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POST-PARETO ANALYSIS FOR MULTIOBJECTIVE PARABOLIC CONTROL SYSTEMS*

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Abstract

In this paper is presented the problem of optimizing a functional over a pareto control set associated with a convex multiobjective control problem in Hilbert spaces, namely parabolic system. This approach generalizes for this setting some results obtained in finite dimensions. Some examples are presented. General optimality results are obtained, and a special attention is paid to the linear-quadratic multi objective parabolic system when is possible to get explicit optimality conditions.

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1 Introduction

Since the legendary paper of H.W. Kuhn and A.W.Tucker (1951), Multi-Objective Optimization Problems (MOP) took progressively an important

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place in Operation Research. However the genesis of the theory goes back to Pareto (1906) inspired by Edgeworth's indifference curves.

If we want to simultaneously minimize p-objectives, problem which we will denote[1]

$$(MOP) \qquad \mathrm{MIN}_{\mathbf{R}^p_+}[f_1(x),\dots,f_p(x)] \quad \text{s.t. } x \in X_{ad},$$

where the objectives $f_1,\dots,f_p$ are real valued functions defined on a set X, and $X_{ad} \subset X$ is the feasible set, it is very unlikely that all theses functions have a common minimizer. That is why we are led to consider feasible solutions which ensure some sort of equilibrium between the objectives, roughly speaking, solutions which are such that none of the objectives can be improved further without deteriorating another. Precisely, we have the following definitions (see e.g. [28]).

Definition 1 *For problem (MOP) a point $\hat{x} \in X_{ad}$ is a*

- Pareto solution *if there is no $x \in X_{ad}$, such that $f_i(x) \le f_i(\hat{x})$ for all i, with at least one inequality strictly satisfied.*
- weakly Pareto solution *if there is no $x \in X_{ad}$, such that $f_i(x) < f_i(\hat{x})$ for all i.*
- properly Pareto solution *if $\hat{x}$ is a Pareto solution, and there exists a real $M > 0$ so that, for every $j \in \{1,\dots,p\}$ and every $x \in X_{ad}$ with $f_j(x) < f_j(\hat{x})$, at least one $k \in \{1,\dots,p\}$ exists with $f_k(x) > f_k(\hat{x})$ and*
$$\frac{f_j(\hat{x}) - f_j(x)}{f_k(x) - f_k(\hat{x})} \le M.$$

Notice that a (MOP) can be naturally considered as the *grand coalition p-player cooperative game* (see Example 2 in the next section).

The drawback is that the (weakly or properly) Pareto set is usually very large (may be infinite and even unbounded), so a decision maker (or a supervisor of a cooperative p-player game) may *select* a Pareto solution *optimizing his own criterion.* Thus we can consider the following problem of Post-Pareto

[1]The subscript $\mathbf{R}^p_+$ stands for the ordering cone in $\mathbf{R}^p$, since more generally one may consider vector optimization problems with respect to different partial order in the outcome space.

Analysis of optimizing a *scalar objective* $f_0 : X \to \mathbf{R}$ over the Pareto set, i.e. we consider the problem

$$\min_{x \in \mathcal{P}} f_0(x), \tag{1}$$

where $\mathcal{P}$ is the (weakly or properly) Pareto set associated with problem (MOP).

This problem of optimizing a scalar function over a Pareto set has been considered the first time in [32], and intensively studied in the last three decades (see e.g. [1, 4, 5, 6, 7, 8, 9, 19, 22, 23, 24, 26, 27] and [34] for a survey). In all these papers the Pareto (or efficient) set is associated with a mathematical programming problem, not with a multiobjective control problem. Some related results concerning generalizations to semivectorial bilevel problems can be found in [11, 14]. Approaches to the case of multiobjective control for ODE are presented in [13, 17], and in [15, 16] for the more general case of semivectorial bilevel problems. A stochastic case has been considered in [12].

This problem is difficult because the Pareto set is not explicitly described, and it is not convex (even for a multi-objective linear programming problem!)

In the present paper is studied the more difficult case of a multiobjective convex control problem in Hilbert spaces, especially parabolic equations.

The paper is organized as follows. In Section 2 the general framework for our problem and the main notations and hypotheses are introduced. Moreover, two concretes examples to justify the interest of our problem in this special setting are presented. In Section 3 some useful preliminary results are obtained. The last Section contains the main results. Using scalarization techniques, i.e. replacing the multiobjective problem with a family of scalar problems obtained by convex combination (weighted sum) of the objectives, we show that the weakly (resp. properly) Pareto set[2] is the union of the minimizers of these scalarized problems when the vector of weights runs over $\mathbf{R}^p_+ \setminus \{0\}$, (resp. $\mathbf{R}^p_{++} := \operatorname{int} \mathbf{R}^p_+$). Thus, theoretically, our problem can be restated as optimizing over the previous union. Considering the set-valued map which associates to each weighting vector the set of the minimizers, the problem becomes a real set valued optimization problem over $\mathbf{R}^p_+ \setminus \{0\}$ (or $\mathbf{R}^p_{++}$). Using set-valued analysis techniques optimality conditions are presented. Next a special attention is paid to the case when the

[2]Note that the set of Pareto points, which is located between the properly and the weakly Pareto sets, cannot be characterized in this way.

previous set-valued map is single-valued and sufficient conditions to ensure this property are presented in Theorem 4. These conditions are satisfied in particular when we deal with the linear-quadratic multi-objective problem. The differentiability of this map is proven and some ideas how to obtain an explicit description of this map are given.

2 Problem statement

We deal with the following post-Pareto optimization control problem

$$(PPOCP) \qquad \min J_0(z,u) \quad \text{s.t.}$$

(z,u) is a weakly (or properly) Pareto control process for the following multi-objective convex control optimization problem in Hilbert spaces

$$(MOCCOP) \qquad \mathrm{MIN}_{\mathbf{R}^p_+}[J_1(z,u),\ldots,J_p(z,u)] \quad \text{s.t.}$$

$$\begin{aligned}
\frac{dz}{dt}(t)+A(t)z(t) &= B(t)u(t) \quad \text{a.e. on }]0,T[&& (2)\\
u(t) &\in U \qquad\quad \text{a.e. on }]0,T[&& (3)\\
z(0) &= z_0 && (4)\\
z(T) &\in Z_T && (5)
\end{aligned}$$

Throughout the paper we will consider the following general assumptions and notations.

