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A NOTE ON MILD SOLUTIONS FOR NONCONVEX FRACTIONAL SEMILINEAR DIFFERENTIAL INCLUSIONS*

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Abstract

We consider a Cauchy problem for a fractional semilinear differential inclusions involving Caputo's fractional derivative in non separable Banach spaces under Filippov type assumptions and we prove the existence of solutions.

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keywords: fractional derivative, fractional semilinear differential inclusion, Lusin measurable multifunctions.

1 Introduction

Differential equations with fractional order have recently proved to be strong tools in the modelling of many physical phenomena. As a consequence there was an intensive development of the theory of differential equations of fractional order ([20, 22, 24] etc.). The study of fractional differential inclusions was initiated by El-Sayed and Ibrahim ([17]). Very recently several qualitative results for fractional differential inclusions were obtained in [1, 3, 7-11,

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13, 23] etc.. Applied problems require definitions of fractional derivative allowing the utilization of physically interpretable initial conditions. Caputo's fractional derivative, originally introduced in [5] and afterwards adopted in the theory of linear visco elasticity, satisfies this demand. For a consistent bibliography on this topic, historical remarks and examples we refer to [1].

The study of theory of abstract differential equations with fractional derivatives in infinite dimensional spaces is also very recent. The main problem consists in how to introduce new concepts of mild solutions. One of the first paper on this topic is [16]. In [19] it is showed that several papers on fractional differential equations in Banach spaces were incorrect and used an approach to treat these equations based on the theory of resolvent operators for integral equations. A suitable definition of mild solutions based on Laplace transform and probability density functions may be found in [25-28].

In this paper we study fractional semilinear differential inclusions of the form

$$D_c^r x(t) \in Ax(t) + F(t, x(t)) \quad t \in I, \quad x(0) = x_0 \tag{1.1}$$

where $I = [0, T]$, X is a Banach space, A is the infinitesimal generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$, $F(.,.) : I \times X \to \mathcal{P}(X)$ is a set-valued map and D_c^r is the Caputo fractional derivative of order $r \in (0, 1]$.

In our recent paper [12] it is shown that Filippov's ideas ([18]) can be suitably adapted in order to prove the existence of solutions to problem (1.1) provided the Banach space X is separable.

De Blasi and Pianigiani ([15]) established the existence of mild solutions for semilinear differential inclusions on an arbitrary, not necessarily separable, Banach space X. Even if the ideas of Filippov are still present, the approach in [15] has a fundamental difference which consists in the construction of the measurable selections of the multifunction. This construction does not use classical selection theorems as Kuratowsky and Ryll-Nardzewski ([21]) or Bressan and Colombo ([4]).

The aim of this note is to obtain an existence result for problem (1.1) similar to the one in [15]. We will prove the existence of solutions for problem (1.1) in an arbitrary space X under assumptions on F of Filippov type. Our result may be interpreted as extension of the result in [15] to fractional semilinear differential inclusions and as an extension of the result in [12] to non separable Banach spaces.

The paper is organized as follows: in Section 2 we present the notations, definitions and the preliminary results to be used in the sequel and in Section 3 we prove the main result.

2 Preliminaries

Consider X an arbitrary real Banach space with norm $|.|$ and with the corresponding metric $d(.,.)$. Let $\mathcal{P}(X)$ be the space of all bounded nonempty subsets of X endowed with the Pompeiu-Hausdorff pseudometric

$$\mathrm{d}_H(A,B)=\max\{\mathrm{d}^*(A,B),\mathrm{d}^*(B,A)\},\quad \mathrm{d}^*(A,B)=\sup_{a\in A}\mathrm{d}(a,B),$$

where $\mathrm{d}(x,A)=\inf_{a\in A}|x-a|$, $A\subset X, x\in X$.

Let $\mathcal{L}$ be the σ-algebra of the (Lebesgue) measurable subsets of R and, for $A\in\mathcal{L}$, let $\mu(A)$ be the Lebesgue measure of A.

Let X be a Banach space and Y be a metric space. An open (resp. closed) ball in Y with center y and radius r is denoted by $B_Y(y,r)$ (resp. $\overline{B}_Y(y,r)$. In what follows $B=B_X(0,1)$.

