}},$$

$$\mathfrak{L}_A(t) : \mathcal{S}_n^{\mathcal{D}} \to \mathcal{S}_n^{\mathcal{D}},$$

are defined as follows

$$\mathcal{L}_A(t)X = A(t)X + XA^T(t) \tag{37}$$

$$\mathfrak{L}_A(t)X = A^T(t)X + XA(t) \tag{38}$$

for all $X = \{X(i)\}_{i\in\mathcal{D}} \in \mathcal{S}_n^{\mathcal{D}}$.

According to the notation introduced at the beginning of this subsection the i^{th} component of (37) and (38) respectively, is:

$$[\mathcal{L}_A(t)X](i) = A(t,i)X(i) + X(i)A^T(t,i)$$

$$[\mathfrak{L}_A(t)X](i) = A^T(t,i)X(i) + X(i)A(t,i)$$

$i \in \mathcal{D}, t \in \mathcal{I}$.

We deduce that $\|\mathcal{L}_A(t)X\|_\infty \leq 2\|A(t)\|_\infty \|X\|_\infty$ and $\|\mathfrak{L}_A(t)X\|_\infty \leq 2\|A(t)\|_\infty \|X\|_\infty$. Hence, $\mathcal{L}_A(t), \mathfrak{L}_A(t) \in \mathbf{B}(\mathcal{S}_n^{\mathcal{D}})$. Moreover $t \to \mathcal{L}_A(t)$ and $t \to \mathfrak{L}_A(t)$ are continuous functions in the topology induced by the operator norm.

Remark 6 *To be sure that the linear differential equations* (39), (44), *respectively, defined by $\mathcal{L}_A(t)$ and $\mathfrak{L}_A(t)$ on $\mathcal{S}_n^{\mathcal{D}}$ have nice properties, would be sufficient to assume that $t \to \mathcal{L}_A(t)$ and $t \to \mathfrak{L}_A(t)$ are strongly continuous operator valued functions. This means that for each $X \in \mathcal{S}_n^{\mathcal{D}}$, $t \to \mathcal{L}_A(t)X$ and $t \to \mathfrak{L}_A(t)X$ are continuous vector valued functions. If we take $X = \{X(i)\}_{i\in\mathcal{D}}$ with $X(i) = I_n$, $\forall i \in \mathcal{D}$ one obtains that $t \to A^T(t) + A(t)$ must be continuous. This condition is not far from our assumption that $t \to A(t)$ is a continuous function.*

Let us consider the extended Lyapunov equation

$$\frac{d}{dt}X(t) = \mathcal{L}_A(t)X(t), \quad t \in \mathcal{I}. \tag{39}$$

Let $T_A(t,t_0)$ $t, t_0 \in \mathcal{I}$ be the linear operator defined by

$$(T_A(t,t_0)X)(i) = \Phi_i(t,t_0)X(i)\Phi_i^T(t,t_0) \tag{40}$$

$\forall i \in \mathcal{D}$ and $X = \{X(i)\}_{i\in\mathcal{D}} \in \mathcal{S}_n^{\mathcal{D}}$, where $\Phi_i(t,t_0)$ is the fundamental matrix solution of the linear differential equation on $\mathbf{R}^n$:

$$\frac{d}{dt}x(t) = A(t,i)x(t).$$

This means that $t \to \Phi_i(t,t_0)$ verifies

$$\begin{aligned} \frac{d}{dt}\Phi_i(t,t_0) &= A(t,i)\Phi_i(t,t_0) \\ \Phi_i(t_0,t_0) &= I_n. \end{aligned} \tag{41}$$

Based on the convention of notations introduced before we may write (40) in the compact form:

$$T_A(t,t_0)X = \Phi(t,t_0)X\Phi^T(t,t_0) \tag{42}$$

for all $t, t_0 \in \mathcal{I}$, where $\Phi(t,t_0) = \{\Phi_i(t,t_0)\}_{i\in\mathcal{D}}$. If $\mathcal{D} = \{1,2,...,d\}$ one may check that $t \to \Phi(t,t_0)$ is a differentiable map and it satisfies:

$$\frac{d}{dt}\Phi(t,t_0) = A(t)\Phi(t,t_0), \quad \Phi(t_0,t_0) = J^d = (I_n...I_n).$$

By direct calculations one obtains from (42) that

$$\begin{aligned} \frac{d}{dt}T_A(t,t_0)X &= \mathcal{L}_A(t)T_A(t,t_0)X \\ T_A(t_0,t_0)X &= X \end{aligned} \tag{43}$$

for all $t, t_0 \in \mathcal{I}, X \in \mathcal{S}_n^d$. Therefore $T_A(t,t_0)$ defined by (40), or equivalently by (42) is just the linear evolution operator on $\mathcal{S}_n^d$ defined by the linear differential equation (39).

It remains to show that (40) defines also the linear evolution operator generated by (39) on $\mathcal{S}_n^\infty$. To this end, let us notice that

$$|\Phi_i(t,s)| \le e^{\gamma(t-s)}$$

for all $i \in \mathbf{Z}_+, t, s \in \mathcal{I}$, where $\gamma = \sup_{t\in\mathcal{I}} \|A(t)\|_\infty$. Using also the fact that $t \to A(t,i)$ are continuous functions uniformly with respect to $i \in \mathbf{Z}_+$ we deduce that

$$\lim_{h\to 0} \frac{1}{|h|}|\Phi_i(t+h,t_0) - \Phi_i(t,t_0) - hA(t,i)\Phi_i(t,t_0)| = 0$$

uniformly with respect to $i \in \mathbf{Z}_+$.

This shows that $t \to \Phi(t,t_0) : \mathcal{I} \to \mathcal{M}_n^\infty$ is a differentiable map and it satisfies:

$$\frac{d}{dt}\Phi(t,t_0) = \mathcal{L}_A(t)\Phi(t,t_0), \;\; \Phi(t_0,t_0) = J^\infty = (I_n...I_n...) \in \mathcal{S}_n^\infty.$$

Thus we may obtain that $T_A(t,t_0)$ defined by (42) for $\mathcal{D} = \mathbf{Z}_+$ is differentiable and satisfies (43).

Remark 7 *From* (40) *one sees that* $T_A(t,t_0)X \in \mathcal{S}_{n+}^{\mathcal{D}}$ *if* $X \in \mathcal{S}_{n+}^{\mathcal{D}}$. *This shows that the operator valued function* $\mathcal{L}_A(\cdot)$ *generates a positive evolution on the Banach space* $\mathcal{S}_n^{\mathcal{D}}$.

Changing $A(t,i)$ with $A^T(t,i)$ in (40), (41) one obtains that the operator valued function $\mathfrak{L}_A(\cdot)$ generates also positive evolution on the Banach space $\mathcal{S}_n^{\mathcal{D}}$. However, concerning the operator valued function $\mathfrak{L}_A(\cdot)$ we are interested by the anticausal evolution operator $T_A^a(t,t_0)$ defined by the linear differential equation

$$\frac{d}{dt}Y(t) + \mathfrak{L}_A(t)Y(t) = 0. \tag{44}$$

Reasoning as in the case of the equation (39) we may conclude that

$$(T_A^a(t,t_0)Y)(i) = \Phi_i^T(t_0,t)Y(i)\Phi_i(t_0,t) \tag{45}$$

for all $i \in \mathcal{D}$, $0 \leq t \leq t_0$, $Y = \{Y(i)\}_{i\in\mathcal{D}} \in \mathcal{S}_n^{\mathcal{D}}$.

From (45) one deduces that the operator valued function $\mathfrak{L}_A(\cdot)$ generates an anticausal positive evolution on the Banach space $\mathcal{S}_n^{\mathcal{D}}$.

4.2 Lyapunov-type differential equations on the space $\mathcal{S}_n^d$

Let $\mathcal{I} \subseteq \mathbf{R}$ be an interval and $A_k : \mathcal{I} \to \mathcal{M}_n^d$, $k = 0,\ldots,r$ be continuous functions

$$A_k(t) = (A_k(t,1),\ldots A_k(t,d)),\ k \in \{0,\ldots,r\},\ t \in \mathcal{I}.$$

Denote by $Q \in \mathbf{R}^{d\times d}$ a matrix which elements q_{ij} verify the condition

$$q_{ij} \geq 0 \text{ if } i \neq j. \tag{46}$$

For each $t \in \mathcal{I}$ we define the linear operator $\mathcal{L}(t), \mathfrak{L}(t) : \mathcal{S}_n^d \to \mathcal{S}_n^d$ by

$$\begin{aligned}(\mathcal{L}(t)S)(i) &= A_0(t,i)S(i) + S(i)A_0^T(t,i) \\ &\quad + \sum_{k=1}^{r} A_k(t,i)S(i)A_k^T(t,i) + \sum_{j=1}^{d} q_{ji}S(j),\end{aligned} \tag{47}$$

$$\begin{aligned}(\mathfrak{L}(t)S)(i) &= A_0^T(t,i)S(i) + S(i)A_0(t,i) \\ &\quad + \sum_{k=1}^{r} A_k^T(t,i)S(i)A_k(t,i) + \sum_{j=1}^{d} q_{ij}S(j)\end{aligned} \tag{48}$$

$i \in \mathcal{D}$, $S \in \mathcal{S}_n^d$. It is easy to see that $t \longmapsto \mathcal{L}(t)$ and $t \longmapsto \mathfrak{L}(t)$ are continuous operator valued functions.