$(A(t))_{t\in]0,T[}$ *is a family of linear continuous operators from a real Hilbert space* V *to its (topological) dual* V'*, i.e.* $A(t)\in\mathcal{L}(V,V')$*,* $0<t<T$*.* $(B(t))_{t\in]0,T[}$ *is a family of linear continuous operators from a real Hilbert space* $\mathcal{U}$ *to* V'*. We suppose that there exists a real Hilbert space* H *such that* $V\subset H$ *with linear continuous and dense embedding. Then we have that* $H'\subset V'$ *with linear continuous and dense embedding. We identify* $H\equiv H'$ *using the canonical isomorphism given by Riesz' theorem, thus the following inclusions are linear, continuous and dense*

$$V\subset H\subset V',$$

and H *is called "pivot space" (see e.g. [18] or [10]).*

For $v' \in V'$ and $v \in V$, we denote by $(v'|\, v)$ the value of the functional v' in v. Note the for $h \in H$ considered as an element of V', and for $v \in V$, the real $(h \,|\, v)$ coincides with the usual scalar product between h and v in H. So there is no confusion to denote also $(\cdot \,|\, \cdot)$ the scalar product of H. The norm of H (respectively V and V') will be denoted by $|\cdot|$ (respectively by $\|\cdot\|$ and $\|\cdot\|_*$).

Suppose that there are some $\alpha \in \mathbf{R}$ and $\omega > 0$ such that, for all $v \in V$, $t \in]0,T[$,

$$(A(t)v \,|\, v) + \alpha|v|^2 \geq \omega\|v\|^2. \tag{6}$$

Moreover, we suppose that for all $v, w \in V$, the function $t \mapsto (A(t)v \,|\, w)$ is measurable on $]0,T[$, and there is a constant $c > 0$, such that

$$\|A(t)\|_{\mathcal{L}(V,V')} \leq c \qquad a.e. \text{ on }]0,T[.$$

Also, suppose that for any $u \in L^2(0,T;\mathcal{U})$, the function $t \mapsto B(t)u(t)$ is measurable on $]0,T[$ and

$$\|B(t)\|_{\mathcal{L}(\mathcal{U},V')} \leq c \qquad a.e. \text{ on }]0,T[.$$

The initial value $z_0 \in H$ is specified. We suppose that U is a nonempty closed convex subset of $\mathcal{U}$. The "target set" Z_T is a nonempty closed convex subset of H.

Each objective $J_i : L^2(0,T;V) \cap C(0,T;H) \times L^2(0,T;\mathcal{U}) \to \mathbf{R} \cup \{+\infty\}$ is given by

$$J_i(z,u) = l_i(z(T)) + \int_0^T L_i(t, z(t), u(t))dt,$$

where $L_i :]0,T[\times V \times \mathcal{U} \to \mathbf{R} \cup \{+\infty\}$ is a Borel function such that for each $t \in]0,T[$, the function $L_i(t,\cdot,\cdot)$ is lower semicontinuous and proper, $l_i : H \to \mathbf{R} \cup \{+\infty\}$ is supposed proper, lower semicontinuous, and there are some real numbers β_i, γ_i and and $a \in L^1(0,T)$ such that

$$\forall (v,u) \in V \times \mathcal{U} \qquad L_i(t,v,u) \geq a_i(t) + \beta_i\|v\|^2 + \gamma_i\|u\|^2_{\mathcal{U}}, \quad t \in]0,T[\tag{7}$$

$i = 0, 1, \ldots, p$.

Moreover, the objectives of (MOCCOP) problem are supposed convex, i.e. for all $i = 1, \ldots, p$, and $t \in]0,T[$, the functions $L_i(t,\cdot,\cdot)$, l_i are convex. J_0 is not necessarily convex.

Other assumptions will be specified when necessary.

Example 1 The abstract problem (MOCCOP) contains as a particular case the following multi-objective parabolic boundary control problem

$$\mathrm{MIN}_{\mathbf{R}_+^p}[J_1(z,u),\ldots,J_p(z,u)] \quad \text{s.t. } (z,u) \text{ verifies}$$

$$\begin{aligned}
\frac{\partial z}{\partial t} - \mathrm{div}_x(k(x)\nabla_x z) - q(x)z &= 0 \quad \text{a.e. in } Q && (8)\\
\frac{\partial z}{\partial n} + \rho(x)z &= u \quad \text{a.e. in } \Sigma && (9)\\
z(x,0) &= z_0(x) \quad \text{a.e. in } \Omega && (10)\\
u(t) &\in U \quad \text{a.e. in }]0,T[&& (11)
\end{aligned}$$

where $\Omega \subset \mathbf{R}^n$ is an open bounded set, its boundary Γ is of class C^1,

$$Q = \Omega\times]0,T[, \quad \Sigma = \Gamma\times]0,T[\,,$$

$k \in C^1(\bar\Omega)$, $k(x) > 0$, $\forall x \in \bar\Omega$, $q \in C(\bar\Omega)$, $\rho \in C(\Gamma)$, $\rho \geq 0$.

The function $z = z(x,t) : \Omega \times [0,T] \to \mathbf{R}$ is the *state,* and the function $u(t) \in L^2(\Gamma)$ is the (boundary) *control* at the moment $t \in]0,T[$, supposed square integrable, i.e. $u \in L^2(0,T;L^2(\Gamma))$. The initial value $z_0 \in L^2(\Omega)$ is specified.

Put $V = H^1(\Omega)$, $H = L^2(\Omega)$, $\mathcal{U} = L^2(\Gamma)$, and define $A(t) \equiv A \in \mathcal{L}(V,V')$, $B(t) \equiv B \in \mathcal{L}(\mathcal{U},V')$ by

$$\forall z,w \in V \ \ (Az\,|\,w) = \int_\Omega (k\nabla z\cdot\nabla w - qzw)dx + \int_\Gamma k\rho zw d\sigma$$

$$\forall u \in \mathcal{U}, w \in V, \ \ (Bu\,|\,w) = \int_\Gamma kuwd\sigma.$$

Note that the last boundary integral is well defined since for each element w of $H^1(\Omega)$ its *trace* on Γ, $w_{|\Gamma}$ is well defined and belongs to $L^2(\Gamma)$ (see e.g. [3] or [29] and the references herein).

