H_2 OPTIMAL FILTERING FOR DISCRETE-TIME LINEAR STOCHASTIC SYSTEMS WITH PERIODIC COEFFICIENTS AND MARKOVIAN JUMPING[*]

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Abstract

The purpose of the paper is to present a design procedure of the optimal filter for discrete-time stochastic linear system with periodic coefficients simultaneously affected by a non-homogeneous but periodic Markov chain and state multiplicative white noise perturbations. The optimal filter minimizes a performance index described by the Cesaro limit of the mean square of the deviations of the signal generated by the filter from the values of the signal which must be estimated. It is proved that the optimal filter with respect to the considered performance criterion has a Luenberger observer form which gain depends on the unique periodic solution of a discrete-time linear equation together with the stabilizing solution of a suitable discrete-time Riccati type

equation with periodic coefficients. The theoretical developments are illustrated by a numerical example.

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1 Introduction.

The estimation of a remote signal when measurements of another observed signal are available is a classical problem. Usually, both the remote signal and the measured signal are outputs of a dynamic system. In the case when the two signals are linear combinations of the states of a dynamic linear system, a significant advanced in the estimation problem was achieved by the Kalman-Bucy filter [15, 16]. In this case both the system describing the evolution of the states as well as the measured output are assumed to be corrupted only by additive noise. Although the first applications of the Kalman filtering were in aerospace domain, these techniques were rapidly disseminated in many other fields of engineering sciences, geophysics, economy and finance. An important issue in the Kalman type filtering intensively investigated over the last few decades is the influence of the modelling uncertainty of the plant which generates the remote signal over the filtering performance. It is a known fact that in the case of classical Kalman filters the performance deteriorates in the presence of modelling errors. Among the papers devoted to the robust filtering for systems subjected to parametric uncertainty one mentions for example in [18] and its references. Besides the deterministic representations of the model uncertainties there are many other cases when the plant parameters variations can be described as random perturbations of their nominal values. An important class of stochastic systems frequently used to approximate such variations are the models with state dependent noise (or multiplicative white noise). These stochastic systems have been intensively studied over the last four decades and many useful theoretical results including stability, robust design and optimal control are available (see e.g. [19] and [9]).

The filtering problem for discrete-time linear stochastic systems with multiplicative white noise perturbations attracted a great deal of interest in

the past decades under different assumptions and performance criteria. So, in [3, 4] was considered the case when the multiplicative noises affect only the measured output of the plant, while, in [2] was investigated the case when the multiplicative noises affect only the state equations. In [5, 20, 6] the estimation problem was considered for a wider class of linear stochastic systems. We refer to linear stochastic systems with Markovian jumping of the coefficients and affected by multiplicative and additive white noise perturbations.

Lately, there is an increasing interest regarding the various control problems for time varying systems with periodic coefficients. A convincing motivation concerning the arising of the mathematical models described by equations with periodic coefficients together with a rich list of references related to this topic, may be found in the monographs [1, 11, 12]. It is worth mentioning that unlike the general time varying context, in the case of controlled systems with periodic coefficients, the derived algorithms for different control problems may be implemented due to the finite memory required by the numerical computations.

The goal of this paper is to extend the results proved in Chapter 12 of [1] to the case of systems having the state space representation described by systems of discrete-time stochastic equations with periodic coefficients simultaneously affected by multiplicative and additive white noise perturbations and Markovian switching. It is known, see for example [10], that in the case of this kind of stochastic systems the Kalman filter constructed based on the stabilizing solution of corresponding Riccati type equation of stochastic control is not implementable because its state space representation contains multiplicative white noise.

In our approach the class of admissible filters consists of all discrete-time linear systems with periodic coefficients and Markovian switching having arbitrary dimension of the state space. To measure the performance achieved by an admissible filter, we introduced a performance index described by the Cesaro limit of the mean square of the deviations of the signal generated by the filter from the values of the signal which must be estimated. We show that the optimal filter with respect to the considered performance criterion has in fact the structure of a Luenberger observer which gain is constructed based on the unique periodic solution of a discrete-time linear equation together with the stabilizing solution of a suitable discrete-time Riccati type equation with periodic coefficients. Unlike the filters constructed

via discrete-time Riccati equation of stochastic control, as in [10], the optimal filter derived in the present paper is implementable. The theoretical developments are illustrated by a numerical example.

2 Problem formulation

Consider the system **G** having the state space representation described by:

$$
\mathbf{G} \begin{cases}
x(t+1) = \left[A_0(t, \eta_t) + \sum_{k=1}^{r} w_k(t) A_k(t, \eta_t) \right] x(t) + B(t, \eta_t) v(t), \\
y(t) = \left[C_0(t, \eta_t) + \sum_{k=1}^{r} w_k(t) C_k(t, \eta_t) \right] x(t) + D(t, \eta_t) v(t), \\
z(t) = C_z(t, \eta_t) x(t).
\end{cases}
\tag{1}
$$

$t \in \mathbb{Z}_+ = \{0, 1, 2, ...\}$, where $x(t) \in \mathbb{R}^n$ are the state vectors, $y(t) \in \mathbb{R}^{n_y}$ are the measurements available at time t, while $z(t)$ is the signal which must be estimated. In (1) the sequence $\{\eta_t\}_{t \geq 0}$ is a nonhomogeneous Markov chain on a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with the set of the states $\mathfrak{S} = \{1, 2, ..., N\}$ and the sequence of probability transition matrices $\{P_t\}_t \geq 0$ (for details regarding the nonhomogeneous Markov chains we refer to [7] or [14]); $\{w(t)\}_{t \geq 0}$ ($w(t) = (w_1(t), w_2(t), ..., w_r(t))^T$), $\{v(t)\}_{t \geq 0}$ are sequences of independent random vectors satisfying the assumptions:

H$_1$) $E[w(t)] = 0$, $E[w(t)w^T(t)] = I_r$ for all $t \in \mathbf{Z}_+$; the stochastic process $\{w(t)\}_{t \geq 0}$ is independent of the stochastic process $\{\eta_t\}_{t \geq 0}$.

H$_2$) $E[v(t)] = 0$, $E[v(t)v^T(t)] = I_{m_v}$ for all $t \in \mathbf{Z}_+$; the stochastic process $\{v(t)\}_{t \geq 0}$ is independent of $\{w(t), \eta_t\}_{t \geq 0}$.

As usual $E[\cdot]$ stands for the mathematical expectation and superscript T denotes the transposition of a matrix or a vector.

