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A TOPOLOGICAL PROPERTY OF THE SOLUTION SET OF A SECOND-ORDER DIFFERENTIAL INCLUSION*

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Abstract

We consider a Cauchy problem for a Sturm-Liouville type differential inclusion involving a nonconvex set-valued map and we prove that the set of selections corresponding to the solutions of the problem considered is a retract of the space of integrable functions on unbounded interval.

MSC: 34A60

keywords: differential inclusion, decomposable set, retract.

1 Introduction

In this paper we study second-order differential inclusions of the form

$$(p(t)x'(t))' \in F(t, x(t)) \quad a.e.\ [0, \infty), \quad x(0) = x_0, \quad x'(0) = x_1, \tag{1.1}$$

where $F : [0, \infty) \times \mathbf{R}^n \to \mathcal{P}(\mathbf{R}^n)$ is a set-valued map, $x_0, x_1 \in \mathbf{R}^n$ and $p(.) : [0, \infty) \to (0, \infty)$ is continuous.

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Even if we deal with an initial value problem instead of a boundary value problem, the differential inclusion (1.1)-(1.2) may be regarded as an extension to the set-valued framework of the classical Sturm-Liouville differential equation. Several qualitative properties and existence results for problem (1.1) may be found in [3-9] etc..

In [6] we proved that the solution set of problem (1.1) is arcwise connected when the set-valued map is Lipschitz in the second variable and the problem is defined on a bounded interval. The aim of this paper is to establish a more general topological property of the solution set of problem (1.1). Namely, we prove that the set of selections of the set-valued map F that correspond to the solutions of problem (1.1) is a retract of $L^1_{loc}([0,\infty),\mathbf{R}^n)$. The result is essentially based on Bressan and Colombo results ([1]) concerning the existence of continuous selections of lower semicontinuous multifunctions with decomposable values.

We note that in the classical case of differential inclusions several topological properties of solution set are obtained using various methods and tools ([2, 10-14] etc.). The result in the present paper extends to Sturm-Liouville differential inclusions the main result in [12] obtained in the case of classical differential inclusions.

The paper is organized as follows: in Section 2 we present the notations, definitions and the preliminary results to be used in the sequel and in Section 3 we prove our main result.

2 Preliminaries

Let $T > 0$, $I := [0, T]$ and denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I. Let X be a real separable Banach space with the norm $|.|$. Denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X and by $\mathcal{B}(X)$ the family of all Borel subsets of X. If $A \subset I$ then $\chi_A(.) : I \to \{0, 1\}$ denotes the characteristic function of A. For any subset $A \subset X$ we denote by $cl(A)$ the closure of A.

The distance between a point $x \in X$ and a subset $A \subset X$ is defined as usual by $d(x, A) = \inf\{|x - a|; a \in A\}$. We recall that Pompeiu-Hausdorff distance between the closed subsets $A, B \subset X$ is defined by $d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}$, $d^*(A, B) = \sup\{d(a, B);\ a \in A\}$.

As usual, we denote by $C(I, X)$ the Banach space of all continuous functions $x : I \to X$ endowed with the norm $|x|_C = \sup_{t\in I}|x(t)|$ and by $L^1(I, X)$

the Banach space of all (Bochner) integrable functions $x : I \to X$ endowed with the norm $|x|_1 = \int_0^T |x(t)|dt$.

We recall first several preliminary results we shall use in the sequel.

A subset $D \subset L^1(I, X)$ is said to be *decomposable* if for any $u, v \in D$ and any subset $A \in \mathcal{L}(I)$ one has $u\chi_A + v\chi_B \in D$, where $B = I\backslash A$.

We denote by $\mathcal{D}(I, X)$ the family of all decomposable closed subsets of $L^1(I, X)$.

Next (S, d) is a separable metric space; we recall that a multifunction $G : S \to \mathcal{P}(X)$ is said to be lower semicontinuous (l.s.c.) if for any closed subset $C \subset X$, the subset $\{s \in S;\ G(s) \subset C\}$ is closed.