A multifunction $F:Y\to\mathcal{P}(X)$ with closed bounded nonempty values is said to be d_H-continuous at $y_0\in Y$ if for every $\varepsilon>0$ there exists $\delta>0$ such that for any $y\in B_Y(y_0,r)$ we have $\mathrm{d}_H(F(y),F(y_0))\le\varepsilon$. F is called d_H-continuous if it is so at each point $y_0\in Y$.

Let $A\in\mathcal{L}$, with $\mu(A)<\infty$. A multifunction $F:Y\to\mathcal{P}(X)$ with closed bounded nonempty values is said to be *Lusin measurable* if for every $\varepsilon>0$ there exists a compact set $K_\varepsilon\subset A$, with $\mu(A\backslash K_\varepsilon)<\varepsilon$ such that F restricted to K_ε is d_H-continuous.

It is clear that if $F,G:A\to\mathcal{P}(X)$ and $f:A\to X$ are Lusin measurable then so are F restricted to B ($B\subset A$ measurable), $F+G$ and $t\to\mathrm{d}(f(t),F(t))$. Moreover, the uniform limit of a sequence of Lusin measurable multifunctions is also Lusin measurable.

We recall next the following definitions. For more details, we refer to [20].

Definition 2.1. a) The fractional integral of order $r>0$ of a Lebesgue integrable function $f:(0,\infty)\to\mathbf{R}$ is defined by

$$I^rf(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(r)}f(s)ds,\quad t>0,r>0$$

provided the right-hand side is pointwise defined on $(0,\infty)$ and $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\alpha)=\int_0^{\infty}t^{\alpha-1}e^{-t}dt$.

b) The Riemann-Liouville derivative of order r of $f(.)\in L^1(I,\mathbf{R})$ is defined by

$$D_L^r f(t)=\frac{1}{\Gamma(n-r)}\frac{d^n}{dt^n}\int_0^t\frac{f(s)}{(t-s)^{r+1-n}}ds,\quad t>0,\quad n-1<r<n.$$

c) The Caputo fractional derivative of order r of $f(.)\in L^1(I,\mathbf{R})$ is defined by

$$D_c^r f(t)=D_L^r(f(t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0))\quad t>0,\quad n-1<r<n.$$

Remark 2.2. a) If $f(.)\in C^n([0,\infty),\mathbf{R})$ then $D_c^r f(t)=I^{n-r}f^{(n)}(t)$, $t>0$, $n-1<r<n$.

b) The Caputo derivative of a constant is equal to zero.

c) If $f:I\to X$, with X a Banach space, then integrals which appears in Definition 2.1 are taken in Bochner's sense.

Let denote by I the interval $[0,T]$, $T>0$, consider $F:I\times X\to\mathcal{P}(X)$ a set-valued map and $x_0\in X$. Consider $A:D(A)\to X$ the infinitesimal generator of a strongly continuous semigroup $\{T(t),t\geq 0\}$ and let $M\geq 0$ be such that $\sup_{t\in I}|T(t)|\leq M$.

Definition 2.3. A continuous function $x(.)\in C(I,X)$ is called a *mild solution* of problem (1.1) if there exists a (Bochner) integrable function $f(.)\in L^1(I,X)$ such that $f(t)\in F(t,x(t))$ *a.e.* (I) and

$$x(t)=S_1(t)x_0+\int_0^t(t-u)^{r-1}S_2(t-u)f(u)du\quad\forall t\in I,\tag{2.1}$$

where

$$S_1(t)=\int_0^{\infty}\xi_r(\theta)T(t^r\theta)d\theta,\quad S_2(t)=r\int_0^{\infty}\theta\xi_r(\theta)T(t^r\theta)d\theta,$$

$$\xi_r(\theta)=\frac{1}{r}\theta^{-1-\frac{1}{r}}\omega_r(\theta^{-\frac{1}{r}})\geq 0,$$

$$\omega_r(\theta)=\frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-rn-1}\frac{\Gamma(nr+1)}{n!}sin(n\pi r),\quad\theta>0$$

and ξ_r is a probability density function defined on $(0,\infty)$, i.e. $\xi_r(\theta) \geq 0$, $\theta \in (0,\infty)$ and $\int_0^\infty \xi_r(\theta)d\theta = 1$.