Regarding the coefficients of the system (1) we assume that the sequences $\{A_k(t, i)\}_{t \geq 0} \subset \mathbb{R}^{n \times n}$, $\{C_k(t, i)\}_{t \geq 0} \subset \mathbb{R}^{n_y \times n}$, $0 \leq k \leq r$, $\{B(t, i)\}_{t \geq 0} \subset \mathbb{R}^{n \times m_v}$, $\{C_z(t, i)\}_{t \geq 0} \subset \mathbf{R}^{n_z \times n}$, $\{D(t, i)\}_{t \geq 0} \subset \mathbb{R}^{n_y \times m_v}$, are periodic with a period $\theta \geq 1$. Throughout this paper, we also assume that the sequence $\{P_t\}_{t \geq 0}$ is periodic with period θ.

Our goal is to design a linear dynamic system (often called *filter*) fed to its input with the measurements $y(s)$, $0 \leq s \leq t$, such that its output $z_F(t)$ to be "a best estimation" of $z(t)$. The class of admissible filters considered

in this paper consists of the family of linear systems $\mathbf{G}_F$ of the form:

$$\mathbf{G}_F \begin{cases} x_F(t+1) = A_F(t, \eta_t)x_F(t) + B_F(t, \eta_t)y(t), \\ z_F(t) = C_F(t, \eta_t)x_F(t). \end{cases} \tag{2}$$

with the properties:

α) the sequences $\{A_F(t,i)\}_{t \in \mathbb{Z}_+} \subset \mathbb{R}^{n_F \times n_F}$, $\{B_F(t,i)\}_{t \in \mathbb{Z}_+} \subset \mathbb{R}^{n_F \times n_y}$, $\{C_F(t,i)\}_{t \in \mathbb{Z}_+} \subset \mathbb{R}^{n_z \times n_F}$ are periodic with period θ.

β) the zero solution of the linear equation

$$x_F(t+1) = A_F(t, \eta_t)x_F(t) \tag{3}$$

is exponentially stable in mean square (ESMS).

Throughout this paper $\mathfrak{F}_s$ stands for the set of all filters $\mathbf{G}_F$ of arbitrary dimension $n_F \geq 1$ of the state space, satisfying the conditions α) and β) from above.

To measure the quality of the estimation achieved by an admissible filter we introduce the performance index $J : \mathfrak{F}_s \to \mathbb{R}_+$ defined by

$$J(\mathbf{G}_F) = \lim_{\tau \to \infty} \frac{1}{\tau} \sum_{t=0}^{\tau} E[|z(t) - z_F(t)|^2]. \tag{4}$$

In Section 4 we shall provide a set of conditions which guarantee the existence of a filter $\tilde{\mathbf{G}}_F \in \mathfrak{F}_s$ satisfying the optimality condition $J(\tilde{\mathbf{G}}_F) = \min_{\mathbf{G}_F \in \mathfrak{F}_s} J(\mathbf{G}_F)$.

Also, we shall provide a state space representation of the optimal filter $\tilde{\mathbf{G}}_F$. In the special case of $N = 1$ (no Markovian jumps) and $A_k(t, 1) = 0$, $C_k(t, 1) = 0$, $1 \leq k \leq r$, $t \in \mathbb{Z}_+$, the optimal filter derived in this paper recovers the one designed in [1]. It is worth mentioning that in our approach, the class of admissible filters is wider than the one considered in [1], where, only filters in state observer form were considered.

3   Computation of the performance of an admissible filter

When a filter $\mathbf{G}_F$ is coupling to the system (1) one obtains the following system on the space $\mathbb{R}^n \times \mathbb{R}^{n_F}$:

$$\begin{aligned} x_{cl}(t+1) &= \left[A_{0cl}(t, \eta_t) + \sum_{k=1}^{r} w_k(t) A_{kcl}(t, \eta_t) \right] x_{cl}(t) + B_{cl}(t, \eta_t)v(t) \\ z_{cl}(t) &= z(t) - z_F(t) = C_{cl}(t, \eta_t)x_{cl}(t) \end{aligned} \tag{5}$$

where we denoted:

$$x_{cl}(t) = \begin{pmatrix} x^T(t) & x_F^T(t) \end{pmatrix}^T, A_{0cl}(t,i) = \begin{pmatrix} A_0(t,i) & 0 \\ B_F(t,i)C_0(t,i) & A_F(t,i) \end{pmatrix},$$
$$A_{kcl}(t,i) = \begin{pmatrix} A_k(t,i) & 0 \\ B_F(t,i)C_k(t,i) & 0 \end{pmatrix}, B_{cl}(t,i) = \begin{pmatrix} B(t,i) \\ B_F(t,i)D(t,i) \end{pmatrix}, \qquad (6)$$
$$C_{cl}(t,i) = \begin{pmatrix} C_z(t,i) & -C_F(t,i) \end{pmatrix}.$$

Consider the linear system

$$x(t+1) = \left[A_0(t,\eta_t) + \sum_{k=1}^{r} w_k(t)A_k(t,\eta_t) \right] x(t) \qquad (7)$$

obtained from (1) for $B(t,i) = 0$. Combining Corollary 3.9 (iii) with Theorem 3.10 in [6] one obtains:

Corollary 3.1. If the zero solution of the linear system (7) is ESMS, then for any admissible filter $\mathbf{G}_F \in \mathfrak{F}_s$ the zero solution of the linear closed-loop system

$$x_{cl}(t+1) = \left[A_{0cl}(t,\eta_t) + \sum_{k=1}^{r} w_k(t)A_{kcl}(t,\eta_t) \right] x_{cl}(t) \qquad (8)$$

is ESMS.

Before stating the main result of this section, let us introduce the notation: $\mathfrak{P}_t = P_0 \cdot P_1 \cdot ... \cdot P_{t-1}$, $t \geq 1$. Since $\mathfrak{P}_\theta$ is a stochastic matrix, then the following matrix is well defined by

$$\mathfrak{Q}(\theta) = \lim_{\tau \to \infty} \frac{1}{\tau} \sum_{k=0}^{\tau} \mathfrak{P}_\theta^k. \qquad (9)$$

Throughout this paper, $\mathcal{S}_d$ stands for the linear space of real symmetric matrices of size $d \times d$. Set, $\mathcal{S}_d^N = \mathcal{S}_d \times \mathcal{S}_d... \times \mathcal{S}_d$. The space $\mathcal{S}_d^N$ equipped with the inner product

$$< \mathbf{X}, \mathbf{Y} > = \sum_{j=1}^{N} Tr[X(j)Y(j)] \qquad (10)$$

is a Hilbert space.