Lemma 2.1. ([1]) *Let $F^* : I \times S \to \mathcal{P}(X)$ be a closed-valued $\mathcal{L}(I) \otimes \mathcal{B}(S)$-measurable multifunction such that $F^*(t, .)$ is l.s.c. for any $t \in I$.*

Then the multifunction $G : S \to \mathcal{D}(I, X)$ defined by

$$G(s) = \{v \in L^1(I, X); \quad v(t) \in F^*(t, s) \quad a.e.\ (I)\}$$

is l.s.c. with nonempty closed values if and only if there exists a continuous mapping $p : S \to L^1(I, X)$ such that

$$d(0, F^*(t, s)) \le p(s)(t) \quad a.e.\ (I),\ \forall s \in S.$$

Lemma 2.2. ([1]) *Let $G : S \to \mathcal{D}(I, X)$ be a l.s.c. multifunction with closed decomposable values and let $\phi : S \to L^1(I, X)$, $\psi : S \to L^1(I, \mathbf{R})$ be continuous such that the multifunction $H : S \to \mathcal{D}(I, X)$ defined by*

$$H(s) = cl\{v(.) \in G(s); \quad |v(t) - \phi(s)(t)| < \psi(s)(t) \quad a.e.\ (I)\}$$

has nonempty values.

Then H has a continuous selection, i.e. there exists a continuous mapping $h : S \to L^1(I, X)$ such that $h(s) \in H(s) \quad \forall s \in S$.

Consider a set-valued map $F : [0, \infty) \times \mathbf{R}^n \to \mathcal{P}(\mathbf{R}^n)$, $x_0, x_1 \in \mathbf{R}^n$ and a continuous mapping $p(.) : [0, \infty) \to (0, \infty)$ that define the Cauchy problem (1.1).

A continuous mapping $x(.) \in C([0, \infty), \mathbf{R}^n)$ is called a solution of problem (1.1) if there exists a integrable function $f(.) \in L^1_{loc}([0, \infty), \mathbf{R}^n)$ such that

$$f(t) \in F(t, x(t)) \quad a.e.\ [0, \infty), \tag{2.1}$$

$$x(t) = x_0 + p(0)x_1 \int_0^t \frac{1}{p(s)} ds + \int_0^t \frac{1}{p(s)} \int_0^s f(u) du ds \quad \forall t \in [0, \infty). \quad (2.2)$$

Note that, if we put $G(t,u) := \int_u^t \frac{1}{p(s)}$, $t \in I$, then (2.2) may be rewritten as

$$x(t) = x_0 + p(0)x_1 G(t,0) + \int_0^t G(t,u) f(u) du \quad \forall t \in [0, \infty). \quad (2.3)$$

We shall call $(x(.), f(.))$ a *trajectory-selection pair* of (1.1) if (2.1) and (2.2) are satisfied.

We shall use the following notations for the solution sets and for the selection sets of problem (1.1).

$$\mathcal{S}(x_0, x_1) = \{x(.) \in C([0,\infty), \mathbf{R}^n); \quad x(.) \text{ is a solution of } (1.1)\}, \quad (2.4)$$

$$\mathcal{T}(x_0, x_1) = \{f(.) \in L^1_{loc}([0,\infty), \mathbf{R}^n);\ f(t) \in F(t, x_0 + p(0)x_1 G(t,0) + \\ + \textstyle\int_0^t G(t,u) f(u) du) \quad a.e.\ [0,\infty)\}. \quad (2.5)$$

3 The main result

In order to prove our topological property of the solution set of problem (1.1) we need the following hypotheses.