It is easy to see that using Green formula, the variational formulation of problem (8, 9, 10, 11) can be written in the abstract form (2, 3, 4), and A, B satisfy all the hypotheses.

Suppose we have p captors, the i^{th} captor being located on the boundary in a measurable zone $\Gamma_i \subset \Gamma$, $i = 1,\ldots,p$, and the desirable state is $z_d \in L^2(0,T;V)$. Suppose that the sets $(\Gamma_i)_{1\leq i\leq p}$ are mutually disjoints and the values of the desired state are known only on the boundary zone Γ_i. Consider $l_i = 0$, and L_i described by

$$\forall (t,z,u) \in]0,T[\times V \times \mathcal{U} \qquad L_i(t,z,u) = \int_{\Gamma_i} (z - z_d(t))^2 d\sigma + \langle R_i u, u\rangle_{\mathcal{U}},$$

where $R_i \in \mathcal{L}(\mathcal{U})$ is a nonnegative symmetric operator.

Finally, let us consider $L_0 = 0$, $\forall x \in H^2$, $l_0(x) = \|x\|$.

Roughly speaking, the meaning of our problem of optimizing $J_0(z,u)$ over the set of weakly (or properly) Pareto processes of the multi-objective control problem is that amongst all the (weakly or properly) Pareto controls, i.e., amongst all the controls which are such that we cannot improve an objective J_i $(i \geq 1)$ without deteriorating further another objective J_k, $(k \geq 1)$, we are looking for the control which realizes the minimal final state norm.

$\triangle$

Example 2 In this example (MOCCOP) problem is stated as a *grand coalition of a p-player cooperative differential game.* Consider the special case when $\mathcal{U}$ is a product of p Hilbert spaces $\mathcal{U} = \mathcal{U}_1 \times \cdots \times \mathcal{U}_p$, and consequently $U = U_1 \times \cdots \times U_p$, $u(t) = (u_1(t), \ldots, u_p(t))$. Thus $B(t)u(t) = B_1(t)u_1(t) + \cdots + B_p(t)u_p(t)$, with $B_i(t) \in \mathcal{L}(\mathcal{U}_i, V')$. The player i has the objective J_i and *interacts with the system* with the control $u_i \in L^2(0,T;\mathcal{U}_i)$. Consider that a *"supervisor"* of the game has its own objective J_0. Thus, amongst all the controls which are such that no player can improve his objective without further deteriorating the performance of another player, the supervisor choses the control which optimizes his objective.

Suppose we have the same diffusion process as in previous example (8-11), but the boundary control is different :

$$\begin{aligned}
\frac{\partial z}{\partial t} - \operatorname{div}_x(k(x)\nabla_x z) - q(x)z &= 0 \quad \text{a.e. in } Q && (12)\\
\frac{\partial z}{\partial n} + \rho(x)z &= \sum_{i=1}^{p} u_i \quad \text{a.e. in } \Sigma && (13)\\
z(x,0) &= z_0(x) \quad \text{a.e. in } \Omega && (14)\\
u(t) &\in U \quad \text{a.e. in }]0,T[&& (15)
\end{aligned}$$

The functional spaces are the same except that we take $\mathcal{U} = \prod_{i=1}^{p} \mathcal{U}_i$ where $\mathcal{U}_i = L^2(\Gamma)$, $i = 1, \ldots, p$, and $U_i = \{u_i \in L^2(\Gamma) | \operatorname{supp} u_i \subset \Gamma_i\}$ where $\Gamma_i \subset \Gamma$ is a closed subset of Γ representing the zone where player (agent) i interacts with the system. Now the control is of the form $u(t) = (u_1(t), \dots, u_p(t))$, $U = U_1 \times \dots \times U_p$. The operator A is the same, but B is now given by

$$\forall u = (u_1, \dots, u_p) \in \mathcal{U} \quad Bu = \sum_{i=1}^{p} B_i u_i,$$

where

$$\forall w \in V \quad B_i u_i = \int_{\Gamma_i} k u_i w d\sigma.$$

Suppose that Ω is sufficiently smooth such that the state at each moment belongs to $H^2(\Omega)$, and $n \leq 3$, hence $z(\cdot, t) \in C(\bar{\Omega})$ (see e.g. [18] for details about solution regularity). The player i observes the systems in some points (point sensors) $x_k^{(i)} \in \bar{\Omega}$, $k = 1, \dots, m_i$. Suppose that each player wants to minimize his energy and the square of the deviation from the desired state z_d in his points of observation i.e.

$$J_i(z,u) = \int_0^1 \Big(\sum_{k=1}^{m_i} |z(t, x_k^{(i)}) - z_d(t, x_k^{(i)})|^2 + \|u_i(t)\|^2_{\mathcal{U}_i} \Big) dt,$$

and the supervisor wants to minimize the final state global deviation, i.e.

$$J_0(z,u) = \|z(T) - z_d(T)\|_{L^2(\Omega)}.$$

$\triangle$

3 Preliminary results

Lemma 1 *For each $z_0 \in H$ and $u \in L^2(0,T;\mathcal{U})$, there exists a unique function $z_u : [0,T] \to H$ such that $z_u \in L^2(0,T;V) \cap C(0,T;H)$, $\frac{dz_u}{dt} \in L^2(0,T;V')$ verifying the abstract Cauchy problem (2), (4).*

Moreover, the correspondence $u \mapsto z_u$ is an affine continuous operator from $L^2(0,T;\mathcal{U})$ to $L^2(0,T;V)$, and from $L^2(0,T;\mathcal{U})$ to $C(0,T;H)$.