We shall call $(x(.), f(.))$ a *trajectory-selection pair* of (1.1).

The results summarized in the next lemmas will be used in the proof of our main results.

Lemma 2.4. ([27,28]) *a) For any fixed $t \geq 0$, $S_1(t)$ and $S_2(t)$ are linear and bounded operators, i.e. for any $x \in X$*

$$|S_1(t)x| \leq M|x|, \quad |S_2(t)x| \leq \frac{M}{\Gamma(r)}|x|.$$

b)$\{S_1(t), t \geq 0\}$ and $\{S_2(t), t \geq 0\}$ are strongly continuous.

c) If $T(t), t \geq 0$ is compact, then $S_1(t), t \geq 0$ and $S_2(t), t \geq 0$ are also compact operators.

In what follows X is a real Banach space and we assume the following hypotheses.

Hypothesis 2.5. i) *$F(.,.) : I \times X \to \mathcal{P}(X)$ has nonempty closed bounded values and for any $x \in X$ $F(.,x)$ is Lusin measurable on I.*

ii) *There exists $l(.) \in L^1(I,(0,\infty))$ with $L := \sup_{t\in I} I^r l(t) < +\infty$ such that, $\forall t \in I$*

$$\mathrm{d}_H(F(t,x_1), F(t,x_2)) \leq l(t)|x_1 - x_2|, \quad \forall\, x_1, x_2 \in X.$$

iii) *There exists $q(.) \in L^1(I,(0,\infty))$ with $Q := \sup_{t\in I} I^r q(t) < +\infty$ such that $\forall t \in I$ we have*

$$F(t,0) \subset q(t)B.$$

Lemma 2.6. ([15]) i) *Let $F_i : I \to \mathcal{P}(X)$, i=1,2 be two Lusin measurable multifunctions and let $\varepsilon_i > 0$, i=1,2 be such that*

$$H(t) := (F_1(t) + \varepsilon_1 B) \cap (F_2(t) + \varepsilon_2 B) \neq \emptyset, \quad \forall t \in I.$$

Then the multifunction $H : I \to \mathcal{P}(X)$ has a Lusin measurable selection $h : I \to X$.

ii) *Assume that Hypothesis* 2.5 *is satisfied. Then for any $x(.) : I \to X$ continuous, $u(.) : I \to X$ measurable and $\varepsilon > 0$ we have*

a) the multifunction $t \to F(t, x(t))$ is Lusin measurable on I.

b) the multifunction $G : I \to \mathcal{P}(X)$ defined by

$$G(t) := (F(t, x(t)) + \varepsilon B) \cap B_X(u(t), d(u(t), F(t, x(t))) + \varepsilon)$$

has a Lusin measurable selection $g : I \to X$.

3 The main results

We are ready now to prove our main result.

Theorem 3.1. *Consider A the infinitesimal generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$ on a Banach space X such that there exists a constant $M \geq 1$ with $\sup_{t \in I} |T(t)| \leq M$. We assume that Hypothesis* 2.5 *is satisfied and $ML < 1$.*

Then, for every $x_0 \in X$ the problem (1.1) *has a solution $x(.) : I \to X$.*

Proof. Let $0 < \varepsilon < 1$, $\varepsilon_n = \frac{\varepsilon}{2^{n+2}}$ and $f_0(.) : I \to X$, $f_0(t) \equiv 0$ and define

$$x_0(t) = S_1(t)x_0 + \int_0^t (t-s)^{r-1} S_2(t-s) f_0(s) \mathrm{d}s = S_1(t)x_0, \quad \forall t \in I$$

Since $x_0(.)$ is continuous, by Lemma 2.6 ii) there exists a Lusin measurable function $f_1(.) : I \to X$ satisfying, for $t \in I$,

$$f_1(t) \in (F(t, x_0(t)) + \varepsilon_1 B) \cap B(f_0(t), \mathrm{d}(f_0(t), F(t, x_0(t))) + \varepsilon_1)$$

Obviously, $f_1(.)$ is Bochner integrable on I. Define $x_1(.) : I \to X$ by

$$x_1(t) = S_1(t)x_0 + \int_0^t (t-s)^{r-1} S_2(t-s) f_1(s) \mathrm{d}s, \quad \forall t \in I$$