Hypothesis 3.1. i) *$F(.,.) : [0,\infty) \times \mathbf{R}^n \to \mathcal{P}(\mathbf{R}^n)$ has nonempty compact values and is $\mathcal{L}([0,\infty)) \otimes \mathcal{B}(\mathbf{R}^n)$ measurable.*

ii) *There exists $L \in L^1_{loc}([0,\infty), \mathbf{R})$ such that, for almost all $t \in [0,\infty)$, $F(t,.)$ is $L(t)$-Lipschitz in the sense that*

$$\mathrm{d}_H(F(t,x), F(t,y)) \le L(t)|x-y| \quad \forall x, y \in \mathbf{R}^n.$$

iii) *There exists $p \in L^1_{loc}([0,\infty), \mathbf{R}^n)$ such that*

$$\mathrm{d}_H(\{0\}, F(t,0)) \le p(t) \quad a.e.\ [0,\infty).$$

In what follows $I = [0,T]$ and let $M := \sup_{t \in I} \frac{1}{p(t)}$. Note that $|G(t,u)| \le Mt$ $\forall t, u \in I, u \le t$. We use the notations

$$\tilde{u}(t) = x_0 + p(0)x_1 G(t,0) + \int_0^t G(t,s) u(s) ds, \quad u \in L^1(I, \mathbf{R}^n) \quad (3.1)$$

and

$$p_0(u)(t) = |u(t)| + p(t) + L(t)|\tilde{u}(t)|, \quad t \in I \tag{3.2}$$

Let us note that

$$d(u(t), F(t, \tilde{u}(t)) \leq p_0(u)(t) \quad a.e.\ (I) \tag{3.3}$$

and, since for any $u_1, u_2 \in L^1(I, \mathbf{R}^n)$

$$|p_0(u_1) - p_0(u_2)|_1 \leq (1 + MT \int_0^T L(s)ds|)|u_1 - u_2|_1$$

the mapping $p_0 : L^1(I, \mathbf{R}^n) \to L^1(I, \mathbf{R}^n)$ is continuous.

Also define

$$\mathcal{T}_I(x_0, x_1) = \{f \in L^1(I, \mathbf{R}^n); \quad f(t) \in F(t, x_0 + p(0)x_1G(t, 0) + \\ + \int_0^t G(t, s)f(s)ds) \quad a.e.\ (I)\}.$$

Proposition 3.2. *Assume that Hypothesis 3.1 is satisfied and let* $\phi : L^1(I, \mathbf{R}^n) \to L^1(I, \mathbf{R}^n)$ *be a continuous map such that* $\phi(u) = u$ *for all* $u \in \mathcal{T}_I(x_0, x_1)$*. For* $u \in L^1(I, \mathbf{R}^n)$*, we define*

$$\Psi(u) = \{u \in L^1(I, \mathbf{R}^n); \quad u(t) \in F(t, \widetilde{\phi(u)}(t)) \quad a.e.\ (I)\},$$

$$\Phi(u) = \begin{cases} \{u\} & \textit{if } u \in \mathcal{T}_I(x_0, x_1), \\ \Psi(u) & \textit{otherwise.} \end{cases}$$

Then the multifunction $\Phi : L^1(I, \mathbf{R}^n) \to \mathcal{P}(L^1(I, \mathbf{R}^n))$ *is lower semicontinuous with closed decomposable and nonempty values.*

Proof. According to (3.3), Lemma 2.1 and the continuity of p_0 we obtain that Ψ has closed decomposable and nonempty values and the same holds for the set-valued map Φ.

Let $C \subset L^1(I, \mathbf{R}^n)$ be a closed subset, let $\{u_m\}_{m \in \mathbf{N}}$ converges to some $u_0 \in L^1(I, \mathbf{R}^n)$ and $\Phi(u_m) \subset C$, for any $m \in \mathbf{N}$. Let $v_0 \in \Phi(u_0)$ and for every $m \in \mathbf{N}$ consider a measurable selection v_m from the set-valued map $t \to F(t, \widetilde{\phi(u_m)}(t))$ such that $v_m = u_m$ if $u_m \in \mathcal{T}_I(x_0, x_1)$ and