Based on the coefficients of the system (8), we introduce the linear operators $\mathcal{L}_{cl}(t) : \mathcal{S}^N_{n+n_F} \to \mathcal{S}^N_{n+n_F}$ by $\mathcal{L}_{cl}(t)\mathbf{Y} = (\mathcal{L}_{cl1}(t)\mathbf{Y}, ..., \mathcal{L}_{clN}(t)\mathbf{Y})$ with

$$\mathcal{L}_{cli}(t)\mathbf{Y} = \sum_{k=0}^{r} \sum_{j=1}^{N} p_t(j,i) A_{kcl}(t,j) Y(j) A_{kcl}^T(t,j) \tag{11}$$

for all $\mathbf{Y} = (Y(1), ..., Y(N)) \in \mathcal{S}^N_{n+n_F}$.

The main result of this section is:

Theorem 3.2. *Under the considered assumptions, if the zero solution of the discrete-time linear system (7) is ESMS, then, for each admissible filter* $\mathbf{G}_F$, *the value of the performance index (4) is given by:*

$$J(\mathbf{G}_F) = \frac{1}{\theta} \sum_{t=0}^{\theta-1} \sum_{j=1}^{N} Tr[C_{cl}(t,j) Y_{cl}^{\pi_0}(t,j) C_{cl}^T(t,j)] \tag{12}$$

where $\mathbf{Y}_{cl}^{\pi_0}(t) = (Y_{cl}^{\pi_0}(t,1), ..., Y_{cl}^{\pi_0}(t,N))$ *is the unique θ-periodic solution of the following discrete-time forward affine equation:*

$$\mathbf{Y}_{cl}^{\pi_0}(t+1) = \mathcal{L}_{cl}(t)\mathbf{Y}_{cl}^{\pi_0}(t) + \mathbf{B}_{cl}^{\pi_0}(t) \tag{13}$$

$\mathbf{B}_{cl}^{\pi_0}(t) = (\mathcal{B}_{cl}^{\pi_0}(t,1), ..., \mathcal{B}_{cl}^{\pi_0}(t,N))$ *with*

$$\mathcal{B}_{cl}^{\pi_0}(t,i) = \sum_{j=1}^{N} p_t(j,i) \mu_t^{\pi_0}(j) B_{cl}(t,j) B_{cl}^T(t,j) \tag{14}$$

$1 \leq i \leq N$, $t \in \mathbb{Z}_+$.

The scalars $\mu_t^{\pi_0}(j)$ involved in (14) are computed via

$$\mu_t^{\pi_0}(j) = \sum_{l=1}^{N} \pi_0(l) \mu_t(l,j) \tag{15}$$

$\mu_t(l,j)$ *being the elements of the matrix*

$$M(t) = \mathfrak{Q}(\theta)\mathfrak{P}_{t+1} \tag{16}$$

$t \in \mathbb{Z}_+$, $\mathfrak{Q}(\theta)$ *being the matrix introduced in (9) and $\pi_0 = (\pi_0(1), ..., \pi_0(N))$ is the initial distribution of the Markov chain.*

Proof. First, let us remark that $J(\mathbf{G}_F) = \lim\limits_{\tau \to \infty} \frac{1}{\tau} \sum\limits_{t=0}^{\tau} E[|C_{cl}(t, \eta_t)x_{cl}(t)|^2]$. Applying Theorem 4.2 in [8], we obtain

$$J(\mathbf{G}_F) = \frac{1}{\theta} \sum_{t=0}^{\theta-1} \sum_{l,j=1}^{N} p_t(j,l)\mu_t^{\pi_0}(j)Tr[B_{cl}^T(t,j)\tilde{X}_{cl}(t+1,l)B_{cl}(t,j)] \qquad (17)$$

where the scalars $\mu_t^{\pi_0}(j)$ are defined by (15)-(16) and

$$\tilde{\mathbf{X}}_{cl}(t) = (\tilde{X}_{cl}(t,1), ..., \tilde{X}_{cl}(t,N))$$

is the unique θ-periodic solution of the following discrete-time backward affine equation:

$$\tilde{\mathbf{X}}_{cl}(t) = \mathcal{L}_{cl}^*(t)\tilde{\mathbf{X}}_{cl}(t+1) + \Xi_{cl}(t) \qquad (18)$$

where $\Xi_{cl}(t) = (C_{cl}^T(t,1)C_{cl}(t,1), ..., C_{cl}^T(t,N)C_{cl}(t,N))$. In (18) $\mathcal{L}_{cl}^*(t)$ is the adjoint operator of (11) with respect to the inner product (10). Further, combining (14) with (10), we may write successively:

$$\sum_{j,l=1}^{N} p_t(j,l)\mu_t^{\pi_0}(j)Tr[B_{cl}^T(t,j)\tilde{X}_{cl}(t+1,l)B_{cl}(t,j)] =$$

$$= \sum_{l=1}^{N} Tr[\tilde{X}_{cl}(t+1,l)\mathcal{B}_{cl}^{\pi_0}(t,l)] =< \tilde{\mathbf{X}}_{cl}(t+1), \mathbf{B}_{cl}^{\pi_0}(t) > . \qquad (19)$$

Based on the equations (13) and (18) we may write:

$$< \tilde{\mathbf{X}}_{cl}(t+1), \mathbf{B}_{cl}^{\pi_0}(t) >=< \tilde{\mathbf{X}}_{cl}(t+1), \mathbf{Y}_{cl}^{\pi_0}(t+1) > -$$

$$- < \tilde{\mathbf{X}}_{cl}(t+1), \mathcal{L}_{cl}(t)\mathbf{Y}_{cl}^{\pi_0}(t) >=< \tilde{\mathbf{X}}_{cl}(t+1), \mathbf{Y}_{cl}^{\pi_0}(t+1) > -$$

$$- < \mathcal{L}_{cl}^*(t)\tilde{\mathbf{X}}_{cl}(t+1), \mathbf{Y}_{cl}^{\pi_0}(t) >=< \tilde{\mathbf{X}}_{cl}(t+1), \mathbf{Y}_{cl}^{\pi_0}(t+1) > -$$

$$- < \tilde{\mathbf{X}}_{cl}(t), \mathbf{Y}_{cl}^{\pi_0}(t) > + < \Xi_{cl}(t), Y_{cl}^{\pi_0}(t) > .$$

So, (19) becomes:

$$\begin{aligned} &\sum_{j,l=1}^{N} p_t(j,l)\mu_t^{\pi_0}(j)Tr[B_{cl}^T(t,j)\tilde{X}_{cl}(t+1,l)B_{cl}(t,j)] \\ &=< \tilde{\mathbf{X}}_{cl}(t+1), \tilde{\mathbf{Y}}_{cl}^{\pi_0}(t+1) > - < \tilde{\mathbf{X}}_{cl}(t), \tilde{\mathbf{Y}}_{cl}^{\pi_0}(t) > + < \Xi_{cl}(t), \mathbf{Y}_{cl}^{\pi_0}(t) > \end{aligned} \qquad (20)$$

$0 \leq t \leq \theta - 1$.