Proof. Denote for a.e $t \in]0,T[$

$$f(t) = B(t)u(t).$$

It is easy to see $f \in L^2(0,T;V')$ and the correspondence $u \mapsto f$ is a linear continuous operator from $L^2(0,T;\mathcal{U})$ to $L^2(0,T;V')$. The existence and uniqueness of the function $z_u : [0,T] \to H$ such that $z_u \in L^2(0,T;V) \cap C(0,T;H)$, $\frac{dz_u}{dt} \in L^2(0,T;V')$ verifying

$$\frac{dz_u}{dt}(t) + A(t)z_u(t) = f(t) \quad \text{a.e. on }]0,T[, \qquad z(0) = z_0$$

is a well known result (see e.g. [29, 3]).

The only thing to prove is the last assertion. Let $z^h : [0,T] \to H$ be the unique function such that $z^h \in L^2(0,T;V) \cap C(0,T;H)$, $\frac{dz^h}{dt} \in L^2(0,T;V')$ verifying

$$\frac{dz^h}{dt}(t) + A(t)z^h(t) = 0 \quad \text{a.e. on }]0,T[, \qquad z^h(0) = z_0,$$

and let $z^f : [0,T] \to H$ be the unique function such that $z^f \in L^2(0,T;V) \cap C(0,T;H)$, $\frac{dz^f}{dt} \in L^2(0,T;V')$ verifying

$$\frac{dz^f}{dt}(t) + A(t)z^f(t) = f(t) \quad \text{a.e. on }]0,T[, \qquad z^f(0) = 0.$$

Obviously $z_u = z^h + z^f$, z^h does not depend on u, so it is sufficient to prove that the correspondence $f \mapsto z^f$ is a linear continuous operator from $L^2(0,T;V')$ to $L^2(0,T;V)$, and from $L^2(0,T;V')$ to $C(0,T;H)$.

Note that the function $t \mapsto \tilde{z}(t) := e^{-\alpha t} z^f(t)$ verifies the Cauchy problem $\frac{d\tilde{z}}{dt}(t) + (A(t) + \alpha I)\tilde{z}(t) = e^{-\alpha t} f(t)$, $\tilde{z}(0) = 0$, and $\tilde{A}(t) := A(t) + \alpha I$ verifies for all $v \in V$ and $t \in]0,T[$, $(\tilde{A}(t)v \mid v) \geq \omega \|v\|^2$, so we can assume without loss of generality that $\alpha = 0$ in (6).

Then we have for a.e. $t \in]0,T[$

$$\begin{aligned} \frac{d}{dt}|z^f(t)|^2 &= 2\left(\frac{dz^f(t)}{dt} \,|\, z^f(t)\right) \\ &= 2(-A(t)z^f(t) + f(t) \mid z^f(t)) \\ &\leq -2\omega\|z^f(t)\|^2 + 2(f(t) \mid z^f(t)). \end{aligned}$$

Since $z^f \in C(0,T;H)$, we obtain for all $t \in [0,T]$ that

$$|z^f(t)|^2 + 2\omega \int_0^t \|z^f(s)\|^2 ds \leq 2 \int_0^t (f(s) \mid z^f(s)) ds.$$

Since

$$\begin{aligned} 2\int_0^t (f(s) \mid z^f(s))ds &\leq 2\left(\int_0^t \|f(s)\|_*^2 ds\right)^{1/2} \left(\int_0^t \|z^f(s)\|^2 ds\right)^{1/2} \\ &\leq \frac{1}{\omega}\|f\|^2_{L^2(0,T,V')} + \omega \int_0^t \|z^f(s)\|^2 ds, \end{aligned}$$

we obtain finally that

$$\max\left\{\|z^f\|^2_{C(0,T;H)}, \omega\|z^f\|^2_{L^2(0,T;V)}\right\} \leq \frac{1}{\omega}\|f\|^2_{L^2(0,T,V')}$$

and the conclusions follow immediately. △

Proposition 1 *Let us define for all $u \in L^2(0,T;\mathcal{U})$ and $i = 0, 1, \ldots, p$*

$$\hat{J}_i(u) := J_i(z_u, u), \tag{16}$$

where the map $u \mapsto z_u$ has been introduced in Lemma 1. Then the function $\hat{J}_i : L^2(0,T;\mathcal{U}) \to \mathbf{R} \cup \{+\infty\}$ is lower semicontinuous.

Proof. Fix $i \in \{0, 1, \ldots, p\}$. By (7) we obtain as in ([2], Example 2 page 14) that the functional $]0,T[\times L^2(0,T;V) \times L^2(0,T;\mathcal{U}) \ni (z,u) \mapsto \int_0^T L_i(t, z(t), u(t))dt \in \mathbf{R} \cup \{+\infty\}$ is lower semicontinuous. Then, according to Lemma 1, by composition with the continuous map $u \mapsto z_u$, we obtain that $u \mapsto \int_0^T L_i(t, z_u(t), u(t))dt$ is lower semicontinuous from $L^2(0,T;\mathcal{U})$ to $\mathbf{R} \cup \{+\infty\}$. Also, the functional $u \mapsto l_i(z_u(T))$ is lower semicontinuous from $L^2(0,T;\mathcal{U})$ to $\mathbf{R} \cup \{+\infty\}$, since, by Lemma 1, for each fixed $t \in [0,T]$, we obtain easily that the map $u \mapsto z_u(t)$ is continuous from $L^2(0,T;\mathcal{U})$ to H. △

4 Main results

Consider the set

$$U_{ad} := \{u \in L^2(0,T;\mathcal{U}) \mid u(t) \in U \quad \text{a.e. on} \quad]0,T[, \quad z_u(T) \in Z_T\}. \tag{17}$$

The set U_{ad} is *closed and convex* in $L^2(0,T;\mathcal{U})$. Indeed, since U and Z_T are convex, and $u \mapsto z_u$ is affine, we obtain that U_{ad} is convex. On the other hand, since U is closed in $\mathcal{U}$, and Z_T is closed in H, using Fischer-Riesz theorem and the fact that $u \mapsto z_u(T)$ is continuous from $L^2(0,T;\mathcal{U})$ to H according to Lemma 1, it is easy to see that U_{ad} is closed in $L^2(0,T;\mathcal{U})$.