$$|v_m(t) - v_0(t)| = d(v_0(t), F(t, \widetilde{\phi(u_m)}(t)) \quad a.e.\ (I)$$

otherwise. One has

$$|v_m(t) - v_0(t)| \leq$$

$$\leq d_H(F(t, \widetilde{\phi(u_m)}(t)), F(t, \widetilde{\phi(u_0)}(t))) \leq L(t)|\widetilde{\phi(u_m)}(t) - \widetilde{\phi(u_0)}(t)|$$

hence

$$|v_m - v_0|_1 \leq MT \int_0^T L(s)ds.|\widetilde{\phi(u_m)} - \widetilde{\phi(u_0)}|_1.$$

Since $\phi : L^1(I, \mathbf{R}^n) \to L^1(I, \mathbf{R}^n)$ is continuous, it follows that v_m converges to v_0 in $L^1(I, \mathbf{R}^n)$. On the other hand, $v_m \in \Phi(u_m) \subset C$ $\forall m \in \mathbf{N}$ and since C is closed we infer that $v_0 \in C$. Hence $\Phi(u_0) \subset C$ and Φ is lower semicontinuous.

In what follows we shall use the following notations

$$I_k = [0, k], \quad k \geq 1, \quad |u|_{1,k} = \int_0^k |u(t)|dt, \quad u \in L^1(I_k, \mathbf{R}^n).$$

We are able now to prove the main result of this paper.

Theorem 3.3. *Assume that Hypothesis 3.1 is satisfied, there exists* $M := \sup_{t\in[0,\infty)} \frac{1}{p(t)}$ *and* $x_0, x_1 \in \mathbf{R}^n$.

Then there exists a continuous mapping $G : L^1_{loc}([0,\infty), \mathbf{R}^n) \to L^1_{loc}([0,\infty), \mathbf{R}^n)$ *such that*

(i) $G(u) \in \mathcal{T}(x_0, x_1), \quad \forall u \in L^1_{loc}([0,\infty), \mathbf{R}^n)$,

(ii) $G(u) = u, \quad \forall u \in \mathcal{T}(x_0, x_1)$.

Proof. We shall prove that for every $k \geq 1$ there exists a continuous mapping $g^k : L^1(I_k, \mathbf{R}^n) \to L^1(I_k, \mathbf{R}^n)$ with the following properties

(I) $g^k(u) = u, \quad \forall u \in \mathcal{T}_{I_k}(x_0, x_1)$

(II) $g^k(u) \in \mathcal{T}_{I_k}(x_0, x_1), \quad \forall u \in L^1(I_k, \mathbf{R}^n)$

(III) $g^k(u)(t) = g^{k-1}(u|_{I_{k-1}})(t), \quad \forall t \in I_{k-1}$

If the sequence $\{g^k\}_{k\geq 1}$ is constructed, we define $G : L^1_{loc}([0,\infty), \mathbf{R}^n) \to L^1_{loc}([0,\infty), \mathbf{R}^n)$ by

$$G(u)(t) = g^k(u|_{I_k})(t), \quad \forall k \geq 1$$

From (III) and the continuity of each $g^k(.)$ it follows that $G(.)$ is well defined and continuous. Moreover, for each $u \in L^1_{loc}([0,\infty), \mathbf{R}^n)$, according to (II) we have

$$G(u)|_{I_k}(t) = g^k(u|_{I_k})(t) \in \mathcal{T}_{I_k}(x_0, x_1), \quad \forall k \geq 1$$

and thus $G(u) \in \mathcal{T}(x_0, x_1)$.