Plugging (20) in (17) and taking into account that $\tilde{\mathbf{X}}_{cl}(\cdot)$ and $\mathbf{Y}_{cl}^{\pi_0}(\cdot)$ are θ-periodic sequences, we obtain:

$$J(\mathbf{G}_F) = \frac{1}{\theta} \sum_{t=0}^{\theta-1} < \Xi_{cl}(t), Y_{cl}(t) > . \tag{21}$$

Invoking again the formula of the inner product from (10) we deduce that (21) coincides with (12). Thus the proof is complete.

From (12)-(15) one sees that the value of the performance $J(\mathbf{G}_F)$ depends upon the initial distribution π_0 of the Markov chain. Based on the monotonicity property of the θ periodic solution of the Lyapunov type equations we may introduce a new performance criterion not depending upon the initial distributions of the Markov chain. If $0 \leq t \leq \theta - 1$ and $1 \leq j \leq N$ we define

$$\nu_t(j) = \max_{1 \leq i \leq N} \mu_t(i,j). \tag{22}$$

From (15) one obtains

$$\mu_t^{\pi_0}(j) \leq \nu_t(j) \tag{23}$$

for all $0 \leq t \leq \theta - 1$, $1 \leq j \leq N$ and for any initial distribution π_0 of the Markov chain.

Let $\tilde{\mathbf{Y}}_{cl}(t) = (\tilde{Y}_{cl}(t,1), ..., \tilde{Y}_{cl}(t,N))$ be the unique θ periodic solution of the discrete-time forward affine equation

$$\mathbf{Y}_{cl}(t+1) = \mathcal{L}_{cl}(t)\mathbf{Y}_{cl}(t) + \tilde{\mathbf{B}}_{cl}(t) \tag{24}$$

where $\tilde{\mathbf{B}}_{cl}(t) = (\tilde{\mathcal{B}}_{cl}(t,1), ..., \tilde{\mathcal{B}}_{cl}(t,N))$ with

$$\tilde{\mathcal{B}}_{cl}(t,i) = \sum_{j=1}^{N} p_t(j,i)\tilde{\nu}_t(j)B_{cl}(t,j)B_{cl}^T(t,j). \tag{25}$$

In (25) the scalars $\tilde{\nu}_t(j)$, $t \in \mathbb{Z}$ are obtained from $\nu_t(j)$ by periodicity, i.e. $\tilde{\nu}_t(j) = \nu_{t-[\frac{t}{\theta}]\theta}(j)$ for all $t \in \mathbb{Z}$, where $[\frac{t}{\theta}]$ is the largest integer, less or equal with $\frac{t}{\theta}$. If the zero solution of (7) is ESMS, then for each admissible filter $\mathbf{G}_F \in \mathfrak{F}_s$, the corresponding equation (24) has a unique bounded on $\mathbb{Z}$ solution $\{\tilde{\mathbf{Y}}_{cl}(t)\}_{t \in \mathbb{Z}}$. Additionally, this solution is θ periodic.

Furthermore, if $\mathbf{Y}_{cl}^{\pi_0}(t) = (Y_{cl}^{\pi_0}(t,1), ..., Y_{cl}^{\pi_0}(t,N))$ is the unique θ-periodic solution of the discrete-time affine equation (13)-(16), we have

$$Y_{cl}^{\pi_0}(t,i) \leq \tilde{Y}_{cl}(t,i) \qquad (26)$$

for all $(t,i) \in \mathbb{Z} \times \mathfrak{S}$. We introduce the performance index

$$\tilde{J}(\mathbf{G}_F) = \frac{1}{\theta} \sum_{t=0}^{\theta-1} \sum_{i=1}^{N} Tr[C_{cl}(t,i)\tilde{Y}_{cl}(t,i)C_{cl}^T(t,i)]. \qquad (27)$$

From (12), (26) and (27) we deduce that

$$J(\mathbf{G}_F) \leq \tilde{J}(\mathbf{G}_F) \qquad (28)$$

for all $\mathbf{G}_F \in \mathfrak{F}_s$. $\tilde{J}(\cdot)$ introduced via (27) does not depend upon the initial distributions of the Markov chain. From (28), we deduce that the minimization of the cost (27) may lead to an acceptable value of the cost (4).

4   Main result

In order to provide a unified approach of the optimization problems asking for the designing of a filter $\mathbf{G}_F \in \mathfrak{F}_s$ minimizing either the cost $J(\cdot)$ introduced via (4) or the cost $\tilde{J}(\cdot)$ introduced in (27) we shall consider a new performance criterion $J_\epsilon : \mathfrak{F}_s \to \mathbb{R}_+$ defined by

$$J_\epsilon(\mathbf{G}_F) = \frac{1}{\theta} \sum_{t=0}^{\theta-1} \sum_{i=1}^{N} Tr[C_{cl}(t,i)Y_{cl}^\epsilon(t,i)C_{cl}^T(t,i)] \qquad (29)$$

where $\mathbf{Y}_{cl}^\epsilon(t) = (Y_{cl}^\epsilon(t,1), ..., Y_{cl}^\epsilon(t,N))$ is the unique θ-periodic solution of the discrete-time forward affine equation

$$\mathbf{Y}_{cl}^\epsilon(t+1) = \mathcal{L}_{cl}(t)\mathbf{Y}_{cl}^\epsilon(t) + \mathbf{B}_{cl}^\epsilon(t), \ \ t \in \mathbb{Z} \qquad (30)$$

where $\mathbf{B}_{cl}^\epsilon(t) = (\mathcal{B}_{cl}^\epsilon(t,1), ..., \mathcal{B}_{cl}^\epsilon(t,N))$ with

$$\mathcal{B}_{cl}^\epsilon(t,i) = \sum_{j=1}^{N} p_t(j,i)\epsilon_t(j)B_{cl}(t,j)B_{cl}^T(t,j) \qquad (31)$$

$\{\epsilon_t(j)\}_{t \in \mathbb{Z}}$, $1 \leq j \leq N$ are some given θ-periodic sequences of non-negative real numbers. One sees that in the special case $\epsilon_t(j) = \mu_t^{\pi_0}(j)$ we have

$J_\epsilon(\mathbf{G}_F) = J(\mathbf{G}_F)$ for all $\mathbf{G}_F \in \mathfrak{F}_s$ while, if $\epsilon_t(j) = \tilde{\nu}_t(j)$ for all $t, j \in \mathbb{Z} \times \mathfrak{S}$, then for any $\mathbf{G}_F \in \mathfrak{F}_s$ we have $J_\epsilon(\mathbf{G}_F) = \tilde{J}(\mathbf{G}_F)$.