From now on we will assume that

(A) $$U_{ad} \neq \emptyset.$$

(B) *the functionals $\hat{J}_i$ take finite values on U_{ad}, $i = 1, 2, \dots, p$.*

Now, according to Proposition 1, our problem (PPOPC) can be written equivalently as

$$\min \hat{J}_0(u) \quad \text{s.t.} \tag{18}$$

$$u \text{ is a (weakly or properly) Pareto solution to}$$

$$\mathrm{MIN}_{\mathbf{R}^p_+}[\hat{J}_1(u), \dots, \hat{J}_p(u)] \quad \text{s.t.} \quad u \in U_{ad}. \tag{19}$$

Let us consider the map $\hat{J} = (\hat{J}_1, \dots, \hat{J}_p) : U_{ad} \to \mathbf{R}^p$. The following result is known as "scalarization theorem" (see e.g. [28, 30, 31]), and allows to replace a convex multiobjective minimization problem with a family of scalar convex minimization problem.

Theorem 1 *Let $\tilde{u} \in U_{ad}$. Then $\tilde{u}$ is a weakly (resp. properly) Pareto solution to problem (19) if, and only if, there exists $\theta = (\theta_1, \dots, \theta_p) \in \mathbf{R}^p_+ \setminus \{0\}$ (resp. $\theta \in \mathbf{R}^p_{++}$) such that $\tilde{u}$ is a minimizer of the functional*

$$U_{ad} \ni u \mapsto \langle \theta, \hat{J}(u) \rangle = \sum_{i=1}^{p} \theta_i \hat{J}_i(u) \tag{20}$$

over U_{ad}.

Let the symbol σ stands for "weak" ($\sigma = w$) or "proper" ($\sigma = p$). Denote

$$\Theta_\sigma = \begin{cases} \mathbf{R}^p_+ \setminus \{0\} & \text{if } \sigma = w \\ \mathbf{R}^p_{++} & \text{if } \sigma = p \end{cases}$$

Then the scalarization theorem can be written as

$$\sigma\text{-}\underset{u \in U_{ad}}{\mathrm{ARGMIN}}_{\mathbf{R}^p_+} \hat{J}(u) = \bigcup_{\theta \in \Theta_\sigma} \underset{u \in U_{ad}}{\mathrm{argmin}} \langle \theta, \hat{J}(u) \rangle, \tag{21}$$

where the left hand side stands for the σ-Pareto set associated with problem (19), and $\sigma \in \{w, p\}$.

4.1 Necessary optimality conditions

In this subsection necessary optimality conditions in a general setting are presented. To do this we need some results from set-valued analysis , and for reader's convenience we will recall briefly some basic facts and notations.

Let $\mathcal{X}$, $\mathcal{Y}$ be real Banach spaces and let $F : \mathcal{X} \to 2^{\mathcal{Y}}$ be a set-valued map, $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$. Denote

$$\mathrm{dom}\, F = \{x \in \mathcal{X} |\ F(x) \neq \emptyset\}, \ \ \mathrm{Gr}\,(F) = \{(x, y) \in \mathcal{X} \times \mathcal{Y} |\ y \in F(x)\},$$

$$F(A) = \bigcup_{x \in A} F(x) \quad F^{-}(B) = \{x \in \mathcal{X} |\ F(x) \cap B \neq \emptyset\}.$$

The *contingent cone* $T(A, x)$ of the set A at the point $x \in A$ is the set of the elements $h \in \mathcal{X}$ such that there exists a sequence $(x_n)_{n \geq 1}$ of elements of A and a sequence $(t_n)_{n \geq 1}$ of positive real numbers such that

$$x = \lim x_n \quad \text{and} \quad h = \lim t_n(x_n - x).$$

The *contingent derivative* $DF(x_0, y_0) : \mathcal{X} \to 2^{\mathcal{Y}}$ of F at $(x_0, y_0) \in \mathrm{Gr}\,(F)$ is defined by

$$\mathrm{Gr}\,(DF(x_0, y_0)) = T(\mathrm{Gr}\,(F), (x_0, y_0)).$$

This is equivalent to say that, for each $x \in \mathcal{X}$, $y \in DF(x_0, y_0)(x) \Longleftrightarrow$

$$\exists t_n > 0,\ (x_n, y_n) \in \mathrm{Gr}\,(F)\ :\ \lim(x_n, y_n) = (x_0, y_0)$$

$$\text{and} \ \ (x, y) = \lim t_n(x_n - x_0, y_n - y_0).$$

Now we go back to our problem. Let $P_\sigma : \mathbf{R}^p \to 2^{L^2(0,T;\mathcal{U})}$ be the set-valued map given by

$$P_\sigma(\theta) := \begin{cases} \operatorname*{argmin}_{u \in U_{ad}} \langle \theta, \hat{J}(u) \rangle & \text{if } \theta \in \Theta_\sigma \\ \emptyset & \text{if } \theta \in \mathbf{R}^p \setminus \Theta_\sigma. \end{cases}$$

It is obvious that P_σ has convex closed values which are subsets of U_{ad}. Moreover (21) can be written as

$$\sigma\text{-}\operatorname*{ARGMIN}_{u \in U_{ad}}{}_{\mathbf{R}^p_+} \hat{J}(u) = P_\sigma(\Theta_\sigma). \tag{22}$$

Consider the following scalar set-valued minimization problem

$$(SSVM_\sigma) \qquad\qquad \min_{\theta\in\Theta_\sigma} \hat{J}_0 \circ P_\sigma(\theta).$$

Recall that a *solution* to this problem is an element $(\tilde{\theta},\tilde{y}) \in \mathrm{Gr}\,(\hat{J}_0 \circ P_\sigma)$ such that

$$\tilde{y} = \min(\hat{J}_0 \circ P_\sigma)(\Theta_\sigma).$$

The following two results have been obtained in [6]

Proposition 2 *Problem (18-19) is equivalent to problem $(SSVM_\sigma)$ in the following sense*

If $\tilde{u}$ solves (18-19), then $P_\sigma^-(\{\tilde{u}\}) \neq \emptyset$, and for each $\tilde{\theta} \in P_\sigma^-(\{\tilde{u}\})$ we have that $(\tilde{\theta}, \hat{J}_0(\tilde{u}))$ is a solution to problem $(SSVM_\sigma)$.