Fix $\varepsilon > 0$ and for $m \geq 0$ set $\varepsilon_m = \frac{m+1}{m+2}\varepsilon$. For $u \in L^1(I_1, \mathbf{R}^n)$ and $m \geq 0$ define $m(t) = \int_0^t L(s)ds$,

$$p_0^1(u)(t) = |u(t)| + p(t) + L(t)|\tilde{u}(t)|, \quad t \in I_1$$

and

$$p_{m+1}^1(u)(t) = M^{m+1}\int_0^t p_0^1(u)(s)\frac{(m(t)-m(s))^m}{m!}ds + M^m\frac{(m(t))^m}{m!}\varepsilon_{m+1}.$$

By the continuity of the map $p_0^1(.) = p_0(.)$, already proved, we obtain that $p_m^1 : L^1(I_1, \mathbf{R}^n) \to L^1(I_1, \mathbf{R}^n)$ is continuous.

We define $g_0^1(u) = u$ and we shall prove that for any $m \geq 1$ there exists a continuous map $g_m^1 : L^1(I_1, \mathbf{R}^n) \to L^1(I_1, \mathbf{R}^n)$ that satisfies

$$g_m^1(u) = u, \quad \forall u \in \mathcal{T}_{I_1}(x_0, x_1), \tag{a_1}$$

$$g_m^1(u)(t) \in F(t, \widetilde{g_{m-1}^1(u)}(t)) \quad a.e.\ (I_1), \tag{b_1}$$

$$|g_1^1(u)(t) - g_0^1(u)(t)| \leq p_0^1(u)(t) + \varepsilon_0 \quad a.e.\ (I_1), \tag{c_1}$$

$$|g_m^1(u)(t) - g_{m-1}^1(t)| \leq L(t)p_{m-1}^1(u)(t) \quad a.e.\ (I_1), \quad m \geq 2. \tag{d_1}$$

For $u \in L^1(I_1, \mathbf{R}^n)$, we define

$$\Psi_1^1(u) = \{v \in L^1(I_1, \mathbf{R}^n); \;\; v(t) \in F(t, \widetilde{u}(t)) \;\; a.e.(I_1)\},$$

$$\Phi_1^1(u) = \begin{cases} \{u\} & \text{if } u \in \mathcal{T}_{I_1}(x_0, x_1), \\ \Psi_1^1(u) & \text{otherwise.} \end{cases}$$

and by Proposition 3.2 (with $\phi(u) = u$) we obtain that $\Phi_1^1 : L^1(I_1, \mathbf{R}^n) \to \mathcal{D}(I_1, \mathbf{R}^n)$ is lower semicontinuous. Moreover, due to (3.3) the set

$$H_1^1(u) = cl\{v \in \Phi_1^1(u); \quad |v(t) - u(t)| < p_0^1(u)(t) + \varepsilon_0 \quad a.e.\ (I_1)\}$$

is not empty for any $u \in L^1(I_1, \mathbf{R}^n)$. So applying Lemma 2.2, we find a continuous selection g_1^1 of H_1^1 that satisfies (a_1)-(c_1).

Suppose we have already constructed $g_i^1(.)$, $i = 1, \ldots m$ satisfying (a_1)-(d_1). Then from (b_1), (d_1) and Hypothesis 3.1 we get

$$\begin{aligned} &d(g_m^1(u)(t), F(t, \widetilde{g_m^1(u)}(t)) \leq L(t)(|\widetilde{g_{m-1}^1(u)}(t) - \widetilde{g_m^1(u)}(t)| \leq \\ &L(t)\textstyle\int_0^T ML(s)p_m^1(u)(s)ds = L(t)(p_{m+1}^1(u)(t) - r_m^1(t)) < L(t)p_{m+1}^1(u)(t), \end{aligned} \tag{3.4}$$

where $r_m^1(t) := M^m \frac{(m(t))^m}{m!}(\varepsilon_{m+1} - \varepsilon_m) > 0$.