Let us introduce the discrete-time forward affine equation on $\mathcal{S}_n^N$:

$$
\begin{aligned}
Q_c(t+1, i) = \sum_{j=1}^{N} p_t(j, i) \Big\{ & \sum_{k=0}^{r} A_k(t, j) Q_c(t, j) A_k^T(t, j) \\
& + \epsilon_t(j) B(t, j) B^T(t, j) \Big\}, \quad 1 \le i \le N.
\end{aligned}
\tag{32}
$$

The equation (32) may be written in a compact form as:

$$
\mathbf{Q}_c(t+1) = \mathcal{L}(t)\mathbf{Q}_c(t) + \mathbf{B}^\epsilon(t)
\tag{33}
$$

where $\mathbf{Q}_c(t) = (Q_c(t, 1), ..., Q_c(t, N))$,

$$
\mathbf{B}^\epsilon(t) = \left(\sum_{j=1}^{N} p_t(j, 1) \epsilon_t(j) B(t, j) B^T(t, j), ..., \sum_{j=1}^{N} p_t(j, N) \epsilon_t(j) B(t, j) B^T(t, j) \right)
\tag{34}
$$

and $\mathcal{L}_t : \mathcal{S}_n^N \to \mathcal{S}_n^N$ is defined by $\mathcal{L}(t)\mathbf{X} = (\mathcal{L}_1(t)\mathbf{X}, ..., \mathcal{L}_N(t)\mathbf{X})$,

$$
\mathcal{L}_i(t)\mathbf{X} = \sum_{j=1}^{N} p_t(j, i) \sum_{k=0}^{r} A_k(t, j) X(j) A_k^T(t, j)
\tag{35}
$$

for all $\mathbf{X} = (X(1), ..., X(N)) \in \mathcal{S}_n^N$.

Combining Theorem 3.10 and Theorem 2.6 (ii) from [6], we obtain:

Corollary 4.1. Under the considered assumptions if the zero solution of the discrete-time linear system (7) is ESMS, then the discrete-time forward affine equation (32) has a unique bounded solution $\{\mathbf{Q}_c(t)\}_{t \in \mathbb{Z}}$. Moreover this solution is a θ-periodic sequence. It has the representation formula:

$$
\tilde{\mathbf{Q}}(t) = \sum_{s=-\infty}^{t-1} T(t, s+1) \mathbf{B}^\epsilon(s)
\tag{36}
$$

where $T(t, s) = \mathcal{L}(t-1)\mathcal{L}(t-2)...\mathcal{L}(t, s)$ if $t > s$ and $T(t, s) = I_{\mathcal{S}_n^N}$ if $t = s$ ($I_{\mathcal{S}_n^N}$ being the identity operator on $\mathcal{S}_n^N$).

Remark 4.1. From (36) one obtains $\tilde{\mathbf{Q}}_c(\theta) = T(\theta, 0)\tilde{\mathbf{Q}}_c(0) + \sum_{s=0}^{\theta-1} T(\theta, s+1)\mathbf{B}^\epsilon(s)$. The periodicity condition $\tilde{\mathbf{Q}}_c(\theta) = \tilde{\mathbf{Q}}_c(0)$ leads to

$$
[I_{\mathcal{S}_n^N} - T(\theta, 0)]\tilde{\mathbf{Q}}_c(0) = \sum_{s=0}^{\theta-1} T(\theta, s+1)\mathbf{B}^\epsilon(s).
\tag{37}
$$

This shows that the initial value $\tilde{\mathbf{Q}}_c(0)$ of the θ-periodic solution of (32) is obtained as solution of the linear equation (37). Since the zero solution of the system (7) is ESMS, iff $\rho[T(\theta, 0)] < 1$ ($\rho[\cdot]$ being the spectral radius) we may conclude that under the assumptions of Corollary 4.1, the linear equation (37) has a unique solution.

Let us associate the following discrete-time Riccati equation of filtering (DTRE-F)

$$
\begin{aligned}
Y(t+1, i) = \sum_{j=1}^{N} p_t(j, i) &\{ A_0(t, j) Y(t, j) A_0^T(t, j) - [A_0(t, j) Y(t, j) C_0^T(t, j) \\
&+ L^\epsilon(t, j)][R^\epsilon(t, j) + C_0(t, j) Y(t, j) C_0^T(t, j)]^{-1} \\
&\times [C_0(t, j) Y(t, j) A_0^T(t, j) + (L^\epsilon(t, j))^T] + M^\epsilon(t, j) \}
\end{aligned}
\tag{38}
$$

where we denoted

$$
R^\epsilon(t, j) = \epsilon_t(j) D(t, j) D^T(t, j) + \sum_{k=1}^{r} C_k(t, j) \tilde{Q}_c(t, j) C_k^T(t, j)
$$

$$
L^\epsilon(t, j) = \epsilon(j) B(t, j) D^T(t, j) + \sum_{k=1}^{r} A_k(t, j) \tilde{Q}_c(t, j) C_k^T(t, j)
\tag{39}
$$

$$
M^\epsilon(t, j) = \epsilon_t(j) B(t, j) B^T(t, j) + \sum_{k=1}^{r} A_k(t, j) \tilde{Q}_c(t, j) A_k^T(t, j)
$$

We recall that a global solution $\{\mathbf{Y}_s(t)\}_{t \in \mathbb{Z}}$ of the DTRE-F (38) is a stabilizing solution if the zero solution of the closed-loop system

$$
x(t + 1) = [A_0(t, \eta_t) + K_s(t, \eta_t) C_0(t, \eta_t)] x(t)
\tag{40}
$$

is ESMS, where

$$
\begin{aligned}
K_s(t, j) = [A_0(t, j) Y_s(t, j) C_0^T(t, j) + L^\epsilon(t, j)][R^\epsilon(t, j) \\
+ C_0(t, j) Y_s(t, j) C_0^T(t, j)]^{-1}, \quad (t, j) \in \mathbb{Z} \times \mathfrak{S}.
\end{aligned}
\tag{41}
$$

Remark 4.2. a) In the definition of the stabilizing solution $\mathbf{Y}(\cdot)$ we tacitly assumed that the matrices $R^\epsilon(t, j) + C_0(t, j) Y_s(t, j) C_0^T(t, j)$ are invertible for all $(t, j) \in \mathbb{Z} \times \mathfrak{S}$.

b) Since the coefficients of the DTRE-F are θ-periodic sequences then the bounded and stabilizing solution $\{\mathbf{Y}_s(t)\}_{t \in \mathbb{Z}}$, if it exists, is also a θ-periodic sequence.