One can see that (47) and (48) may be written as:

$$\mathcal{L}(t)S = \mathcal{L}_A(t)S + \Pi(t)S$$

$$\mathfrak{L}(t)S = \mathfrak{L}_A(t)S + \tilde{\Pi}(t)S$$

where $\mathcal{L}_A(t)$ and $\mathfrak{L}_A(t)$ are the extended Lyapunov operators introduced via (37) and (38), respectively, for $\mathcal{D} = \{1, 2, ..., d\}$ and $A(t, i) = A_0(t, i) + \frac{1}{2} q_{ii} I_n$,

$$(\Pi(t)S)(i) = \sum_{k=1}^{r} A_k(t, i)S(i)A_k^T(t, i) + \sum_{j=1, j \neq i}^{d} q_{ji} S(j),$$

$$(\tilde{\Pi}(t)S)(i) = \sum_{k=1}^{r} A_k^T(t, i)S(i)A_k(t, i) + \sum_{j=1, j \neq i}^{d} q_{ij} S(j).$$

Based on (46) one obtains that $\Pi(t)S \geq 0, \tilde{\Pi}(t)S \geq 0$ if $S \geq 0$. Therefore, combining Remark 7 and Corollary 2 (i) we conclude that the operator valued function $\mathcal{L}(\cdot)$ introduced via (47) generates a positive evolution on $\mathcal{S}_n^d$. Also, combining (45) and Corollary 2 (ii) we infer that the operator valued function $\mathfrak{L}(\cdot)$ defines an anticausal positive evolution on $\mathcal{S}_n^d$. Moreover, by direct calculation one obtains that $\mathfrak{L}(t) = \mathcal{L}^*(t)$ where $\mathcal{L}^*(t)$ is the adjoint operator with respect to the inner product (36) of $\mathcal{L}(t)$.

The Lyapunov operator $\mathcal{L}(t)$ defines the following linear differential equation on $\mathcal{S}_n^d$:

$$\frac{d}{dt} S(t) = \mathcal{L}(t) S(t), \ t \in \mathcal{I} \tag{49}$$

while, the linear operator $\mathfrak{L}(t)$ defines the following backward differential equation

$$\frac{d}{dt} S(t) + \mathfrak{L}(t) S(t) = 0, \ t \in \mathcal{I}. \tag{50}$$

Criteria for exponential stability of the zero solution of differential equations (49) and (50) may be found in Chapter 2 in [9].

4.3 Lyapunov-type differential equations on the space $\mathcal{S}_n^\infty$

4.3.1 Definition and basic properties

Let $A_k : \mathcal{I} \to \mathcal{M}_n^\infty, 0 \leq k \leq r$ be continuous and bounded functions. This means that $A_k(t) = \{A_k(t,i)\}_{i\in\mathbf{Z}_+}$ are such that $t \to A_k(t,i)$ are continuous functions on $\mathcal{I}$ uniformly with respect to $i \in \mathbf{Z}_+$ and $\sup_{t\in\mathcal{I}} \|A_k(t)\|_\infty < \infty$. Let $Q = (q_{ij})_{i,j\in\mathbf{Z}_+}$ be an infinite real matrix whose elements satisfy the conditions:

$$q_{ij} \geq 0, \quad \text{if} \quad i \neq j \tag{51}$$

and

$$\sup_{i\in\mathbf{Z}_+} (|q_{ii}| + \sum_{j=0, j\neq i}^{\infty} q_{ij}) = \nu < \infty. \tag{52}$$

It is worth mentioning that the conditions (51) and (52) are satisfied by the generator matrix of a standard homogeneous Markov process with an infinite countable number of states $(\eta(t), P, \mathbf{Z}_+)$ (see Section 7 in [11] for more details).

Based on the functions $t \to A_k(t,i)$ and the elements q_{ij} of the matrix Q, one constructs the operators $\mathcal{L}$ and $\mathfrak{L}$ by:

$$(\mathcal{L}(t)X)(i) = A_0(t,i)X(i) + X(i)A_0^T(t,i) + \sum_{k=1}^{r} A_k(t,i)X(i)A_k^T(t,i) + \sum_{j=0}^{\infty} q_{ji}X(j) \tag{53}$$

$$(\mathfrak{L}(t)X)(i) = A_0^T(t,i)X(i) + X(i)A_0(t,i) + \sum_{k=1}^{r} A_k^T(t,i)X(i)A_k(t,i) + \sum_{j=0}^{\infty} q_{ij}X(j) \tag{54}$$

for all sequences $X = \{X(i)\}_{i\in\mathbf{Z}_+}$.

Lemma 1 *If the real numbers q_{ij} satisfy conditions* (51) *and* (52) *then for each $t \in \mathcal{I}$, $\mathcal{L}(t) \in \mathbf{B}(\ell^1(\mathbf{Z}_+, \mathcal{S}_n))$ and $\mathfrak{L}(t) \in \mathbf{B}(\mathcal{S}_n^\infty)$.*

Proof. If $X \in \ell^1(\mathbf{Z}_+, \mathcal{S}_n)$ then one obtains via (32), (51)-(53) that:

$$\tilde{\|}\mathcal{L}(t)X\tilde{\|}_1 = \sum_{i=0}^{\infty} |(\mathcal{L}(t)X)(i)| \leq \gamma(t)\tilde{\|}X\tilde{\|}_1$$

where

$$\gamma(t) = 2\|A_0(t)\|_\infty + \sum_{k=1}^{r} \|A_k(t)\|_\infty^2 + \nu. \tag{55}$$

Based on (33) we may write $\|\mathcal{L}(t)X\|_1 \leq n\tilde{\|}\mathcal{L}(t)X\tilde{\|}_1 \leq n\gamma(t)\tilde{\|}X\tilde{\|}_1$ which yields $\|\mathcal{L}(t)X\|_1 \leq n\gamma(t)\|X\|_1$. This shows that $\mathcal{L}(t)$ introduced by (53) defines a linear and bounded operator on $\ell^1(\mathbf{Z}_+, \mathcal{S}_n)$ and $\|\mathcal{L}(t)\|_1 \leq n\gamma(t)$, $t \geq 0$.

Similarly, if $X \in \mathcal{S}_n^\infty$ one obtains via (51), (52), (54) that

$$\|\mathfrak{L}(t)X\|_\infty \leq \gamma(t)\|X\|_\infty$$

where $\gamma(t)$ is defined by (55). This completes the proof.

In the developments of this paper the linear operator $\mathcal{L}(t)$ introduced via (53) will be named *the Lyapunov type operator on the space* $\ell^1(\mathbf{Z}_+, \mathcal{S}_n)$ defined by the system $(A_0, A_1, ..., A_r; Q)$ while $\mathfrak{L}(t)$ will be named *the Lyapunov type operator on the space* $\mathcal{S}_n^\infty$ defined by the system $(A_0, A_1, ..., A_r; Q)$.

Proposition 3 *[11] Under the considered assumptions, the operator valued function* $\mathcal{L}(\cdot)$ *introduced by* (53) *defines a positive evolution on* $\ell^1(\mathbf{Z}_+, \mathcal{S}_n)$ *while, the operator valued function* $\mathfrak{L}(\cdot)$ *introduced by* (54) *defines an anticausal positive evolution on the Banach space* $\mathcal{S}_n^\infty$*.*

Let $T(t,\tau)$, $(t,\tau) \in \mathcal{I} \times \mathcal{I}$ be the linear evolution operator on $\ell^1(\mathbf{Z}_+, \mathcal{S}_n)$ defined by the linear differential equation

$$\frac{d}{dt}X(t) = \mathcal{L}(t)X(t). \tag{56}$$

This means that $\frac{d}{dt}T(t,\tau) = \mathcal{L}(t)T(t,\tau), \quad T(\tau,\tau) = I_{\ell^1(\mathbf{Z}_+, \mathcal{S}_n)}$.

Consider, also $T^a(t,\tau)$ the anticausal linear evolution operator on $\mathcal{S}_n^\infty$ defined by the backward linear differential equation

$$\frac{d}{dt}X(t) + \mathfrak{L}(t)X(t) = 0. \tag{57}$$

This means that

$$\begin{aligned} \tfrac{\partial}{\partial t}T^a(t,\tau) &= -\mathfrak{L}(t)T^a(t,\tau), \\ T^a(\tau,\tau) &= I_{\mathcal{S}_n^\infty}. \end{aligned} \tag{58}$$

Remark 8 *Under the considered assumptions the operator valued functions $t \to \mathcal{L}(t)$ and $t \to \mathfrak{L}(t)$ are continuous in the topology induced by the norms of Banach algebras $\mathbf{B}(\ell^1(\mathbf{Z}_+, \mathcal{S}_n))$ and $\mathbf{B}(\mathcal{S}_n^\infty)$, respectively.*

In the previous subsection we saw that in the case $\mathcal{D} = \{1, 2, ..., d\}$ the operator defined by (48) which is the analogous of the operator $\mathfrak{L}(t)$ (introduced by (54)) coincides with the adjoint $\mathcal{L}^*(t)$ of the operator $\mathcal{L}(t)$.

In the case $\mathcal{D} = \mathbf{Z}_+$, such an equality is not possible because the operator $\mathcal{L}(t)$ and $\mathfrak{L}(t)$ act on different linear spaces.

In the next developments we shall see that under some additional assumptions the restriction of the operator $\mathfrak{L}(t)$ to the Hilbert space $(\ell_2(\mathbf{Z}_+, \mathcal{S}_n), \|\cdot\|_2)$ coincides with the adjoint operator of $\mathcal{L}(t)$.