For $u \in L^1(I_1, \mathbf{R}^n)$, we define

$$\Psi_{m+1}^1(u) = \{v \in L^1(I_1, \mathbf{R}^n);\ \ v(t) \in F(t, \widetilde{g_m^1(u)}(t)) \quad a.e.\ (I_1)\},$$

$$\Phi_{m+1}^1(u) = \begin{cases} \{u\} & \text{if } u \in \mathcal{T}_{I_1}(x_0, x_1), \\ \Psi_{m+1}^1(u) & \text{otherwise.} \end{cases}$$

We apply Proposition 3.2 (with $\phi(u) = g_m^1(u)$) and obtain that $\Phi_{m+1}^1(.)$ is lower semicontinuous with closed decomposable and nonempty values. Moreover, by (3.4), the set

$$H_{m+1}^1(u) = cl\{v \in \Phi_{m+1}^1(u);\ |v(t) - g_{m+1}^1(u)(t)| < L(t)p_{m+1}^1(u)(t)\ a.e.\ (I_1)\}$$

is nonempty for any $u \in L^1(I_1, \mathbf{R}^n)$. With Lemma 2.2, we find a continuous selection g_{m+1}^1 of H_{m+1}^1, satisfying (a_1)-(d_1).

Therefore we obtain that

$$|g_{m+1}^1(u) - g_m^1(u)|_{1,1} \le \frac{(Mm(1))^m}{m!}(M|p_0^1(u)|_{1,1} + \varepsilon)$$

and this implies that the sequence $\{g_m^1(u)\}_{m\in\mathbf{N}}$ is a Cauchy sequence in the Banach space $L^1(I_1, \mathbf{R}^n)$. Let $g^1(u) \in L^1(I_1, \mathbf{R}^n)$ be its limit. The function $s \to |p_0^1(u)|_{1,1}$ is continuous, hence it is locally bounded and the Cauchy condition is satisfied by $\{g_m^1(u)\}_{m\in\mathbf{N}}$ locally uniformly with respect to u. Hence the mapping $g^1(.) : L^1(I_1, \mathbf{R}^n) \to L^1(I_1, \mathbf{R}^n)$ is continuous.

From (a_1) it follows that $g^1(u) = u, \quad \forall u \in \mathcal{T}_{I_1}(x_0, x_1)$ and from (b_1) and the fact that F has closed values we obtain that

$$g^1(u)(t) \in F(t, \widetilde{g^1(u)}(t)), \quad a.e.\,(I_1) \quad \forall u \in L^1(I_1, \mathbf{R}^n).$$

In the next step of the proof we suppose that we have already constructed the mappings $g^i(.) : L^1(I_i, \mathbf{R}^n) \to L^1(I_i, \mathbf{R}^n),\ i = 2, ..., k-1$ with the properties (I)-(III) and we shall construct a continuous map $g^k(.) : L^1(I_k, \mathbf{R}^n) \to L^1(I_k, \mathbf{R}^n)$ satisfying (I)-(III).

Let $g_0^k : L^1(I_k, \mathbf{R}^n) \to L^1(I_k, \mathbf{R}^n)$ be defined by

$$g_0^k(u)(t) = g^{k-1}(u|_{I_{k-1}})(t)\chi_{I_{k-1}} + u(t)\chi_{I_k \setminus I_{k-1}}(t) \tag{3.5}$$

Let us note, first, that $g_0^k(.)$ is continuous. Indeed, if $u_0, u \in L^1(I_k, \mathbf{R}^n)$ one has

$$|g_0^k(u) - g_0^k(u_0)|_{1,k} \le |g^{k-1}(u|_{I_{k-1}}) - g^{k-1}(u_0|_{I_{k-1}})|_{1,k-1} + \int_{k-1}^{k} |u(t) - u_0(t)|dt$$

So, using the continuity of $g^{k-1}(.)$ we get the continuity of $g_0^k(.)$.