By induction, we construct a sequence $x_n : I \to X$, $n \geq 2$ given by

$$x_n(t) = S_1(t)x_0 + \int_0^t (t-s)^{r-1} S_2(t-s) f_n(s) \mathrm{d}s, \quad \forall t \in I, \tag{3.1}$$

where $f_n(.) : I \to X$ a Lusin measurable function satisfying, for $t \in I$,

$$f_n(t) \in (F(t, x_{n-1}(t)) + \varepsilon_n B) \cap B(f_{n-1}(t), \mathrm{d}(f_{n-1}(t), F(t, x_{n-1}(t))) + \varepsilon_n). \tag{3.2}$$

From (3.2), for $n \geq 2$, and $t \in I$ we obtain

$|f_n(t) - f_{n-1}(t)| \leq \mathrm{d}(f_{n-1}(t), F(t, x_{n-1}(t))) + \varepsilon_n \leq \mathrm{d}(f_{n-1}(t), F(t, x_{n-2}(t))) +$

$\mathrm{d}_H(F(t, x_{n-2}(t)), F(t, x_{n-1}(t))) + \varepsilon_n \leq \varepsilon_{n-1} + l(t)|x_{n-1}(t) - x_{n-2}(t)| + \varepsilon_n.$

Since $\varepsilon_{n-1} + \varepsilon_n < \varepsilon_{n-2}$ we deduce, for $n \geq 2$, that

$$|f_n(t) - f_{n-1}(t)| \leq \varepsilon_{n-2} + l(t)|x_{n-1}(t) - x_{n-2}(t)|. \tag{3.3}$$

Denote $p_0(t) := \mathrm{d}(f_0(t), F(t, x_0(t))) = \mathrm{d}(0, F(t, x_0(t))), t \in I$. One has

$$\begin{aligned} &p_0(t) \leq \mathrm{d}(0, F(t, 0)) + \mathrm{d}_H(F(t, 0), F(t, x_0(t))) \leq \\ &\leq q(t) + l(t)|x_0(t)| \leq q(t) + Ml(t)|x_0|. \end{aligned}$$

Therefore

$$I^r p_0(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} p_0(s)\mathrm{d}s \leq Q + ML|x_0|.$$

Denote $k = Q + ML|x_0| + \frac{T^r}{\Gamma(r+1)}\varepsilon$

Next we prove, by recurrence, that, for $n \geq 2$ and $t \in I$ we have

$$|x_n(t) - x_{n-1}(t)| \leq \frac{T^r}{\Gamma(r+1)} \sum_{j=0}^{n-2} \varepsilon_{n-2-j} M^{j+1} L^j + M^n L^{n-1} k \tag{3.4}$$

We start with $n = 2$. In view of (3.1), (3.2) and (3.3), for $t \in I$, one has

$$|x_2(t) - x_1(t)| \leq \int_0^t (t-s)^{r-1} |S_2(t-s)(f_2(s) - f_1(s))| \mathrm{d}s \leq$$

$$\leq \frac{M}{\Gamma(r)} \int_0^t (t-s)^{r-1} |f_2(s) - f_1(s)| \mathrm{d}s \leq \frac{M}{\Gamma(r)} \int_0^t (t-s)^{r-1} [\varepsilon_0 +$$

$$l(s)|x_1(s) - x_0(s)|]\mathrm{d}s \leq \frac{MT^r}{\Gamma(r+1)}\varepsilon_0 + \frac{M}{\Gamma(r)} \int_0^t (t-s)^{r-1} l(s)|x_1(s) - x_0(s)|\mathrm{d}s$$

$$\leq \frac{MT^r}{\Gamma(r+1)}\varepsilon_0 + \frac{M^2}{(\Gamma(r))^2} \int_0^t (t-s)^{r-1} l(s) \left(\int_0^s (s-u)^{r-1} |f_1(u) - f_0(u)| \mathrm{d}u\right)\mathrm{d}s$$

$$\leq \frac{MT^r}{\Gamma(r+1)}\varepsilon_0 + \frac{M^2}{(\Gamma(r))^2} \int_0^t (t-s)^{r-1} l(s) \left(\int_0^s (s-u)^{r-1} [p_0(u) + \varepsilon_1] \mathrm{d}u\right)\mathrm{d}s$$

$$\leq \frac{MT^r}{\Gamma(r+1)}\varepsilon_0 + M^2Lk,$$

i.e, (3.4) is verified for $n = 2$.