Reasoning as in the proof of Theorem 26 in [21] (see also Theorem A5 in [8]), we obtain the following set of necessary and sufficient conditions for the existence of the stabilizing and θ-periodic solution of DTRE (38).

Theorem 4.2. *Assume: a) the zero solution of the system (7) is ESMS.*

b) $\sum_{i=1}^{N} p_t(i,j) > 0$, $0 \le t \le \theta - 1$, $1 \le j \le N$.

Under these conditions, the following are equivalent:

(i) the DTRE-F (38) has a stabilizing solution $\{\mathbf{Y}_s(t)\}_{t\in\mathbb{Z}}$ which is θ-periodic and satisfies the following sign conditions:

$$R^\epsilon(t,i) + C_0(t,i)Y_s(t,i)C_0^T(t,i) > 0 \tag{42}$$

for $0 \le t \le \theta - 1$, $1 \le i \le N$;

(ii) there exist the symmetric matrices $Z(t,i) \in \mathbb{R}^{n\times n}$, $0 \le t \le \theta - 1$, $1 \le i \le N$ satisfying the following system of LMIs:

$$\begin{pmatrix} A_0(t,i) \\ C_0(t,i) \end{pmatrix} \sum_{j=1}^{N} p_{t-1}(j,i)Z(t-1,j) \begin{pmatrix} A_0(t,i) \\ C_0(t,i) \end{pmatrix}^T -$$

$$diag[Z(t,i),0] + \begin{pmatrix} M^\epsilon(t,i) & L^\epsilon(t,i) \\ (L^\epsilon(t,i))^T & R^\epsilon(t,i) \end{pmatrix} > 0$$

$0 \le t \le \theta - 1$, $1 \le i \le N$ *with $Z(-1,i) = Z(\theta - 1,i)$ and $p_{-1}(i,j) = p_{\theta-1}(i,j)$ $\forall i,j \in \{1,2,...,N\}$.*

The dependence of the stabilizing solution of (38) and of the stabilizing gain (41) with respect to ϵ was suppressed for the sake of simplicity.

The main result of this paper is:

Theorem 4.3. *a) The assumptions H_1) and H_2) are fulfilled;*

b) the zero solution of the linear system (7) is ESMS;

c) the DTRE-F (38) has a θ-periodic and stabilizing solution $\{\mathbf{Y}_s(t)\}_{t\in\mathbb{Z}}$ which satisfy the sign condition (42).

Consider the filter $\tilde{\mathbf{G}}_F$ having the state space representation given by

$$\begin{aligned} x_F(t+1) &= [A_0(t,\eta_t) + K_s(t,\eta_t)C_0(t,\eta_t)]x_F(t) - K_s(t,\eta_t)y(t) \\ z_F(t) &= C_z(t,\eta_t)x_F(t) \end{aligned} \tag{43}$$

where $K_s(t,i)$ is introduced via (41).

Under the considered assumptions, the filter $\tilde{\mathbf{G}}_F$ lies in $\mathfrak{F}_s$ and satisfies the optimality condition: $J_\epsilon(\tilde{\mathbf{G}}_F) = \min_{\mathbf{G}_F\in\mathfrak{F}_s} J_\epsilon(\mathbf{G}_F)$.

The optimal value achieved by the cost performance (29) is:

$$J_\epsilon(\tilde{\mathbf{G}}_F) = \frac{1}{\theta} \sum_{t=0}^{\theta-1} \sum_{i=1}^{N} Tr[C_z(t,i)Y_s(t,i)C_z^T(t,i)]. \qquad (44)$$

Proof. The fact that $\tilde{\mathbf{G}}_F \in \mathfrak{F}_s$ is obtained using Corollary 3.1. To this end, one takes into account that, in this special case, the system (3) coincides with (40). Let $\mathbf{G}_F \in \mathfrak{F}_s$ be an arbitrary admissible filter and let $\mathbf{Y}_{cl}^\epsilon(t) = (Y_{cl}^\epsilon(t,1), ..., Y_{cl}^\epsilon(t,N))$ be the θ-periodic solution of the corresponding discrete-time forward affine equation (30)-(31).

Let $\begin{pmatrix} Y_{11}(t,i) & Y_{12}(t,i) \\ Y_{12}^T(t,i) & Y_{22}(t,i) \end{pmatrix}$ be the partition of the matrix $Y_{cl}^\epsilon(t,i)$ compatible with the structure of the coefficients given in (6), i.e. $Y_{11}(t,i) \in \mathcal{S}_n$ and $Y_{22}(t,i) \in \mathcal{S}_{n_F}$. Based on (6) and (11) we obtain the following partition of (30)-(31):

$$
\begin{aligned}
Y_{11}(t+1,i) &= \sum_{j=1}^{N} p_t(j,i)\left\{\sum_{k=0}^{r} A_k(t,j)Y_{11}(t,j)A_k^T(t,j) + \epsilon_t(j)B(t,j)B^T(t,j)\right\} \\
Y_{12}(t+1,i) &= \sum_{j=1}^{N} p_t(j,i)\left\{\sum_{k=0}^{r} A_k(t,j)Y_{11}(t,j)C_k^T(t,j)B_F^T(t,j)\right. \\
&\qquad\qquad\left. + A_0(t,j)Y_{12}(t,j)A_F^T(t,j) + \epsilon(j)B(t,j)D^T(t,j)B_F^T(t,j)\right\} \\
Y_{22}(t+1,i) &= \sum_{j=1}^{N} p_t(j,i)\left\{B_F(t,j)C_0(t,j)Y_{11}(t,j)C_0^T(t,j)B_F^T(t,j)\right. \\
&\qquad\qquad + A_F(t,j)Y_{22}(t,j)A_F^T(t,j) \\
&\qquad\qquad + \sum_{k=1}^{r} B_F(t,j)C_k(t,j)Y_{11}(t,j)C_k^T(t,j)B_F^T(t,j) \\
&\qquad\qquad\left. + \epsilon(j)B_F(t,j)D(t,j)D^T(t,j)B_F^T(t,j)\right\}.
\end{aligned}
\qquad (45)
$$