To this end we need the following auxiliary result which could be also of interest in itself.

Lemma 2 *If $A, M \in \mathbf{R}^{n\times n}$ are given matrices, then $|AM|_2 \leq \min\{|A||M|_2, |A|_2|M|\}$ where $|\cdot|$ and $|\cdot|_2$ are the norms introduced by (20) and (21).*

Theorem 1 *Assume that beside the conditions* (51)-(52) *the real numbers q_{ij} satisfy the condition:*

$$\sup_{i\in\mathbf{Z}_+} \sum_{j=0}^{\infty} |q_{ji}| = \tilde{q} < +\infty. \tag{59}$$

Let $\tilde{\mathfrak{L}}(t) = \mathfrak{L}(t)|_{\ell^2(\mathbf{Z}_+,\mathcal{S}_n)}$ be the restriction of the operator $\mathfrak{L}(t)$ to $\ell^2(\mathbf{Z}_+, \mathcal{S}_n) \subset \mathcal{S}_n^\infty$. Under these conditions, for each $t \in \mathcal{I}$, the following hold:

(i) $\tilde{\mathfrak{L}}(t) \in \mathbf{B}(\ell^2(\mathbf{Z}_+, \mathcal{S}_n))$.

(ii) $\mathcal{L}(t) \in \mathbf{B}(\ell^2(\mathbf{Z}_+, \mathcal{S}_n))$.

(iii) $\tilde{\mathfrak{L}}(t) = \mathcal{L}^*(t)$.

Proof. (*i*) Let $X = \{X(i)\}_{i\in\mathbf{Z}_+} \in \ell^2(\mathbf{Z}_+, \mathcal{S}_n)$ be arbitrary but fixed. Based on (54) we obtain

$$|(\mathfrak{L}(t)X)(i)|_2^2 \leq 4[|A_0^T(t,i)X(i)|_2^2 + |X(i)A_0(t,i)|_2^2 + \\ |\sum_{k=1}^{r} A_k^T(t,i)X(i)A_k(t,i)|_2^2 + |\sum_{j=0}^{\infty} q_{ij}X(j)|_2^2].$$

Based on Lemma 2 we deduce

$$|(\mathfrak{L}(t)X)(i)|_2^2 \leq 4\left[\gamma_1(t)|X(i)|_2^2 + \left(\sum_{j=0}^{\infty}|q_{ij}||X(j)|_2\right)^2\right] \tag{60}$$

where $\gamma_1(t) = 2\|A_0(t)\|_\infty^2 + r\sum\limits_{k=1}^{r}\|A_k(t)\|_\infty^4$.

Let $N \in \mathbf{Z}_+$, $N \geq 1$ be arbitrary but fixed. We have

$$\left(\sum_{j=0}^{N}|q_{ij}||X(j)|_2\right)^2 \leq \sum_{j=0}^{N}|q_{ij}|\sum_{j=0}^{N}|q_{ij}||X(j)|_2^2. \tag{61}$$

Using (52) we obtain:

$$\left(\sum_{j=0}^{N}|q_{ij}||X(j)|_2\right)^2 \leq \nu\sum_{j=0}^{N}|q_{ij}||X(j)|_2^2. \tag{62}$$

Further we have $\sum\limits_{i=0}^{N_1}\left(\sum\limits_{j=0}^{N}|q_{ij}||X(j)|_2\right)^2 \leq \nu\sum\limits_{j=0}^{N}\left(\sum\limits_{i=0}^{N_1}|q_{ij}||X(j)|_2^2\right)$ for all $N_1 \in \mathbf{Z}_+$, $N_1 \geq 1$. Using (59) one gets:

$$\sum_{i=0}^{N_1}\left(\sum_{j=0}^{N}|q_{ij}||X(i)|_2\right)^2 \leq \nu\tilde{q}\|X\|_2^2$$

for all $N_1, N \in \mathbf{Z}_+$.

Taking the limit for $N \to \infty$, $N_1 \to \infty$ one obtains

$$\sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty}|q_{ij}||X(j)|_2\right)^2 \leq \sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty}|q_{ij}||X(j)|_2\right)^2 \leq \nu\tilde{q}\|X\|_2^2 \tag{63}$$

for all $i \in \mathbf{Z}_+$.

So, we have shown that the right hand side of (60) is finite. Further, from (60)-(63) we deduce:

$$\sum_{i=0}^{\infty}|(\mathfrak{L}(t)X)(i)|_2^2 \leq 4(\gamma_1(t) + \nu\tilde{q})\|X\|_2^2.$$

This shows that $(\mathfrak{L}(t)X) \in \ell^2(\mathbf{Z}_+, \mathcal{S}_n)$ if $X \in \ell^2(\mathbf{Z}_+, \mathcal{S}_n)$. Furthermore we have $\|\mathfrak{L}(t)X\|_2 \leq \gamma_2(t)\|X\|_2$ $\forall X \in \ell^2(\mathbf{Z}_+, \mathcal{S}_n)$, with

$$\gamma_2(t) = 2(\gamma_1(t) + \nu\tilde{q})^{\frac{1}{2}}. \tag{64}$$

Thus (i) is proved.

Further we show that (53) is well defined if $X = \{X(i)\}_{i\in\mathbf{Z}_+} \in \ell^2(\mathbf{Z}_+, \mathcal{S}_n)$. Proceeding as in the proof of (i), we show that

$$|(\mathcal{L}(t)X)(i)|_2^2 \leq 4\left(\gamma_1(t)|X(i)|_2^2 + \left(\sum_{j=0}^{\infty}|q_{ji}||X(i)|_2\right)^2\right) \tag{65}$$

$i \in \mathbf{Z}_+$, $\gamma_1(t)$ being as in (60).

For each $N \geq 1$ we have $(\sum\limits_{j=0}^{N}|q_{ji}||X(j)|_2)^2 \leq \sum\limits_{j=0}^{N}|q_{ji}|\sum\limits_{j=0}^{N}|q_{ji}||X(j)|_2^2$ which yields $(\sum\limits_{j=0}^{N}|q_{ji}||X(j)|_2)^2 \leq \tilde{q}\sum\limits_{j=0}^{N}|q_{ji}||X(j)|_2^2$.

Further we obtain $\sum\limits_{i=0}^{N_1}\left(\sum\limits_{j=0}^{N}|q_{ji}||X(j)|_2\right)^2 \leq \nu\tilde{q}\|X\|_2^2$.

Taking the limits for $N \to \infty$ and $N_1 \to \infty$ we deduce

$$\sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty}|q_{ji}||X(j)|_2\right)^2 \leq \sum_{i=0}^{\infty}(\sum_{j=0}^{\infty}|q_{ji}||X(j)|_2)^2 \leq \nu\tilde{q}\|X\|_2^2$$

for all $i \in \mathbf{Z}_+$, $X \in \ell^2(\mathbf{Z}_+, \mathcal{S}_n)$.

This shows that the right hand side of (65) is finite for all $i \in \mathbf{Z}_+$. Furthermore we obtain that

$$\sum_{i=0}^{\infty}|(\mathcal{L}(t)X)(i)|_2^2 \leq \gamma_2(t)\|X\|_2^2, \;\; (\forall)\;\; X \in \ell^2(\mathbf{Z}_+, \mathcal{S}_n)$$

where $\gamma_2(t)$ is defined as in (64). Thus we have proved that $\mathcal{L}(t)\in\mathbf{B}(\ell^2(\mathbf{Z}_+, \mathcal{S}_n)$.

In order to prove (iii) one employs (34), (53), (54) to show that the equality $\langle\tilde{\mathfrak{L}}(t)X, Y\rangle_2 = \langle X, \mathcal{L}(t)Y\rangle_2$ holds for all $X, Y \in \ell^2(\mathbf{Z}_+, \mathcal{L}_n)$. Thus the proof is complete.

Remark 9 *Under the assumptions of Lemma 1, the condition* (59) *is satisfied if there exist* $h_1 \geq 0$, $h_2 \geq 0$ *such that* $q_{ij} = 0$ *if* $i < j - h_1$ *or* $i > j + h_2$. *In this case* (59) *is satisfied with* $\tilde{q} = (h_1 + h_2 + 1)\nu$ *where* ν *is the constant from* (52).

By direct calculation one shows that $\tilde{\mathfrak{L}} : \mathcal{I} \to \mathbf{B}(\ell^2(\mathbf{Z}_+, \mathcal{S}_n))$ is a strongly continuous operator valued function. This function defines the linear differential equation:

$$\frac{d}{dt}Y(t) + \tilde{\mathfrak{L}}(t)Y(t) = 0 \tag{66}$$

$t \in \mathcal{I}$ on the space $(\ell^2(\mathbf{Z}_+, \mathcal{S}_n), \|\cdot\|_2)$.

Let $T^a_{\tilde{\mathfrak{L}}}(t, \tau)$, $t, \tau \in \mathcal{I}$, be the anticausal linear evolution operator on $\ell^2(\mathbf{Z}_+, \mathcal{S}_n)$ defined by the linear differential equation (66).

Corollary 5 *Under the assumptions of the Theorem* 1 *we have:*

$$T^a_{\tilde{\mathfrak{L}}}(\tau, t) = T^*(t, \tau), \quad \forall\ t, \tau \in \mathcal{I}.$$

$T(t, \tau)$ *being the linear evolution operator defined by* $\mathcal{L}(t) \in \mathbf{B}(\ell^2(\mathbf{Z}_+, \mathcal{S}_n))$.