On the other hand, since $g^{k-1}(u) = u, \ \forall u \in \mathcal{T}_{I_{k-1}}(x_0, x_1)$ from (3.5) it follows that

$$g_0^k(u) = u, \quad \forall u \in \mathcal{T}_{I_k}(x_0, x_1).$$

For $u \in L^1(I_k, \mathbf{R}^n)$, we define

$$\Psi_1^k(u) = \{w \in L^1(I_k, \mathbf{R}^n); \quad w(t) = g^{k-1}(u|_{I_{k-1}})(t)\chi_{I_{k-1}}(t) + v(t)\chi_{I_k \setminus I_{k-1}}(t), \quad v(t) \in F(t, \widetilde{g_0^k(u)(t)}) \quad a.e. \ ([k-1, k])\},$$

$$\Phi_1^k(u) = \begin{cases} \{u\} & \text{if } u \in \mathcal{T}_{I_k}(x_0, x_1), \\ \Psi_1^k(u) & \text{otherwise.} \end{cases}$$

We apply Proposition 3.2 (with $\phi(u) = g_0^k(u)$) and we obtain that $\Phi_1^k(.) : L^1(I_k, \mathbf{R}^n) \to \mathcal{D}(I_k, \mathbf{R}^n)$ is lower semicontinuous. Moreover, for any $u \in L^1(I_k, \mathbf{R}^n)$ one has

$$d(g_0^k(t), F(t, \widetilde{g_0^k(u)}(t)) = d(u(t), F(t, \widetilde{g_0^k(u)}(t))\chi_{I_k \setminus I_{k-1}} \leq p_0^k(u)(t) \quad a.e.(I_k), \tag{3.6}$$

where

$$p_0^k(u)(t) = |u(t)| + p(t) + L(t)|\widetilde{g_0^k(u)}(t)|.$$

Obviously, $p_0^k : L^1(I_k, \mathbf{R}^n) \to L^1(I_k, \mathbf{R}^n)$ is continuous. For $m \geq 0$ set

$$p_{m+1}^k(u) = (Mk)^{m+1} \int_0^t p_0^k(u)(s) \frac{(m(t) - m(s))^m}{m!} ds + (Mk)^m \frac{(m(t))^m}{m!} \varepsilon_{m+1}.$$

and by the continuity of $p_0^k(.)$ we infer that $p_m^k : L^1(I_k, \mathbf{R}^n) \to L^1(I_k, \mathbf{R}^n)$ is continuous.

We shall prove, next, that for any $m \geq 1$ there exists a continuous map $g_m^k : L^1(I_k, \mathbf{R}^n) \to L^1(I_k, \mathbf{R}^n)$ such that

$$g_m^k(u)(t) = g^{k-1}(u|_{I_{k-1}})(t) \quad \forall t \in I_{k-1}, \tag{a_k}$$

$$g_m^k(u) = u \quad \forall u \in \mathcal{T}_{I_k}(x_0, x_1), \tag{b_k}$$

$$g_m^k(u)(t) \in F(t, \widetilde{g_{m-1}^k(u)(t)}) \quad a.e. \ (I_k), \tag{c_k}$$

$$|g_1^k(u)(t) - g_0^k(u)(t)| \leq p_0^k(u)(t) + \varepsilon_0 \quad a.e. \ (I_k), \tag{d_k}$$

$$|g_m^k(u)(t) - g_{m-1}^k(u)(t)| \leq L(t)p_{m-1}^k(u)(t) \quad a.e. \ (I_k), \quad m \geq 2. \tag{e_k}$$

Define

$$H_1^k(u) = cl\{v \in \Phi_1^k(u); \quad |v(t) - g_0^k(u)(t)| < p_0^k(u)(t) + \varepsilon_0 \quad a.e.\ (I_k)\}.$$

From (3.6), $H_1^k(u) \neq \emptyset \quad \forall u \in L^1(I_1, \mathbf{R}^n)$. Using the continuity of g_0^k, p_0^k and Lemma 2.2, we obtain a continuous selection g_1^k of H_1^k that satisfies (a_k)-(d_k).