One sees that the first equation of (45) coincides with (32). From the uniqueness of θ-periodic solution of the equation (32) we deduce that $Y_{11}(t,i) = \tilde{Q}_c(t,i)$ for all $(t,i) \in \mathbb{Z} \times \mathfrak{S}$. We set, $U(t,i) = \begin{pmatrix} \tilde{Q}_c(t,i) - Y_s(t,i) & Y_{12}(t,i) \\ Y_{12}^T(t,i) & Y_{22}(t,i) \end{pmatrix}$, $(t,i) \in \mathbb{Z} \times \mathfrak{S}$. By direct calculations, involving (38) together with (45) with $Y_{11}(t,i)$ replaced by $\tilde{Q}_c(t,i)$ one obtains that $\mathbf{U}(t) = (U(t,1), ..., U(t,N))$ is

a θ-periodic solution of the following discrete-time forward affine equation:

$$
U(t+1, i) = \sum_{j=1}^{N} p_t(j, i)\{A_{0cl}(t, j)U(t, j)A_{0cl}^T(t, j) + \tag{46}
$$

$$
+\check{B}_{cl}(t, j)[R^\epsilon(t, j) + C_0(t, j)Y_s(t, j)C_0^T(t, j)]\check{B}_{cl}^T(t, j)\}
$$

for all $(t, i) \in \mathbb{Z} \times \mathfrak{S}$, where $\check{B}_{cl}(t, i) = \begin{pmatrix} K_s(t, i) \\ -B_F(t, i) \end{pmatrix}$.

Since $\mathbf{G}_F \in \mathfrak{F}_s$ we deduce via Corollary 3.1 that the zero solution of the corresponding system (8) is ESMS. Consequently, the zero solution of the discrete-time linear equation $x_{cl}(t+1) = A_{0cl}(t, \eta_t)x_{cl}(t)$ is also ESMS.

Invoking (42) we may conclude that the unique θ-periodic solution of (46) satisfies

$$
U(t, i) \geq 0 \tag{47}
$$

for all $(t, i) \in \mathbb{Z} \times \mathfrak{S}$.

Further, we rewrite (29) in the form

$$
J_\epsilon(\mathbf{G}_F) = \frac{1}{\theta}\sum_{t=0}^{\theta-1}\sum_{i=1}^{N} Tr[C_z(t, i)Y_s(t, i)C_z^T(t, i)] +
$$

$$
+\frac{1}{\theta}\sum_{t=0}^{\theta-1}\sum_{i=1}^{N} Tr[C_{cl}(t, i)U(t, i)C_{cl}^T(t, i)]. \tag{48}
$$

Combining (47) and (48) we obtain

$$
J_\epsilon(\mathbf{G}_F) \geq \frac{1}{\theta}\sum_{t=0}^{\theta-1}\sum_{i=1}^{N} Tr[C_z(t, i)Y_s(t, i)C_z^T(t, i)] \tag{49}
$$

for all $\mathbf{G}_F \in \mathfrak{F}_s$.

It remains to show that in the case of the filter described by (43) we have equality in (49). To this end, let us remark that in the special case of the filter $\tilde{\mathbf{G}}_F$ given by (43) we may write:

$$
\begin{aligned}
C_{cl}(t, i)U(t, i)C_{cl}^T(t, i) &= C_z(t, i)\begin{pmatrix} I_n & -I_n \end{pmatrix}U(t, i)\begin{pmatrix} I_n & -I_n \end{pmatrix}^T \\
&\times C_z^T(t, i) = C_z(t, i)\hat{U}_{11}(t, i)C_z^T(t, i)
\end{aligned} \tag{50}
$$

where $\hat{U}_{11}(t,i)$ is the (1,1)-block of the matrix $\hat{U}(t,i) = \mathcal{T}U(t,i)\mathcal{T}^T$, $\mathcal{T} = \begin{pmatrix} I_n & -I_n \\ 0 & I_n \end{pmatrix}$. Multiplying (46) by $\mathcal{T}$ to the left and $\mathcal{T}^T$ to the right one obtains that $\{\hat{U}_{11}(t,i)\}_{t\in\mathbb{Z}}$ and $i \in \mathfrak{S}$ is the bounded solution of the equation

$$\hat{U}(t+1,i) = \sum_{j=1}^{N} p_t(j,i)[A_0(t,j) + K_s(t,j)C_0(t,j)]\hat{U}_{11}(t,j)[A_0(t,j) +$$
$$+K_s(t,j)C_0^T(t,j)]. \tag{51}$$

If we take into account that the zero solution of (40) is ESMS, we may conclude that, the unique bounded solution of the equation (51) is $U_{11}(t,i) = 0$ for all $(t,i) \in \mathbb{Z} \times \mathfrak{S}$.

Plugging this last equality in (50) we deduce that (48) reduces to (44) if $\mathbf{G}_F$ is replaced by $\tilde{\mathbf{G}}_F$ given by (43). Thus the proof is complete.

Remark 4.3. In the special case $\epsilon_t(j) = \mu_t^{\pi_0}(j)$ the filter $\tilde{\mathbf{G}}_F$ designed as in (43) minimizes the cost functional $J(\cdot)$ introduced in (4).

5   A numerical example

In order to illustrate the above theoretical developments, the following periodic stochastic system of form (1) with $n = 2$, $N = 2$, $r = 1$ and $\theta = 3$ is considered

$$A_0(t,1) = \begin{pmatrix} -0.1 - 0.3(t-1) & 0.3 \\ 0 & 0.1 + 0.1(t-1) \end{pmatrix},$$
$$A_0(t,2) = \begin{pmatrix} -0.4 - 0.1(t-1) & 0.3 \\ 0.1 & -0.1 + 0.3(t-1) \end{pmatrix},$$
$$B(t,1) = \begin{pmatrix} 1 + 0.1(t-1) \\ 1 - 0.2(t-1) \end{pmatrix}, B(t,2) = \begin{pmatrix} -3 + 0.1(t-1) \\ 1 + 0.1(t-1) \end{pmatrix},$$
$$A_1(t,1) = \begin{pmatrix} 0.1 + 0.2(t-1) & 0.2 \\ 0.3 - 0.1(t-1) & 1 \end{pmatrix},$$
$$A_1(t,2) = \begin{pmatrix} 0.2 - 0.1(t-1) & 0.3 \\ 0.3 & 0.1 + 0.2(t-1) \end{pmatrix},$$
$$C_0(t,1) = \begin{pmatrix} 1 + 0.1(t-1) & 2 \end{pmatrix}, C_0(t,2) = \begin{pmatrix} 1 - 0.2(t-1) & 1 \end{pmatrix},$$
$$C_1(t,1) = \begin{pmatrix} 0.2 + 0.1(t-1) & 0.1 \end{pmatrix}, C_0(t,2) = \begin{pmatrix} 0.1 + 0.1(t-1) & 0.2 \end{pmatrix},$$

$$D(t,1) = 1 + 0.1(t-1),\; D(t,2) = 3 - 0.2(t-1),$$
$$C_z(t,1) = \begin{pmatrix} 3 + 0.3(t-1) & 9 \\ 3 & 1 + 0.2(t-1) \end{pmatrix},$$
$$C_z(t,2) = \begin{pmatrix} 2 - 0.2(t-1) & 3 \\ 1 & 1 + 0.4(t-1) \end{pmatrix},\; t = 1,2,3.$$