Proof follows from Theorem 1 (iii) and the equality (18).

Let $\mathfrak{L}(t) = \mathfrak{L}_A(t) + \tilde{\Pi}(t)$ be the partition of the linear operator $\mathfrak{L}(t)$ where

$$(\tilde{\Pi}(t)X)(i) = \sum_{k=1}^{r} A_k^T(t,i)X(i)A_k(t,i) + \sum_{j=0, j\neq i}^{\infty} q_{ij}X(j). \tag{67}$$

We prove:

Lemma 3 *For any monotone and bounded sequence* $\{X_k\}_{k\in\mathbf{Z}_+} \subset \mathcal{S}_n^\infty$ *we have:*

(i) $\lim_{k\to\infty} (\tilde{\Pi}(t)[X_k])(i) = (\tilde{\Pi}(t)[X])(i)$ *for all* $i \in \mathbf{Z}_+$, $t \in \mathcal{I}$.

(ii) $\lim_{k\to\infty} (T^a(t, t_0)X_k)(i) = (T^a(t, t_0)X)(i)$ *for all* $i \in \mathbf{Z}_+$, $t \leq t_0$, $t, t_0 \in \mathcal{I}$, *where* $X = \{X(i)\}_{i\in\mathbf{Z}_+} \in \mathcal{S}_n^\infty$ *is defined by* $X(i) = \lim_{k\to\infty} X_k(i)$, $i \in \mathbf{Z}_+$.

Proof. Without loss of generality we may assume that $\{X_k\}_{k\in\mathbf{Z}}$ is an increasing and bounded sequence. This means that there exist $\mu_j \in \mathbf{R}$, $j = 1, 2$ such that

$$\mu_1 I_n \leq X_k(i) \leq X_{k+1}(i) \leq \mu_2 I_n, \quad \forall (k,i) \in \mathbf{Z}_+ \times \mathbf{Z}_+ \tag{68}$$

Therefore, for each $i \in \mathbf{Z}_+$, $X(i) \in \mathcal{S}_n$ is well defined by

$$X(i) = \lim_{k\to\infty} X_k(i). \tag{69}$$

Based on (68) we infer that $X = \{X(i)\}_{i\in\mathbf{Z}_+} \in \mathcal{S}_n^\infty$. From (67) we obtain

$$(\tilde{\Pi}(t)X_k)(i) - (\tilde{\Pi}(t)X)(i) = \sum_{l=1}^{r} A_l^T(t,i)(X_k(i) - X(i))A_l(t,i) + $$

$$+ \sum_{\substack{j=0\\ j\neq i}}^{\infty} q_{ij}(X_k(j) - X(j)). \tag{70}$$

First, from (68) we obtain

$$\lim_{k\to\infty} \sum_{k=1}^{r} A_l^T(t,i)(X_k(i) - X(i))A_l(t,i) = 0 \tag{71}$$

On the other hand, applying Corollary 7 from the Appendix for $a_k(j) = q_{ij}|X_k(j) - X(j)|$, we deduce that $lim_{k\to\infty} \sum_{j=0, j\neq i}^{\infty} q_{ij}|X_k(j) - X(j)| = 0$ which leads to

$$\lim_{k\to\infty} \sum_{j=0, j\neq i}^{\infty} q_{ij}(X_k(j) - X(j)) = 0. \tag{72}$$

Combining (70)-(72) we obtain that (i) is true.

Let us now prove that (ii) holds. To this end, let us denote $Y_k(t) = T^a(t,t_0)X_k$, $t \in (-\infty, t_0] \cap \mathcal{I}$, $t_0 \in \mathcal{I}$ being fixed. Since $T^a(t,t_0)$ is a positive operator, if $t \leq t_0$ the inequalities (68) yield

$$\mu_1(T^a(t,t_0)J^\infty)(i) \leq Y_k(t,i) \leq Y_{k+1}(t,i) \leq \mu_2(T^a(t,t_0)J^\infty)(i) \tag{73}$$

for all $i \in \mathbf{Z}_+$, $t \leq t_0$, $t \in \mathcal{I}$. From (73) we obtain that the matrices $Z(t,i)$ are well defined by

$$Z(t,i) = \lim_{k\to\infty} Y_k(t,i), i \in \mathbf{Z}_+, t \in (-\infty; t_0] \cap \mathcal{I}.$$

Furthermore (73) yields $|Z(t,i)| \le \mu_3 |T^a(t,t_0)J^\infty| = \mu_3\|T^a(t,t_0)\|$ for all $i \in \mathbf{Z}_+$. This leads to

$$|Z(t)| \le \mu_3\|T^a(t,t_0)\|, \quad \forall t \in \mathcal{I}, t \le t_0,$$

where $Z(t) = \{Z(t,i)\}_{i\in\mathbf{Z}_+}$.

Since $t \to \|\mathfrak{L}(t)\|_\infty$ is a bounded function we deduce that $\|T^a(t,t_0)\| \le e^{c(t_0-t)}$, for all $t \in \mathcal{I}$, $t \le t_0$. This leads to

$$|Z(t)| \le \mu_3 e^{c(t_0-t)}. \tag{74}$$

Reasoning in the same way we obtain from (73)

$$|Y_k(t)| \le \mu_3 e^{c(t_0-t)} \tag{75}$$

for all $t \in \mathcal{I}$, $t \le t_0$, $k \in \mathbf{Z}_+$.

Let $T^a_A(t,s)$ be the anticausal linear evolution operator on $\mathcal{S}_n^\infty$ defined by the extended Lyapunov operator $\mathfrak{L}_A(t)$. We have the representation formula

$$Y_k(t) = T^a_A(t,t_0)X_k + \int_t^{t_0} T^a_A(t,s)\tilde{\Pi}(s)Y_k(s)ds$$

for all $t \le t_0$, $t \in \mathcal{I}$.

Based on (45) written for $A(t,i)$ replaced by $A_0(t,i) + \frac{1}{2}q_{ii}I_n$, we obtain the component wise representation formula

$$Y_k(t,i) = \Phi_i^T(t_0,t)X_k(i)\Phi_i(t_0,t) + \int_t^{t_0} \Phi_i^T(s,t)(\tilde{\Pi}(s)Y_k(s))(i)\Phi_i(s,t)ds \tag{76}$$

for all $i \in \mathbf{Z}_+$, $t \le t_0$, $t \in \mathcal{I}$, where $\Phi_i(s,t)$ is the fundamental matrix solution of the differential equation

$$\frac{d}{dt}x(t) = (A_0(t,i) + \frac{1}{2}q_{ii}I_n)x(t).$$

Using the result proved in the part (i) of the lemma, we obtain that

$$\lim_{k\to\infty} \Phi_i^T(s,t)(\tilde{\Pi}(s)Y_k(s))(i)\Phi_i(s,t) = \Phi_i^T(s,t)(\tilde{\Pi}(s)Z(s))(i)\Phi_i(s,t) \tag{77}$$

for all $i \in \mathbf{Z}_+$, $t \le s \le t_0$.

We recall that the boundedness of the function $s \to \|A_0(s)\|_\infty$ together with (52) allow us to deduce that

$$|\Phi_i(s,t)| \le e^{c_1(s-t)}, \tag{78}$$

$\forall t \le s \le t_0$, $t \in \mathcal{I}$, where $c_1 > 0$ is a constant not depending upon s, t.

Further, from (76), (78) together with the boundedness of the functions $s \to \|A_l(s)\|_\infty$, $0 \le l \le r$ yield

$$x^T\Phi_i^T(s,t)(\tilde{\Pi}(s)Y_k(s))(i)\Phi_i(s,t)x \le \tilde{\beta}e^{\tilde{c}(s-t)}|x|^2 \tag{79}$$

for all $t \le s \le t_0$, where $\tilde{\beta}$, $\tilde{c}$ are positive constants. Applying Lebesque's Theorem we obtain via (77) and (79) that

$$\lim_{k\to\infty}\int_t^{t_0} x^T\Phi_i^T(s,t)(\tilde{\Pi}(s)Y_k(s))(i)\Phi_i(s,t)xds$$

$$=\int_t^{t_0} x^T\Phi_i^T(s,t)(\tilde{\Pi}(s)Z(s))(i)\Phi_i(s,t)xds$$

for all $x \in \mathbf{R}^n$. By a standard procedure, one obtains finally that

$$\lim_{k\to\infty}\int_t^{t_0} \Phi_i^T(s,t)(\tilde{\Pi}(s)Y_k(s))(i)\Phi_i(s,t)ds = \int_t^{t_0} \Phi_i^T(s,t)(\tilde{\Pi}(s)Z(s))(i)\Phi_i(s,t)ds$$

for all $t \le t_0$, $t \in \mathcal{I}$. Taking the limit for $k \to \infty$ in (76) we obtain that

$$Z(t,i) = \Phi_i^T(t_0,t)X(i)\Phi_i(t_0,t) + \int_t^{t_0} \Phi_i^T(s,t)(\tilde{\Pi}(s)Z(s))(i)\Phi_i(s,t)ds$$

for all $i \in \mathbf{Z}_+$, $t \le t_0$, $t \in \mathcal{I}$.