Assume we have constructed $g_i^k(.)$, $i = 1, \dots m$ satisfying (a_k)-(e_k). Then from (e_k) we have

$$\begin{aligned}&d(g_m^k(u)(t), F(t, \widetilde{g_m^k(u)}(t)) \leq L(t)(|\widetilde{g_{m-1}^k}(u)(t) - \widetilde{g_m^k(u)}(t)| \leq L(t)\cdot \\ &\textstyle\int_0^T MkL(s)p_m^k(u)(s)ds = L(t)(p_{m+1}^k(u)(t) - r_m^k(t)) < L(t)p_{m+1}^k(u)(t),\end{aligned} \tag{3.7}$$

where $r_m^k(t) := (Mk)^m \frac{(m(t))^m}{m!}(\varepsilon_{m+1} - \varepsilon_m) > 0$.

For $u \in L^1(I_k, \mathbf{R}^n)$, we define

$$\begin{aligned}&\Psi_{m+1}^k(u) = \{w \in L^1(I_k, \mathbf{R}^n); \quad w(t) = g^{k-1}(u|_{I_{k-1}})(t)\chi_{I_{k-1}}(t) + \\ &v(t)\chi_{I_k \setminus I_{k-1}}(t), \quad v(t) \in F(t, \widetilde{g_m^k(u)}(t)) \quad a.e.\ ([k-1, k])\},\end{aligned}$$

$$\Phi_{m+1}^k(u) = \begin{cases} \{u\} & \text{if } u \in \mathcal{T}_{I_k}(x_0, x_1), \\ \Psi_{m+1}^k(u) & \text{otherwise.} \end{cases}$$

With Proposition 3.2 we infer that $\Phi_{m+1}^k(.) : L^1(I_k, \mathbf{R}^n) \to \mathcal{P}(L^1(I_k, \mathbf{R}^n))$ is lower semicontinuous with closed decomposable and nonempty values. By (3.7) the set

$$H_{m+1}^k(u) = cl\{v \in \Phi_{m+1}^k(u); \ |v(t) - g_{m+1}^k(u)(t)| < L(t)p_{m+1}^k(u)(t) \ a.e.\ (I_k)\}$$

is nonempty for any $u \in L^1(I_k, \mathbf{R}^n)$. So, applying Lemma 2.2, we deduce a continuous selection g_{m+1}^k of H_{m+1}^k, satisfying (a_k)-(e_k).

By (e_k) one has

$$|g_{m+1}^k(u) - g_m^k(u)|_{1,k} \leq \frac{(Mkm(k))^m}{m!}(Mk|p_0^k(u)|_{1,1} + \varepsilon].$$

Therefore, with a similar proof as in the case $k = 1$, we find that the sequence $\{g_m^k(u)\}_{m \in \mathbf{N}}$ converges to some $g^k(u) \in L^1(I_k, \mathbf{R}^n)$ and the mapping $g^k(.) : L^1(I_k, \mathbf{R}^n) \to L^1(I_k, \mathbf{R}^n)$ is continuous.

By (a_k) we have that

$$g^k(u)(t) = g^{k-1}(u|_{I_{k-1}})(t) \quad \forall t \in I_{k-1},$$

by (b_k) $g^k(u) = u, \quad \forall u \in \mathcal{T}_{I_k}(x_0, x_1)$ and from (c_k) and the fact that F has closed values we obtain that

$$g^k(u)(t) \in F(t, \widetilde{g^k(u)}(t)), \quad a.e.\ (I_k) \quad \forall u \in L^1(I_k, \mathbf{R}^n).$$

Therefore $g^k(.)$ satisfies the properties (I), (II) and (III).

Remark 3.4. We recall that if Y is a Hausdorff topological space, a subspace X of Y is called retract of Y if there is a continuous map $h : Y \to X$ such that $h(x) = x, \ \forall x \in X$.

Therefore, by Theorem 3.3, for any $x_0, x_1 \in \mathbf{R}^n$, the set $\mathcal{T}(x_0, x_1)$ of selections that correspond to solutions of (1.1) is a retract of the Banach space $L^1_{loc}([0, \infty), \mathbf{R}^n)$.

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