Conversely, if $(\tilde{\theta},\tilde{y})$ is a solution to problem $(SSVM_\sigma)$, then there exists $\tilde{u} \in P_\sigma(\tilde{\theta})$ such that $\tilde{u}$ solves problem (18-19) and $\tilde{y} = \hat{J}_0(\tilde{u})$.

Theorem 2 (Necessary optimality conditions) *Suppose that $\hat{J}_0$ is Fréchet differentiable on an open set containing U_{ad}. Let $\tilde{u}$ solve problem (18-19). Then $P_\sigma^-(\tilde{u}) \neq \emptyset$, and for each $\tilde{\theta} \in P_\sigma^-(\tilde{u})$*

$$\forall\theta \in \mathbf{R}^p \qquad \nabla\hat{J}_0(\tilde{u}) \cdot DP_\sigma(\tilde{\theta},\tilde{u})(\theta) \subset [0,+\infty[\tag{23}$$

where $\nabla\hat{J}_0(\tilde{u})$ stands for the Fréchet derivative of $\hat{J}_0$ at the point $\tilde{u}$.

Remark 1 *If we identify $L^2(0,T;\mathcal{U})$ to its dual by Riesz canonical isomorphism, we can consider $\nabla\hat{J}_0(\tilde{u}) \in L^2(0,T;\mathcal{U})$, and then (23) can be restated as*

$$\forall\theta \in \mathbf{R}^p,\ \ \forall u \in DP_\sigma(\tilde{\theta},\tilde{u})(\theta) \qquad \langle\nabla\hat{J}_0(\tilde{u}), u\rangle_{L^2(0,T;\mathcal{U})} \geq 0. \tag{24}$$

We end up this subsection with an existence result for the scalarized problem. Consider the following coercivity hypothesis

(CH_σ) for all $i \in \{1,\ldots,p\}$, l_i is bounded from below, and in relation (7) we have

$$\beta_i = 0 \quad \text{ and } \quad \begin{cases} \gamma_i > 0 & \text{if } \sigma = w \\ \gamma_i \geq 0,\ \sum_{j=1}^p \gamma_j > 0 & \text{if } \sigma = p. \end{cases}$$

Theorem 3 *Let $\sigma \in \{w, p\}$. Suppose that at least one of the following assumptions is fulfilled*

(i) (CH_σ)

(ii) *The set U is bounded in $\mathcal{U}$.*

Then

$$\text{dom}\,(P_\sigma) = \Theta_\sigma,$$

in other words, for each $\theta \in \Theta_\sigma$, the scalarized problem $\min\limits_{u \in U_{ad}} \langle \theta, \hat{J}(u) \rangle$ admits at least a solution.

Proof. It is obvious that for each $\theta \in \Theta_\sigma$ the functional $u \mapsto \langle \theta, \hat{J}(u) \rangle$ is convex, finite valued and lower semicontinuous on the closed convex set U_{ad}, hence lower semicontinuous for the weak topology of $L^2(0, T; \mathcal{U})$. On the other hand if (CH_σ) holds then $\langle \theta, \hat{J}(\cdot) \rangle$ is coercive and the conclusion is a well known result. If U is bounded in $\mathcal{U}$, then it is easy to see that U_{ad} is bounded in $L^2(0, T; \mathcal{U})$, hence weakly compact, and the conclusion follows from Weierstrass' Theorem. △

4.2 The case when P_σ is single valued

We will make the following assumption to ensure the strict convexity of the functional $\langle \theta, \hat{J}_0(\cdot) \rangle$

$$(SC_\sigma) \qquad \begin{cases} \forall i \in \{1, \ldots, p\} & L_i \text{ is strictly convex if } \sigma = w \\[2ex] \exists i \in \{1, \ldots, p\} \mid & L_i \text{ is strictly convex if } \sigma = p. \end{cases}$$

Then we can state the following result.

Theorem 4 (Existence and uniqueness for the scalarized problem) *Let $\sigma \in \{w, p\}$. Under the hypotheses of Theorem 3 and (SC_σ), for any $\theta \in \Theta_\sigma$ there exists a unique minimizer of $\langle \theta, \hat{J}(\cdot) \rangle$ over U_{ad} denoted from now on $\tilde{u}(\theta)$.*

Conversely, for each σ-Pareto solution to problem (19) $u \in \sigma\text{-}\underset{u \in U_{ad}}{\text{ARGMIN}}_{\mathbf{R}^p_+} \hat{J}(u)$ there exists at least an element $\theta \in \Theta_\sigma$ such that $u = \tilde{u}(\theta)$.

Proof. The existence of $\tilde{u}(\theta)$ follows from Theorem 3 and the uniqueness follows from (SC_σ) which ensures the strict convexity of $\langle \theta, \hat{J}(\cdot)\rangle$.

The last part follows from Theorem 1. △

Thus we obtain immediately the following:

Corollary 1 *Under the hypotheses of Theorem 4 the map P_σ is single valued*

$$\forall \theta \in \Theta_\sigma \qquad P_\sigma(\theta) = \{\tilde{u}(\theta)\}$$

and establishes a surjection *from Θ_σ to* $\sigma\text{-}\underset{u\in U_{ad}}{\mathrm{ARGMIN}}_{\mathbf{R}^p_+} \hat{J}(u)$.

From now on we will keep the hypotheses of Theorem 4.

Consider the function $\tilde{J}_0 : \Theta_\sigma \to \mathbf{R}$ given for all $\theta \in \Theta_\sigma$ by

$$\tilde{J}_0(\theta) := \hat{J}_0(\tilde{u}(\theta)). \tag{25}$$

It is clear that our problem $(PPOCP)$ is equivalent to the following *finite dimensional scalar minimization problem*

$$(SMFD) \qquad \min_{\theta\in\Theta_\sigma} \tilde{J}_0(\theta)$$

in the sense that

(z,u) is a solution to $(PPOCP)$ if, and only if, there exists a solution θ to $(SMFC)$ such that $u = \tilde{u}(\theta)$ and $z = z_u$.