The above equality may be rewritten in a compact form:

$$Z(t) = T_A^a(t,t_0)X + \int_t^{t_0} T_A^a(t,s)\tilde{\Pi}(s)Z(s)ds. \tag{80}$$

Under the considered assumptions the identity (80) allows us to deduce that $t \to Z(t)$ is differentiable and additionally it solves the problem with given terminal condition:

$$\begin{aligned} \frac{d}{dt}Z(t) + \mathfrak{L}(t)Z(t) &= 0, \quad t \le t_0 \\ Z(t_0) &= X. \end{aligned} \tag{81}$$

From the uniqueness of the solution of the problem (81) we conclude that

$$Z(t,i) = (T^a(t,t_0)X)(i) \tag{82}$$

for all $i \in \mathbf{Z}_+$, $t \le t_0$, $t \in \mathcal{I}$.

The conclusion follows now from (82). So, the proof is complete. $\square$

Lemma 4 *Assume that the assumptions of Theorem* 1 *are fulfilled. Let* $\mathbf{H}_i^x = \{H_i^x(j)\}_{j \in \mathbf{Z}_+}$ *be defined by*

$$H_i^x(j) = \begin{cases} 0, & \textit{if } j \ne i \\ xx^T, & \textit{if } j = i. \end{cases} \tag{83}$$

where $x \in \mathbf{R}^n$ *and* $i \in \mathbf{Z}_+$ *are arbitrary but fixed.*

Under the considered assumptions we have:

$$\|T(t,\tau)\mathbf{H}_i^x\|_1 = x^T[(T^a(\tau,t)J^\infty)(i)]x \tag{84}$$

for all $t \ge \tau$, $t,\tau \in \mathcal{I}$.

Proof. First we notice that $\mathbf{H}_i^x \in \ell^1(\mathbf{Z}_+, \mathcal{S}_n)$ and $\|\mathbf{H}_i^x\|_1 = |x|^2$. Therefore $T(t,\tau)\mathbf{H}_i^x$ is well defined and we have

$$\|T(t,\tau)\mathbf{H}_i^x\|_1 = \sum_{j=0}^{\infty} Tr[(T(t,\tau)H_i^x)(j)] \tag{85}$$

for all $t,\tau \in \mathcal{I}$.

For each $k \in \mathbf{Z}_+$ we consider $J_k^\infty = \{J_k^\infty(j)\}_{j \in \mathbf{Z}_+}$ where

$$J_k^\infty(j) = \begin{cases} I_n, \text{if } 0 \le j \le k \\ 0, \text{if } j > k. \end{cases}$$

It is obvious that $J_k^\infty \in \ell^2(\mathbf{Z}_+, \mathcal{S}_n) \subset \mathcal{S}_n^\infty$ and we have $\|J_k^\infty\|_2 = (k+1)\sqrt{n}$ and $\|J_k^\infty\|_\infty = 1$. Also we have $J_k^\infty \le J_{k+1}^\infty \le J^\infty$ for all $k \in \mathbf{Z}_+$. This yields: $T^a(\tau,t)J_k^\infty \leq T^a(\tau,t)J_{k+1}^\infty \leq T^a(\tau,t)J^\infty$, for all $k \in \mathbf{Z}_+$, $t \geq \tau$, $t,\tau \in \mathcal{I}$ because $T^a(\tau,t) \geq 0$ for all $t \geq \tau$

This allows us to obtain

$$x^T[(T^a(\tau,t)J_k^\infty)(i)]x \leq x^T[(T^a(\tau,t)J_{k+1}^\infty)(i)]x \leq x^T[(T^a(\tau,t)J^\infty)(i)]x \quad (86)$$

for all $k \in \mathbf{Z}_+$. Moreover, applying Lemma 3 (ii) for $X_k = J_k^\infty$ we obtain that

$$\lim_{k\to\infty} x^T(T^a(\tau,t)J_k^\infty)(i)x = x^T(T^a(\tau,t)J^\infty)(i)x. \quad (87)$$

On the other hand, from Theorem 1 (i) we deduce that $T^a(\tau,t)J_k^\infty \in \ell^2(\mathbf{Z}_+,\mathcal{S}_n)$. Therefore we may write:

$$x^T[(T^a(\tau,t)J_k^\infty)(i)]x = Tr[(T^a(\tau,t)J_k^\infty)(i)xx^T] =$$

$$= \sum_{j=0}^{\infty} Tr[(T^a(\tau,t)J_k^\infty)(j)H_i^x(j)] = \langle T^a(\tau,t)J_k^\infty, \mathbf{H}_i^x\rangle_2.$$

Further, the equality proved in Theorem 1 (iii) together with (18) yield:

$$\langle T^a(\tau,t)J_k^\infty, \mathbf{H}_i^x\rangle_2 = \langle T^*(t,\tau)J_k^\infty, \mathbf{H}_i^x\rangle_2 =$$

$$= \langle J_k^\infty, T(t,\tau)\mathbf{H}_i^x\rangle_2 = \sum_{j=0}^{k} Tr[(T(t,\tau)\mathbf{H}_i^x)(j)].$$

Thus we obtain

$$x^T[(T^a(\tau,t)J_k^\infty)(i)]x = \sum_{j=0}^{k} Tr[(T(t,\tau)H_i^x)(j)]. \quad (88)$$

Based on (88) we get

$$\|T(t,\tau)\mathbf{H}_i^x\|_1 = \lim_{k\to\infty} \sum_{j=0}^{k} Tr[(T(t,\tau)\mathbf{H}_i^x)(j)] = \lim_{k\to\infty} x^T[(T^a(\tau,t)J_k^\infty)(i)]x. \quad (89)$$

The conclusion follows from (89) and (87). Thus the proof is complete. □

Theorem 2 *Assume that the assumptions of Theorem* 1 *are fulfilled. Then we have*

$$\|T(t,\tau)\|_1 \leq \|T^a(\tau,t)\| \quad \forall\ t \geq \tau,\ t,\tau \in \mathcal{I}. \quad (90)$$

Proof. Let $i \in \mathbf{Z}_+$ be arbitrary but fixed and $\psi_i : \ell^1(\mathbf{Z}_+, \mathcal{S}_n) \to \ell^1(\mathbf{Z}_+, \mathcal{S}_n)$ be defined by

$$\psi_i(\mathbf{X})(j) = \begin{cases} 0, \text{if } j \neq i \\ X(i), \text{if } j = i \end{cases} \tag{91}$$

for any $\mathbf{X} = \{X(j)\}_{j \in \mathbf{Z}_+} \in \ell^1(\mathbf{Z}_+, \mathcal{S}_n)$. We have $\left\| \mathbf{X} - \sum\limits_{i=0}^{k} \psi_i(\mathbf{X}) \right\|_1 = \sum\limits_{i=k+1}^{\infty} |X(i)|_1$ which leads to $\lim\limits_{k \to \infty} \left\| \mathbf{X} - \sum\limits_{i=0}^{k} \psi_i(\mathbf{X}) \right\|_1 = 0$.

Hence $\mathbf{X} = \sum\limits_{i=0}^{\infty} \psi_i(\mathbf{X})$ for all $\mathbf{X} \in \ell^1(\mathbf{Z}_+, \mathcal{S}_n)$.

Further we have

$$T(t,\tau)\mathbf{X} = \sum_{i=0}^{\infty} T(t,\tau)\psi_i(\mathbf{X}) \tag{92}$$

because $T(t,\tau) \in \mathbf{B}(\ell^1(\mathbf{Z}_+, \mathcal{S}_n))$.

Let $\lambda_{i1}, \lambda_{i2}, ..., \lambda_{in}$ be real numbers and $e_{i1}, e_{i2}, ..., e_{in} \in \mathbf{R}^n$ be orthogonal vectors such that $|e_{ij}| = 1$, $1 \leq j \leq n$ and $X(i) = \sum\limits_{j=1}^{n} \lambda_{ij} e_{ij} e_{ij}^T$. Combining (83) and (91) we deduce:

$$\psi_i(\mathbf{X}) = \sum_{j=1}^{n} \lambda_{ij} \mathbf{H}_i^{e_{ij}} \tag{93}$$

where $\mathbf{H}_i^{e_{ij}}$ is defined as in (83) with e_{ij} instead of x.
For each $k \geq 1$ we write

$$\left\| \sum_{i=0}^{k} T(t,\tau)\psi_i(\mathbf{X}) \right\|_1 \leq \sum_{i=0}^{k} \|T(t,\tau)\psi_i(\mathbf{X})\|_1 \leq \sum_{i=0}^{k} \sum_{j=1}^{n} |\lambda_{ij}| \left\| T(t,\tau)\mathbf{H}_i^{e_{ij}} \right\|_1.$$

Applying Lemma 4 we obtain

$$\sum_{i=0}^{k} \|T(t,\tau)\Psi_i(\mathbf{X})\|_1 \leq \sum_{i=0}^{k} \sum_{j=1}^{n} |\lambda_{ij}| e_{ij}^T [(T^a(\tau,t)J^\infty)(i)] e_{ij} \leq$$
$$\leq \sum_{i=0}^{k} \sum_{j=1}^{h} |\lambda_{ij}| |(T^a(\tau,t)J^\infty)(i)|.$$

Invoking (25) we infer $\sum_{i=0}^{k} \|T(t,\tau)\psi_i(\mathbf{X})\|_1 \leq \|T^a(\tau,t)J^\infty\|_\infty \sum_{i=0}^{\infty} |X(i)|_1$ for all $k \geq 1$.