Based on Theorem 4.3 and using iterative algorithms to compute the solution of (32) and the stabilizing solutions of the SDTRE–F (38) respectively (for details, see [6], one obtains for $P = \begin{pmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{pmatrix}$ and for $\varepsilon_1 = \varepsilon_2 = 0.5$

$$Q(1,1) = \begin{pmatrix} 3.2321 & -0.3757 \\ -0.3757 & 1.040Q \end{pmatrix},\; Q(1,2) = \begin{pmatrix} 5.5729 & -0.4059 \\ -0.4059 & 1.8243 \end{pmatrix},$$
$$Q(2,1) = \begin{pmatrix} 2.6092 & -0.3507 \\ -0.3507 & 0.9163 \end{pmatrix},\; Q(2,2) = \begin{pmatrix} 4.0908 & -0.3310 \\ -0.3310 & 1.7786 \end{pmatrix},$$
$$Q(3,1) = \begin{pmatrix} 2.6932 & -0.3475 \\ -0.3475 & 0.8404 \end{pmatrix},\; Q(3,2) = \begin{pmatrix} 4.3945 & -0.3354 \\ -0.3354 & 1.5700 \end{pmatrix},$$
$$Y(1,1) = \begin{pmatrix} 2.1277 & 0.4185 \\ 0.4185 & 0.7618 \end{pmatrix},\; Y(1,2) = \begin{pmatrix} 1.7786 & 0.2967 \\ 0.2967 & 0.4544 \end{pmatrix},$$
$$Y(2,1) = \begin{pmatrix} 0.8567 & 0.2148 \\ 0.2148 & 0.6700 \end{pmatrix},\; Y(2,2) = \begin{pmatrix} 1.4148 & 0.4867 \\ 0.4867 & 1.3812 \end{pmatrix},$$
$$Y(3,1) = \begin{pmatrix} 1.1782 & 0.1700 \\ 0.1700 & 0.4779 \end{pmatrix},\; Y(3,2) = \begin{pmatrix} 1.9872 & 0.3955 \\ 0.3955 & 0.9875 \end{pmatrix},$$

for which the optimal gains K_s are

$$K_s(1,1) = \begin{pmatrix} 0.1151 \\ 0.1192 \end{pmatrix},\; K_s(1,2) = \begin{pmatrix} -0.6402 \\ 0.2358 \end{pmatrix},$$
$$K_s(2,1) = \begin{pmatrix} 0.1367 \\ 0.1627 \end{pmatrix},\; K_s(2,2) = \begin{pmatrix} -0.5826 \\ 0.3295 \end{pmatrix},$$
$$K_s(3,1) = \begin{pmatrix} 0.0632 \\ 0.1363 \end{pmatrix},\; K_s(3,2) = \begin{pmatrix} -0.6961 \\ 0.4708 \end{pmatrix}.$$

References

[1] S. Bittanti, P.Colaneri, *Periodic Systems, Filtering and Control*, Springer- Verlag, London, 2009.

[2] F. Carravetta, A. Germani, M. Raimondi, Polynomial filtering of discrete-time stochastic linear systems with multiplicative state noise, *IEEE Transactions on Automatic Control*, 42, 1106-1126, 1997.

[3] B.S. Chow, W.P. Birkemeier, A new structure of recursive estimator, *IEEE Trans. Autom. Control*, 34, 5, 568-574, 1989.

[4] B. S. Chow, W.P. Birkemeier, A new recursive filter for systems with multiplicative noise, *IEEE Transactions on Information Theory*, 36, 1430–1435, 1990.

[5] O. L.V. Costa, G.R.A.M. Benites, Linear minimum mean square filter for discrete-time linear systems with Markov jumps and multiplicative noises, *Automatica*, 47, 466-476, 2011.

[6] V. Dragan, T. Morozan, A.M. Stoica, *Mathematical Methods in Robust Control of Discrete-Time Linear Stochastic Systems,* Springer, New York, 2010.

[7] J.L.Doob, *Stochastic processes*, Wiley, New-York, 1967.

[8] V. Dragan, T. Morozan, A.M. Stoica, Output based H_2 optimal controllers for a class of discrete-time stochastic linear systems with periodic coefficients, (under preparation).

[9] E. Gershon, U. Shaked, and I. Yaesh. H_∞ control and estimation of state multiplicative linear systems, Springer, 2005.

[10] A. Halanay, T. Morozan, Optimal stabilizing compensators for linear discrete-time systems under independent perturbations, *Revue Roum. Math. Pure et Appl.*, 37, 3, (1992), 213–224.

[11] A. Halanay, V. Ionescu, *Time varying discrete linear systems*, Berlin, Birkhauser, 1994.

[12] A. Halanay, Vl. Rasvan, *Discrete Time Systems. Stability and Stable Oscillations*, Gordon and Breach, 2000.

[13] J.J. Hench, A.J. Laub, Numerical solution of the discrete time periodic Riccati equation, *IEEE Trans. on Autom. Control,* 39, 6, 1197–1210, 1994.

[14] M. Iosifescu, *Finite Markov processes and their applications*, John Wiley and sons, 1980.

[15] R. Kalman, A new approach to linear filtering and prediction problems, *ASME-Trans.-Part D, J. Basic Engineering,* 82, 3445, 1960.

[16] R. Kalman, R.S. Bucy, New results in linear filtering and prediction theory, *ASME Trans.Part D, J. Basic Engineering*, 83, 95-108, 1961.

[17] D. L. Kleinman, On an iterative technique for Riccati equation computation, *IEEE Transactions on Automatic Control*, 13, 1968, 114-115.

[18] R.S. Mangoubi, *Robust Estimation and Failure Detection*, Springer, 1998.

[19] I.R. Petersen, V.A.Ugrinovskii, A.V. Savkin, *Robust Control Design Using H_∞ Methods*, Springer, 2000.

[20] A.M. Stoica, V. Dragan, I. Yaesh, Kalman type filtering for stochastic systems with state dependent noise and Markovian jumps, *Proceedings of The 15th IFAC Symposium on System Identification, Saint-Malo, France, 2009.*

[21] V. M. Ungureanu, V. Dragan, T. Morozan, Global solutions of a class of discrete-time backward nonlinear equations on ordered Banach spaces with applications to Riccati equations of stochastic control, *Optimal Control, Applications and Methods, article first published online: 6 MAR 2012 — DOI: 10.1002/oca.2015.*