Thus the main practical problem is to be able to *find in closed form,* (or at least to have the maximum of information about) the map $\theta \mapsto \tilde{u}(\theta)$.

In the general setting we can apply Pontryagin maximum principle for the control problem having the scalar objective $(z,u)\mapsto\langle\theta,(J_1(z,u),\ldots,J_p(z,u))\rangle$ and satisfying (2, 3, 4, 5) as e.g. in [3], but this approach will be very difficult for applications, and could be the subject of a subsequent paper. We will restrain our study here to a particular but important case of the linear-quadratic multi-objective control problem in Hilbert spaces.

4.3 The case of a linear-quadratic multi-objective parabolic control system

In this section we consider the particular case when, $U = \mathcal{U}$, $Z_T = H$, and for all $i = 1,\ldots,p$,

$l_i = 0, \ \forall (t,v,u) \in]0,T[\times V \times \mathcal{U}, \ L_i(t,v,u) = \|C_i(t)(v - z_d(t))\|_W^2 + \langle R_i(t)u, u\rangle,$

where W is a real Hilbert space, and for all $t \in]0,T]$, $C_i(t) \in \mathcal{L}(V,W)$, $R_i(t) \in \mathcal{L}(\mathcal{U},\mathcal{U})$, is a symmetric positive operator. $z_d \in L^2(0,T;V)$ is the desired state. Moreover, we suppose that $t \mapsto \langle C_i(t)v, w\rangle_W$, $t \mapsto \langle R_i(t)u_1, u_2\rangle_{\mathcal{U}}$, are measurable for all $v \in V, w \in W, u_1, u_2 \in \mathcal{U}$, and $\|C_i(t)\|_{\mathcal{L}(V,W)} \leq c$, $\|R_i(t)\|_{\mathcal{L}(\mathcal{U},\mathcal{U})} \leq c$ for all $t \in]0,T[$.

Hence, hereafter for all $i = 1, \ldots, p$,

$$J_i(z,u) = \int_0^T (\|C_i(t)(z_u(t) - z_d(t))\|_W^2 + \langle R_i(t)u(t), u(t)\rangle_{\mathcal{U}})dt,$$

and problem (MOCCOP) becomes (MOLQP)

$$\mathrm{MIN}_{\mathbf{R}_+^p} \left(\int_0^T (\|C_i(t)(z_u(t) - z_d(t))\|_W^2 + \langle R_i(t)u(t), u(t)\rangle_{\mathcal{U}})dt \right)_{1 \leq i \leq p} \qquad s.t.$$

$$\begin{aligned} \frac{dz}{dt}(t) + A(t)z(t) &= B(t)u(t) \quad \text{a.e. on }]0,T[\\ z(0) &= z_0 \in H \end{aligned}$$

All the hypotheses about the families $(A(t))_{t\in]0,T}$ and $(B(t))_{t\in]0,T}$ are kept in this section.

Moreover, we will make the following assumption (HLQ)

$$\begin{cases} \forall i \in [1,p] \ \exists \alpha_i > 0 \ \forall u \in \mathcal{U}, \ \forall t \in]0,T[\quad \langle R_i(t)u, u\rangle \geq \alpha_i \|u\|_{\mathcal{U}}^2 \quad \text{if } \sigma = w. \\ \\ \exists i \in [1,p] \ \exists \alpha_i > 0 \ \forall u \in \mathcal{U}, \ \forall t \in]0,T[\quad \langle R_i(t)u, u\rangle \geq \alpha_i \|u\|_{\mathcal{U}}^2 \quad \text{if } \sigma = p. \end{cases}$$

Notice that the adjoint of $A^*(t)$ of $A(t)$ will be considered for all $t \in]0,T[$ as $A^*(t) \in \mathcal{L}(V,V')$ given by

$$\forall v, z \in V \qquad (A(t)v \mid z) = (A^*(t)z \mid v).$$

Also, thanks to Riesz' theorem, we will consider for all $t \in]0,T[$, $B^*(t) \in \mathcal{L}(V,\mathcal{U})$ given by[3]

$$\forall (u,v) \in \mathcal{U} \times V \quad (B(t)u \,|\, v) = \langle u, B^*(t)v\rangle_{\mathcal{U}},$$

and $C_i^*(t) \in \mathcal{L}(W,V)$ given by

$$\forall (v,w) \in V \times W \quad \langle C_i(t)v, w\rangle_W = \langle v, C_i^*(t)w\rangle_V.$$

Let $\theta = (\theta_1, \ldots, \theta_p) \in \Theta_\sigma$. We have for all $u \in L^2(0,T;\mathcal{U})$,

$$\begin{aligned} \langle \theta, \hat{J}(u)\rangle &= \sum_{i=1}^{p} \theta_i \int_0^T (\|C_i(t)(z_u(t) - z_d(t))\|_W^2 + \langle R_i(t)u(t), u(t)\rangle_{\mathcal{U}})dt \\ &= \int_0^T \Big(\langle D(\theta)(t)z_u(t), z_u(t)\rangle_V - 2\langle D(\theta)z_d(t), z_u(t)\rangle_V \\ &\quad + \langle R(\theta)(t)u(t), u(t)\rangle_{\mathcal{U}}\Big)dt + c_0(\theta), \end{aligned}$$

where

$$D(\theta) := \sum_{i=1}^{p} \theta_i C_i^* C_i \in \mathcal{L}(V,V), \quad R(\theta) := \sum_{i=1}^{p} \theta_i R_i, \tag{26}$$

$$c_0(\theta) = \int_0^T \langle D(\theta)z_d(t), z_d(t)\rangle dt.$$

It is obvious that the hypotheses of Theorems 3 and 4 are fulfilled, hence for each $\theta \in \Theta_\sigma$, there exists a unique minimizer $\tilde{u}(\theta)$ of $\langle \theta, \hat{J}(\cdot)\rangle$ over $U_{ad} = \mathcal{U}$.