Hence we have shown that

$$\sum_{i=0}^{\infty} \|T(\tau,t)\psi_i(\mathbf{X})\|_1 \leq \|T^a(\tau,t)J^\infty\|_\infty \|\mathbf{X}\|_1. \tag{94}$$

Further, from (92)-(94) we get:

$$\|T(t,\tau)\mathbf{X}\|_1 \leq \|T^a(\tau,t)J^\infty\|_\infty \|\mathbf{X}\|_1, \quad \forall\ \mathbf{X} \in \ell^1(\mathbf{Z}_+, \mathcal{S}_n),\ \ t \geq \tau,\ \ t,\tau \in \mathcal{I}.$$

So, we may conclude that $\|T(t,\tau)\|_1 \leq \|T^a(\tau,t)J^\infty\|_\infty$ for all $t \geq \tau$, $t,\tau \in \mathcal{I}$. To show that the last inequality coincides with (90) we apply Theorem 5 in the special case of the positive operator $T^a(\tau,t)$ together with Corollary 4 (ii) and obtain that $\|T^a(\tau,t)J^\infty\|_\infty = \|T^a(\tau,t)\|$. This ends the proof. □

4.3.2 Detectability and exponential stability

Let $\mathcal{L}$ and $\mathfrak{L}$ be the Lyapunov type operators defined by (53) and (54), respectively. In this section we discuss the exponential stability of the anticausal evolution generated by $\mathcal{L}$ under detectability conditions and under the assumptions of Theorem 1.
Since the cone $\ell^1(\mathbf{Z}_+, \mathcal{S}_{n+})$ has empty interior we cannot apply the developments from Section 4 in [11] in order to derive criteria for exponential stability of the linear differential equation (56) on $\ell^1(\mathbf{Z}_+, \mathcal{S}_n)$. The next corollary shows that the criteria for anticausal exponential stability of (57) could be used as necessary and sufficient conditions for the exponential stability of (56).

Corollary 6 *Under the assumptions of Theorem 1 the following are equivalent:*

(i) the operator valued function $\mathfrak{L}(\cdot)$ defines an exponentially stable anticausal evolution on $\mathcal{S}_n^\infty$;

(ii) the operator valued function $\mathcal{L}(\cdot)$ defines an exponentially stable evolution on $\ell^1(\mathbf{Z}_+, \mathcal{S}_n)$.

Proof. $(i) \to (ii)$. If (i) holds then there exist $\beta \geq 1$, $\alpha > 0$ such that $\|T^a(\tau,t)\| \leq \beta e^{-\alpha(t-\tau)}$ for all $\tau, t \in \mathcal{I}$, $t \geq \tau$. Then from Theorem 2 we get

$$\|T(t,\tau)\|_1 \leq \beta e^{-\alpha(t-\tau)} \tag{95}$$

for all $t \geq \tau$, $t, \tau \in \mathcal{I}$. This shows that (ii) is true.

Let us prove now that $(ii) \to (i)$. If (ii) holds, then there exist $\beta \geq 1$, $\alpha > 0$ such that (95) is true. From Lemma 4 we have

$$x^T (T^a(\tau,t) J^\infty)(i) x \leq \|T(t,\tau)\|_1 \|\mathbf{H}_i^x\|_1$$

which yields $x^T (T^a(\tau,t) J^\infty)(i) x \leq \beta e^{-\alpha(t-\tau)} |x|^2$ for all $t \geq \tau \in \mathcal{I}$, $x \in \mathbf{R}^n$, $i \in \mathbf{Z}_+$. Therefore $|(T^a(\tau,t) J^\infty)(i)| \leq \beta e^{\alpha(\tau - t)}$, $\quad (\forall) \quad i \in \mathbf{Z}_+$, which leads to

$$\|T^a(\tau,t) J^\infty\|_\infty \leq \beta e^{\alpha(\tau-t)}. \tag{96}$$

Applying Theorem 5 to the positive operator $T^a(\tau,t)$ and using Corollary 4 (ii) we obtain from (96) that $\|T^a(\tau,t)\| \leq \beta e^{\alpha(\tau-t)}$, for all $t \geq \tau$, $t, \tau \in \mathcal{I}$. This confirms that the implication $(ii) \to (i)$ is true. So the proof is complete. □

Definition 3 *Let $C : \mathcal{I} \to \mathcal{M}_{pn}^\infty$ be a continuous, bounded function. We say that the pair $(\mathcal{L}, C)$ is detectable if there is a continuous and bounded function $F : \mathcal{I} \to \mathcal{M}_{np}^\infty$ such that*

$$\mathcal{L}_F(t)(X) = \mathcal{L}(t)(X) + F(t) C(t) X + X C(t)^T F(t)^T, X \in \ell^1(\mathbf{Z}_+, \mathcal{S}_n) \tag{97}$$

generates a positive and exponentially stable causal evolution on $\ell^1(\mathbf{Z}_+, \mathcal{S}_n)$.

A standard computation shows that $t \in \mathcal{I} \to \mathcal{L}_F(t) \in B\left(\ell^1(\mathbf{Z}_+, \mathcal{S}_n)\right)$ is a well defined, strongly continuous mapping, which generates a causal evolution operator $T_{\mathcal{L}_F}(t,s)$ on $\ell^1(\mathbf{Z}_+, \mathcal{S}_n)$. Since $\mathcal{L}_F$ and $\mathcal{L}$ have the same form, we can apply Proposition 3 to deduce that $T_{\mathcal{L}_F}(t,s)$ is a positive operator. Moreover, $T_{\mathcal{L}_F}(t,s)$ is exactly the restriction to $\ell^1(\mathbf{Z}_+, \mathcal{S}_n)$ of the positive causal evolution operator $\widehat{T}_{\mathcal{L}_F}(t,s)$ generated by $\mathcal{L}_F$ on $\ell^2(\mathbf{Z}_+, \mathcal{S}_n)$ ([18], [3]). Let

$$\mathfrak{L}_F(t)(X) = \mathfrak{L}(t)(X) + C(t)^T F(t)^T X + X F(t) C(t),$$

for all $X \in \mathcal{S}_n^\infty$ and $t \in \mathcal{I}$. Obviously, the conclusions of Corollary 6 remain true if we replace the operators $\mathfrak{L}$ and $\mathcal{L}$ with $\mathfrak{L}_F$ and $\mathcal{L}_F$, respectively. Therefore $\mathfrak{L}_F$ generates an anticausal exponentially stable evolution on $\mathcal{S}_n^\infty$ if and only if $\mathcal{L}_F$ generates a causal exponentially stable evolution on $\ell^1(\mathbf{Z}_+, \mathcal{S}_n)$.

Remark 10 *The pair* $(\mathcal{L}, C)$ *is detectable if and only if there is* $F : \mathcal{I} \rightarrow \mathcal{M}_{n\times p}^{\infty}$ *a continuous and bounded function such that* $\mathfrak{L}_F(t), t \in \mathcal{I}$ *generates a positive and exponentially stable anticausal evolution on* $\mathcal{S}_n^{\infty}$*.*

Let $\mathcal{I} = \mathbf{R}_+ := [0, \infty)$. A mapping $P : \mathcal{I} \rightarrow \mathcal{S}_{n+}^{\infty}$ will be called nonnegative. The next theorem is the main result of this section. It gives sufficient conditions for the exponential stability of the anticausal evolution generated by the Lyapunov type operator $\mathfrak{L}$, in terms of global solvability of an associated affine equation.

Theorem 3 *Assume that* $(\mathcal{L}, C)$ *is detectable and the backward differential equation*

$$\frac{dP(t)}{dt} + \mathfrak{L}(t) P(t) + C^T(t) C(t) = 0 \tag{98}$$

has a nonnegative solution in the class of all bounded C^1*-mappings* $P : \mathcal{I} \rightarrow \mathcal{S}_n^{\infty}$*. Then* $\mathfrak{L}$ *generates an exponentially stable anticausal evolution on* $\mathcal{S}_n^{\infty}$*.*

Proof. If $i \in \mathbf{Z}_+$ and $x \in \mathbf{R}^n$ are fixed, then the unique solution in $\ell^1(\mathbf{Z}_+, \mathcal{S}_n)$ of the equation

$$\frac{dZ(t)}{dt} = \mathcal{L}(t)(Z(t)), t \geq s \geq 0 \tag{99}$$

$$Z(s) = H_i^x, i \in \mathbf{Z}_+, \tag{100}$$

exists and is given by $Z(t, s; (H_i^x)) = T_{\mathcal{L}}(t, s)(H_i^x) \geq 0, t \geq s$. Let us prove that there is $\gamma > 0$ such that

$$\int_s^{\infty} \|Z(\tau, s; H_i^x)\|_1 d\tau \leq \gamma x^T x \tag{101}$$

for all $s \in \mathbf{R}_+$,$i \in \mathbf{Z}_+$ and $x \in H$. Then, Lemma 4 shows that

$$\int_s^{\infty} x^T [(T_{\mathfrak{L}}^a(s, \tau) J^{\infty})(i)] x d\tau \leq \gamma x^T x.$$

From Theorem 4.4 from [11] it follows that $\mathfrak{L}$ generates an anticausal exponentially stable evolution on $\mathcal{S}_n^{\infty}$ and the proof is complete.

It remains to prove (101).