Using again (3.3) and (3.4) we have

$$|x_{n+1}(t) - x_n(t)| \leq \frac{M}{\Gamma(r)} \int_0^t (t-s)^{r-1}|f_{n+1}(s) - f_n(s)|\mathrm{d}s \leq$$

$$\leq \frac{M}{\Gamma(r)} \int_0^t [\varepsilon_{n-1} + l(s)|x_n(s) - x_{n-1}(s)|]\mathrm{d}s \leq \frac{MT^r}{\Gamma(r+1)}\varepsilon_{n-1} +$$

$$\frac{M}{\Gamma(r)} \int_0^t (t-s)^{r-1} l(s)|x_n(s) - x_{n-1}(s)|\mathrm{d}s \leq \frac{MT^r}{\Gamma(r+1)}\varepsilon_{n-1} +$$

$$\frac{M}{\Gamma(r)} \int_0^t (t-s)^{r-1} l(s)[\frac{T^r}{\Gamma(r+1)} \sum_{j=0}^{n-2} \varepsilon_{n-2-j} M^{j+1} L^j + M^n L^{n-1} k]\mathrm{d}s$$

$$\leq \frac{MT^r}{\Gamma(r+1)}\varepsilon_{n-1} + \frac{T^r}{\Gamma(r+1)} \sum_{j=0}^{n-2} \varepsilon_{n-2-j} M^{j+2} L^{j+1} + M^{n+1} L^n k$$

$$= \frac{T^r}{\Gamma(r+1)} \sum_{j=0}^{n-1} \varepsilon_{n-2-j} M^{j+1} L^j + M^{n+1} L^n k$$

and the statement (3.4) is true for $n + 1$.

From (3.4) it follows that, for $n \geq 2$ and $t \in I$ one has

$$|x_n(t) - x_{n-1}(t)| \leq a_n, \tag{3.5}$$

where

$$a_n = \frac{T^r}{\Gamma(r+1)} \sum_{j=0}^{n-2} \varepsilon_{n-2-j} M^{j+1} L^j + M^n L^{n-1} (Q + ML|x_0| + \frac{T^r}{\Gamma(r+1)}\varepsilon).$$

Obviously, the series whose n-th term is a_n is convergent. So, from (3.5) we have that $x_n(.)$ converges uniformly on I to a continuous function, $x(.) : I \to X$.

On the other hand, in view of (3.3) we have

$$|f_n(t) - f_{n-1}(t)| \leq \varepsilon_{n-2} + l(t)a_{n-1}, \quad t \in I, n \geq 3$$

which implies that the sequence $f_n(.)$ converges to a Lusin measurable function $f(.) : I \to X$.

One may write successively,

$$|\int_0^t (t-s)^{r-1} S_2(t-s) f_n(s) \mathrm{d}s - \int_0^t (t-s)^{r-1} S_2(t-s) f(s) \mathrm{d}s| \leq$$

$$\frac{M}{\Gamma(r)} \int_0^t (t-s)^{r-1} |f_n(s) - f(s)| \mathrm{d}s \leq \frac{M}{\Gamma(r)} \int_0^t (t-s)^{r-1} l(s) |x_{n-1}(s)$$

$$-x(s)| \mathrm{d}s \leq \frac{M}{\Gamma(r)} L |x_{n-1}(.) - x(.)|_C.$$

So, passing with $n \to \infty$ in (3.1) we obtain

$$x(t) = S_1(t) x_0 + \int_0^t (t-u)^{r-1} S_2(t-u) f(u) du \quad \forall t \in I.$$

On the other hand, from (3.2) we get

$$f_n(t) \in F(t, x_n(t)) + \varepsilon_n B, \quad t \in I, n \geq 1$$

and letting $n \to \infty$ we have

$$f(t) \in F(t, x(t)), \quad t \in I.$$

and the proof is complete.

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