Proposition 3 *The map $\Theta_w \ni \theta \mapsto \tilde{u}(\theta) \in L^2(0,T;\mathcal{U})$ is indefinitely Fréchet differentiable on $\Theta_p =$ int(Θ_w).*

Proof. Denote $J(\theta,u) = \langle \theta, \hat{J}(u)\rangle$, thus $J : \Theta_w \times L^2(0,T;\mathcal{U} \to \mathbf{R}$, and, using Lemma 1 it is easy to see that we can write

$$J(\theta,u) = \frac{1}{2}\langle N(\theta)u, u\rangle_{L^2(0,T;\mathcal{U})} - \langle l(\theta), u\rangle_{L^2(0,T;\mathcal{U})} + k_0(\theta),$$

[3]In general, if we do *not* identify a Hilbert space with its dual, the adjoint of $B(t)$ is the operator $B^*(t) \in \mathcal{L}(V',\mathcal{U}')$ which verifies $\langle B(t)u, , v'\rangle_{VV'} = \langle u, B^*(t)v'\rangle_{\mathcal{U}\mathcal{U}'}$ for all $u \in \mathcal{U}$, $v' \in V'$. Thus, if we denote by $I_V : V \to V'$ the canonical isomorphism given by Riesz' theorem, in order to simplify notations, we denote in fact by $B^*(t)$ the operator $I_{\mathcal{U}}^{-1} B^*(t) I_V$.

where $N(\theta) \in \mathcal{L}(L^2(0,T;\mathcal{U}),\, L^2(0,T;\mathcal{U}))$ is a selfadjoint operator such that there is some $\beta > 0$ independent of θ verifying $\langle N(\theta)u, u\rangle_{L^2(0,T;\mathcal{U})} \geq \beta\|u\|^2_{L^2(0,T,\mathcal{U}}$ for all u, $l(\theta) \in L^2(0,T;\mathcal{U})$ and $k_0(\theta) \in \mathbf{R}$ are independent of u. Moreover the maps $\theta \mapsto N(\theta)$, $\theta \mapsto l(\theta)$, $\theta \mapsto k_0(\theta)$ are (restrictions to Θ_w of) linear maps, more precisely

$$N(\theta) = \sum_{i=1}^{p} \theta_i N_i, \quad l(\theta) = \sum_{i=1}^{p} \theta_i l_i,$$

where N_i are selfadjoints and positive, and $l_i \in L^2(0,T;\mathcal{U})$, $i = 1, \ldots, p$. Thus it is obvious that J is C^∞ and its Fréchet derivative (gradient) with respect to u is given by

$$\frac{\partial J}{\partial u}(\theta, u) = N(\theta)u - l(\theta),$$

identifying $L^2(0,T;\mathcal{U})$ with its dual thanks to Riesz' theorem. Fermat's rule implies that for all $\theta \in \Theta_p$ we mast have

$$N(\theta)\tilde{u}(\theta) = l(\theta),$$

hence

$$\tilde{u}(\theta) = N^{-1}(\theta)l(\theta). \tag{27}$$

The RHS term is well defined and the map $\theta \mapsto N^{-1}(\theta)l(\theta)$ is C^∞ on Θ_p. Moreover, for each $\theta \in \Theta_p$, the partial derivative of $\tilde{u}$ at θ is given by

$$\frac{\partial \tilde{u}}{\partial \theta_i}(\theta) = -N^{-1}(\theta)N_i N^{-1}(\theta)l(\theta) + N^{-1}(\theta)l_i, \qquad i = 1, \ldots, p. \tag{28}$$

$\triangle$

Remark 2 *The last result has only a theoretical value because it is a complicated task to give an explicit expression as function of data in (27, 28). That is why we will briefly present other ways to find $\tilde{u}(\theta)$.*

4.4 The adjoint system

We can characterize $\tilde{u}(\theta)$ coupling the initial evolution equation with the adjoint equation (see e.g. [29]). Let $\theta \in \Theta_\sigma$ be fixed. Then $\tilde{u}(\theta)$ is given by

$$\tilde{u}(\theta)(t) = -R^{-1}(\theta)B^*(t)p(t) \quad \text{a.e. on }]0,T[, \tag{29}$$

where $p \in W(0,T) := \{p \in L^2(0,T;V) \mid \frac{dp}{dt} \in L^2(0,T;V')\}$ verifies the following system (limit problem)

$$\begin{cases} \dfrac{dz}{dt}(t) + A(t)z(t) = -B(t)R^{-1}(\theta)B^*(t)p(t) & \text{a.e. on }]0,T[\\ -\dfrac{dp}{dt}(t) + A^*(t)p(t) = D(\theta)(z(t) - z_d(t)) & \text{a.e. on }]0,T[\\ z_u(0) = z_0, \quad p(T) = 0. \end{cases} \tag{30}$$

4.5 Riccati equation

Consider for a fixed $\theta \in \Theta_\sigma$ the following *formal* backward Cauchy problem in the operator space $\mathcal{L}(H,H)$

$$\begin{cases} -\dfrac{dP}{dt} + PA + A^*P + PBR(\theta)^{-1}B^*P = D(\theta) & \text{in }]0,T[\\ P(T) = 0, \end{cases} \tag{31}$$

as well as the backward Cauchy problem in $L^2(0,T;H)$

$$\begin{cases} -\dfrac{dr}{dt} + A^*r + PBR^{-1}(\theta)B^*r = -D(\theta) & \text{in }]0,T[\\ r(T) = 0. \end{cases} \tag{32}$$

The precise meaning of these problems can be found in [29]. Thus, we have (see [29])

$$p(t) = P(t)z(t) + r(t) \qquad \forall t \in]0,T[, \tag{33}$$

where z and p verify (30). Of course we will obtain $\tilde{u}(\theta)$ as a feedback of the state replacing the value of p given by (33) in (29). But for us *the problem is to find also the optimal value of θ!* To do this it could be possible to express $\tilde{u}$ and $z_{\tilde{u}}$ in closed form as functions of θ if we are able to solve a *linear* differential equation in the product operator space $\mathcal{L}(H,H)\times\mathcal{L}(H,H)$ generalizing some ideas from [17] to infinite dimensional case, but this will be the object of a subsequent paper.

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