First, we establish a sufficient condition for (101). From the detectability hypothesis, there is a bounded and continuous function $F : \mathcal{I} \to \mathcal{M}_{np}^{\infty}$ such that the causal evolution operator $T_{\mathcal{L}_F}(t_0, t)$ is exponentially stable on $\ell^1(\mathbf{Z}_+, \mathcal{S}_n)$. We define

$$\Omega^{F,\varepsilon}(t)(X) = \mathcal{L}_F(t)(X) + \varepsilon^2 X, X \in \ell^1(\mathbf{Z}_+, \mathcal{S}_n).$$

Using Gronwall's Lemma and a standard computation we deduce that there is $\varepsilon_0 \in (0,1)$ such that the causal evolution operator $T_{\Omega^{F,\varepsilon}}(t,s)$, generated by $\Omega^{F,\varepsilon}(t)$, $t \in \mathbf{R}_+$, is exponentially stable for all $0 < \varepsilon < \varepsilon_0$. Then there are $\beta_1 \geq 1$ and $\alpha_1 \in (0,1)$ such that $\|T_{\Omega^{F,\varepsilon}}(t,s)\|_1 \leq \beta_1 \alpha_1^{t-s}$ for all $t \geq s$. Moreover, by Proposition 3.3 from [11] (see also Corollary 2), $T_{\Omega^{F,\varepsilon}}(t,s)$ is a positive operator. Now, for such an $\varepsilon \in (0, \varepsilon_0)$, we consider the equation

$$\frac{dY(t)}{dt} = \Omega^{F,\varepsilon}(t)(Y(t)) + \frac{1}{\varepsilon^2} F(t) C(t) Z(t) C(t)^T F(t)^T, t \geq s \geq 0, \tag{102}$$

$$Y(s) = H_i^x, \tag{103}$$

where $Z(t)$ is the solution of (99). We obtain

$$\frac{d(Y(t) - Z(t))}{dt} = \Omega^{F,\varepsilon}(Y(t) - Z(t)) + \Psi(t), t \geq s, t \in \mathbf{N}, \tag{104}$$

$$Y(s) - Z(s) = 0, \tag{105}$$

where

$$\Psi(t) = \left(\varepsilon + \frac{1}{\varepsilon} FC\right) Z \left(\varepsilon + \frac{1}{\varepsilon} FC\right)^T (t).$$

By a standard way it follows that $Y(t) - Z(t) \geq 0$ for all $t \in \mathbf{R}_+$ and, consequently, $\|Y(t)\|_1 \geq \|Z(t)\|_1$. Therefore the existence of $\gamma > 0$ such that

$$\int_s^{\infty} \|Y(t, s; H_i^x)\|_1 \, dt \leq \gamma x^T x, \tag{106}$$

is a sufficient condition for (101) to hold. Then, let us prove (106).

Applying (10) for (102), (103) we obtain

$$Y(t) = T_{\Omega^{F,\varepsilon}}(t,s)(H_i^x) + \frac{1}{\varepsilon^2} \int_s^t T_{\Omega^{F,\varepsilon}}(t,r) \left((FCZ)(r)(FC)(r)^T\right) dr,$$

for any $t > s$. Hence

$$\|Y(t)\|_1 \leq \beta_1 \alpha_1^{t-s} x^T x + \frac{1}{\varepsilon^2} \int_s^t \alpha_1^{t-r} \left\| (FCZ)(r)(FC)(r)^T \right\|_1 dr. \tag{107}$$

By virtue of (26), the conclusions of Lemma 2 remains valid for $|.|_1$ replacing $|.|_2$ and we get

$$\begin{aligned}
\left\| (FCZ)(r)(FC)(r)^T \right\|_1 &= \sum_{j\in\mathbf{Z}_+} \left| \left[(FCZ)(r)(FC)(r)^T \right](j) \right|_1 \\
&= \sum_{j\in\mathbf{Z}_+} Tr\left[\left((FCZ)(r)(FC)(r)^T \right)(j) \right] \leq \\
l^2 \sum_{j\in\mathbf{Z}_+} Tr\left[\left(CZC^T\right)(r)(j) \right] &= l^2 \sum_{j\in\mathbf{Z}_+} Tr\left[\left(C^T CZ\right)(r)(j) \right] := (*).
\end{aligned}$$

The restrictions of the operators $\mathcal{L}$ and $\mathfrak{L}$ to $\ell^2(\mathbf{Z}_+, \mathcal{S}_n)$ will be still denoted by $\mathcal{L}$ and $\mathfrak{L}$. So, Corollary 5 ensures that

$$T_{\mathcal{L}}^*(r,s) = T_{\mathfrak{L}}^a(s,r) \tag{108}$$

in $\ell^2(\mathbf{Z}_+, \mathcal{S}_n)$ and

$$Z(r) = T_{\mathcal{L}}(r,s)(H_i^x) = \left(T_{\mathfrak{L}}^a(s,r)\right)^*(H_i^x).$$

From Lemma 21 in [19], we have

$$\begin{aligned}
(*) &= l^2 \sum_{j\in\mathbf{Z}_+} Tr\left[\left(C^T C\right)(r) \left(T_{\mathfrak{L}}^a(s,r)\right)^* (H_i^x)(j) \right] \\
&\leq l^2 \sum_{j\in\mathbf{Z}_+} Tr\left[T_{\mathfrak{L}}^a(s,r) \left(C^T(r) C(r)\right) H_i^x(j) \right] \\
&= l^2 x^T T_{\mathfrak{L}}^a(s,r) \left(C^T(r) C(r)\right)(i)\, x
\end{aligned} \tag{109}$$

Let $P : \mathcal{I} \to \mathcal{M}_{n+}^{\infty}$ be a nonnegative solution of (98) in the class of all bounded C^1-mappings. From (11), P is also the unique solution of the equation

$$P(s) = T_{\mathfrak{L}}^a(s,t)(P(t)) + \int_s^t T_{\mathfrak{L}}^a(s,r) \left(C^T(r) C(r)\right) dr, s \leq t.$$

By inequality (109) and the positiveness of $T_{\mathfrak{L}}^{a}(t,s)$, we get successively

$$\begin{aligned}\int_s^t \left\|(FCZ)(r)(FC)(r)^T\right\|_1 dr &\le \int_s^t x^T T_{\mathfrak{L}}^{a}(s,r)\left(C^T(r)C(r)\right)(i)\,x dr\\ &= x^T \int_s^t T_{\mathfrak{L}}^{a}(s,r)\left(C^T(r)C(r)\right)(i)\,dr x\\ &= x^T P(s)(i)\,x - x^T T_{\mathfrak{L}}^{a}(s,t)(P(t))(i)\,x\\ &\le x^T P(s)(i)\,x.\end{aligned}$$

Setting $m_1 = l^2 \sup_{s\in\mathbf{R}_+} \|P(s)\|_\infty < \infty$ we obtain

$$\int_s^t \left\|FCZ(r)(CF)(r)^T\right\|_1 dr \le l^2 x^T P(s)(i)\,x \le m_1 x^T x,$$

Taking the integral from $t = s$ to ∞ in (107), we see that there are the positive constants d_1 and d_2 such that

$$\begin{aligned}\int_s^\infty \|Y(t)\|_1 dt &\le d_1 x^T x + \frac{d_2}{\varepsilon^2}\int_s^\infty \left\|FCZ(r)(CF)(r)^T\right\|_1 dr \qquad (110)\\ &\le d_1 x^T x + \frac{d_2}{\varepsilon^2} m_1 x^T x \le \left(d_2 m_1/\varepsilon^2 + d_1\right) x^T x.\end{aligned}$$

So, there is $\gamma > 0$ such that

$$\int_s^\infty \|Y(t)\|_1 dt \le \gamma x^T x$$

and (106) follows. The proof is complete. □

5 Appendix

A. Convex cones. In the sequel we collect several basic definitions regarding the convex cones and ordered Banach spaces. For more details concerning the convex cones and ordered linear spaces we refer to [4, 12, 16, 17] and references therein.

Let $(\mathcal{X}, \|\cdot\|)$ be a real normed space.

Definition 4 *A nonempty subset* $\mathcal{C} \subset \mathcal{X}$ *is called convex cone if:*
(i) $\mathcal{C} + \mathcal{C} \subset \mathcal{C}$
(ii) $\alpha\mathcal{C} \subset \mathcal{C}$ *for all* $\alpha \in \mathbf{R}, \alpha \geq 0$.

It is easy to see that a cone $\mathcal{C}$ is a convex subset and thus we shall say *convex cone* when we refer to a cone.

Definition 5 (i) *A cone* $\mathcal{C}$ *is called a* pointed cone *if* $\mathcal{C} \bigcap (-\mathcal{C}) = \{0\}$.
(ii) *A cone* $\mathcal{C}$ *is called a* solid cone *if its interior* $Int\mathcal{C}$ *is not empty.*
(iii) *A cone* $\mathcal{C}$ *is called* normal cone *if there exists a real number* $\tilde{b} > 0$ *such that* $\|x\| \leq \tilde{b}\|y\|$ *if* $0 \leq x \leq y$.

A convex cone $\mathcal{C} \subset \mathcal{X}$ induces an ordering " $\leq$ " on $\mathcal{X}$, by $x \leq y$ (or equivalently $y \geq x$) if and only if $y - x \in \mathcal{C}$. If $\mathcal{C}$ is a solid cone then $x < y$ (or equivalently $y > x$) if and only if $y - x \in Int\mathcal{C}$. Hence $\mathcal{C} = \{x \in \mathcal{X} | x \geq 0\}$ and $Int\mathcal{C} = \{x \in \mathcal{X} | x > 0\}$. That is why, in this paper, we shall use the notation $\mathcal{X}_+$ for the convex cone which induces the order relation on $\mathcal{X}$.

Remark 11 (i) *If in the definition of a normal cone we may take* $\tilde{b} = 1$ *we shall say that the norm* $\|\cdot\|$ *is monotone with respect to the convex cone* $\mathcal{C}$.
(ii) *If* $\mathcal{C}$ *is a normal cone then it is a pointed cone. Indeed, if* x *is such that* x *and* $-x$ *are in* $\mathcal{C}$ *then from* $(1 + \frac{1}{n})x \in \mathcal{C}$ *we have* $0 \leq -x \leq \frac{1}{n}x$. *Hence* $\|x\| \leq \frac{\tilde{b}}{n}\|x\|$. *Taking the limit for* $n \to \infty$ *we deduce that* $\|x\| = 0$ *hence* $x = 0$. *Thus we obtained that* $\mathcal{C}$ *is pointed cone.*

B. Minkovski norms. Linear positive operators

We assume that $\mathcal{X}$ is a real Banach space ordered by an order relation induced by a solid convex cone $\mathcal{X}_+$ where $\mathcal{X}_+ \neq \mathcal{X}$. For a fixed $\xi \in Int\mathcal{X}_+$ we consider the open and convex subset

$$B_\xi = \{x \in \mathcal{X}; -\xi < x < \xi\}. \tag{111}$$

The Minkovski functional $|\cdot|_\xi : \mathcal{X} \to \mathbf{R}$ associated to the subset B_ξ is

$$|x|_\xi = inf\left\{t > 0; \frac{1}{t}x \in B_\xi\right\}. \tag{112}$$

The main properties of the Minkovski functional introduced by (112) are collected in the next theorem.

Theorem 4 *The Minkovski functional introduced in (112) has the properties:*

(i) $|x|_\xi \geq 0$ *and* $|0|_\xi = 0$.
(ii) $|\alpha x|_\xi = |\alpha||x|_\xi$ *for all* $\alpha \in \mathbf{R}, x \in \mathcal{X}$.
(iii) $|x|_\xi < 1$ *if and only if* $x \in B_\xi$.
(iv) $|x+y|_\xi \leq |x|_\xi + |y|_\xi$ *for all* $x, y \in \mathcal{X}$.
(v) *There exists* $\beta(\xi) > 0$ *such that* $|x|_\xi \leq \beta(\xi)\|x\|, (\forall) x \in \mathcal{X}$.
(vi) $|x|_\xi = 1$ *if and only if* $x \in \partial B_\xi$.
(vii) $|x|_\xi \leq 1$ *if and only if* $x \in \bar{B}_\xi$.
(viii) *If* $\mathcal{X}_+$ *is closed then* $\bar{B}_\xi = \{x \in \mathcal{X}; -\xi \leq x \leq \xi\}$.
(ix) $|\xi|_\xi = 1$.
(x) *The set* $\mathcal{T}(x) = \{t > 0; \frac{1}{t}x \in B_\xi\}$ *coincides with the interval* $(|x|_\xi, \infty)$.
(xi) *If* $x, y, z \in \mathcal{X}$ *are such that* $y \leq x \leq z$ *then* $|x|_\xi \leq max\{|y|_\xi, |z|_\xi\}$.

Proof. Properties (i)-(iv), (vi) and (vii), can be proved in a more general setting of Minkovski functionals, associated to some open and convex subsets in linear topological spaces (see [12]). The other properties are based on the special form of the set B_ξ given in (111). For details see [10].

From (i) and (iv) in Theorem 4 one obtains that the Minkovski functional is a seminorm.

The next result provides a sufficient condition such that the Minkovski seminorm becomes a norm.

Proposition 4 [10] *If* B_ξ *is a bounded set then the Minkovski seminorm* $|\cdot|_\xi$ *defined by (112) is a norm. Moreover there exists* $\alpha_\xi > 0$ *such that* $\|x\| \leq \alpha_\xi |x|_\xi$ *for all* $x \in \mathcal{X}$.

Proposition 5 [10] *If the cone* $\mathcal{X}_+$ *is normal then for all* $\xi \in Int\mathcal{X}_+$ *the set* B_ξ *is bounded.*

Let $(\mathcal{X}, ||\cdot||)$ be a real Banach space ordered by the closed, solid, normal, convex cone $\mathcal{X}_+$.
If $(\mathcal{Y}, ||\cdot||)$ is another Banach space, then $\mathbf{B}(\mathcal{X}, \mathcal{Y})$ stands for the space of linear and bounded operators defined on $\mathcal{X}$ and taking values in $\mathcal{Y}$.
When $\mathcal{X} = \mathcal{Y}$ we shall write $\mathbf{B}(\mathcal{X})$ instead of $\mathbf{B}(\mathcal{X}, \mathcal{X})$.

Under the considered assumptions, the Minkovski functional $|\cdot|_\xi$ is a norm equivalent with the norm $\|\cdot\|$ on $\mathcal{X}$.

If $T \in \mathbf{B}(\mathcal{X})$ then $\|T\|$ and $\|T\|_\xi$ are the operator norms of T, induced by $\|\cdot\|$ and $|\cdot|_\xi$, respectively. This means that

$$\|T\| = \sup_{\|x\| \le 1} \|Tx\| \tag{113}$$

$$\|T\|_\xi = \sup_{|x|_\xi \le 1} |Tx|_\xi. \tag{114}$$

Definition 6 *Let $(\mathcal{X}, \mathcal{X}_+)$, $(\mathcal{Y}, \mathcal{Y}_+)$ be two ordered linear spaces with the order relation induced by the convex cones $\mathcal{X}_+$ and $\mathcal{Y}_+$, respectively. An operator $T \in \mathbf{B}(\mathcal{X}, \mathcal{Y})$ is called positive operator if $T\mathcal{X}_+ \subset \mathcal{Y}_+$. In this case we shall write $T \ge 0$.*

By definition, if $T_1, T_2 \in \mathbf{B}(\mathcal{X})$ then $T_1 \le T_2$ or equivalently $T_2 \ge T_1$ if and only if $T_2 - T_1 \ge 0$.

Remark 12 *If $T : \mathcal{X} \to \mathcal{X}$ is a linear bounded and positive operator then T is a monotone operator. This means that $Tx \le Ty$ if $x \le y$.*

The next result provides a simple formula of the operator norm of a bounded linear positive operator induced by the Minkovski norm.

Theorem 5 *Let $(\mathcal{X}, \|\cdot\|)$ be a real Banach space ordered by a solid, closed, normal, convex cone $\mathcal{X}_+$. Let $\xi \in Int\mathcal{X}_+$ be fixed. Then for every positive operator $T \in \mathbf{B}(\mathcal{X})$ we have $\|T\|_\xi = |T\xi|_\xi$.*

Proof may be done in a standard way using (114).

C. Lebesque's theorem for discrete measures

In the sequel, we provide some useful applications of Lebesque's Theorem to the study of the series of real numbers.

Let $(\mathbf{Z}_+, \mathbf{2}^{(\mathbf{Z}_+)}, \mu)$ be the space with measure, where $\mathbf{Z}_+$ is the set of nonnegative integers, $\mathbf{2}^{(\mathbf{Z}_+)}$ is the family of all subsets of $\mathbf{Z}_+$ and $\mu : \mathbf{2}^{(\mathbf{Z}_+)} \to \overline{\mathbf{R}}_+$ is defined by $\mu(A)$ is the number of elements of A if A is a finite subset, $\mu(A) = +\infty$ if A is an infinite subset and $\mu(\emptyset) = 0$. It is obvious that $\mu(\{i\}) = 1$ if $i \in \mathbf{Z}_+$. A function $\mathbf{a} : \mathbf{Z}_+ \to \mathbf{R}$ is a sequence of real numbers $\mathbf{a} = \{\mathbf{a}(i)\}_{i \in \mathbf{Z}_+}$. It is easy to see that every function $\mathbf{a} : (\mathbf{Z}_+, \mathbf{2}^{(\mathbf{Z}_+)}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ is a measurable function. The Definition 7, (ii), Chapter 1 in [9] specialized to this framework allows us to say that a function $\mathbf{a} = \{\mathbf{a}(i)\}_{i\in\mathbf{Z}_+}$ is integrable if and only if $\sum_{i=0}^{\infty} |\mathbf{a}(i)| < \infty$. We have

$$\int_{\mathbf{Z}_+} \mathbf{a} d\mu = \sum_{i=0}^{\infty} \mathbf{a}(i)$$

if the right hand side is well defined.

Applying Theorem 11, Chapter 1, in [9] one obtains.

Corollary 7 *Let $\mathbf{a}_k, k \geq 0$ be a sequence of functions $\mathbf{a}_k = \{\mathbf{a}_k(i)\}_{i\in\mathbf{Z}_+}$ with the properties:*

(a) $\lim_{k\to\infty} \mathbf{a}_k(i) = \mathbf{x}(i)$ *for all* $i \in \mathbf{Z}_+$*;*

(b) $|\mathbf{a}_k(i)| \leq m(i)$, $k \geq 0$, $i \geq 0$ *where* $\sum_{i=0}^{\infty} m(i) < \infty$.

Under these conditions the following hold:

(i) *The series* $\sum_{i=0}^{\infty} |\mathbf{a}_k(i)|, k \geq 0, \sum_{i=0}^{\infty} |\mathbf{x}(i)|$ *are convergent.*

(ii) $\lim_{k\to\infty} \sum_{i=0}^{\infty} |\mathbf{a}_k(i) - \mathbf{x}(i)| = 0$.

(iii) $\lim_{k\to\infty} \sum_{i=0}^{\infty} \mathbf{a}_k(i) = \sum_{i=0}^{\infty} \mathbf{x}(i)$.

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