ISSN 2066 - 6594

Ann. Acad. Rom. Sci.
Ser. Math. Appl.
Vol. 4, No. 1 / 2012

A HISTORY-DEPENDENT CONTACT PROBLEM WITH UNILATERAL CONSTRAINT*

ANCA FARCAS† FLAVIUS PATRULESCU ‡
MIRCEA SOFONEA§

Abstract

We consider a mathematical model which describes the quasistatic contact between a viscoplastic body and a foundation. The contact is frictionless and is modelled with a new and nonstandard condition which involves both normal compliance, unilateral constraint and memory effects. We derive a variational formulation of the problem then we prove its unique weak solvability. The proof is based on arguments on history-dependent variational inequalities.

MSC: 74M15, 74G25, 74G30, 35Q74

Keywords: viscoplastic material, frictionless contact, unilateral constraint, history-depdendent variational inequality, weak solution.

---

*Accepted for publication on April 2, 2012.

†anca.farcas@ubbcluj.ro, University Babeş-Bolyai, 400110 Cluj-Napoca, Romania

‡fpatrulescu@ictp.acad.ro, Tiberiu Popoviciu Institute of Numerical Analysis P.O. Box 68-1 and University Babeş-Bolyai, 400110 Cluj-Napoca, Romania

§sofonea@univ-perp.fr, Laboratoire de Mathématiques et Physique, Université de Perpignan Via Domitia, 52 Avenue Paul Alduy, 66860 Perpignan, France

1 The model

We consider a viscoplastic body which occupies the domain $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$) with a Lipschitz continuous boundary Γ, divided into three measurable parts Γ_1, Γ_2 and Γ_3, such that $\text{meas}(\Gamma_1) > 0$. We use the notation $\boldsymbol{x} = (x_i)$ for a typical point in $\Omega \cup \Gamma$ and we denote by $\boldsymbol{\nu} = (\nu_i)$ the outward unit normal at Γ. Here and below the indices i, j, k, l run between 1 and d and an index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $v_{i,j} = \partial v_i / \partial x_j$. The body is subject to the action of body forces of density $\boldsymbol{f}_0$, is fixed on Γ_1, and surface tractions of density $\boldsymbol{f}_2$ act on Γ_2. On Γ_3, the body is in frictionless contact with a deformable obstacle, the so-called foundation. We assume that the problem is quasistatic and the time interval of interest is $\mathbb{R}_+ = [0, \infty)$. Everywhere in this paper the dot above a variable represents derivative with respect to the time variable, $\mathbb{S}^d$ denotes the space of second order symmetric tensors on $\mathbb{R}^d$ and r^+ is the positive part of r, i.e. $r^+ = \max\{0, r\}$. The classical formulation of the problem is the following.

Problem $\mathcal{P}$. *Find a displacement field* $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ *such that, for all* $t \in \mathbb{R}_+$,

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t))) \quad \text{in} \quad \Omega, \tag{1}$$

$$\text{Div}\,\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in} \quad \Omega, \tag{2}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_1, \tag{3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on} \quad \Gamma_2, \tag{4}$$

$$\left.\begin{array}{l} u_\nu(t) \le g,\ \ \sigma_\nu(t) + p(u_\nu(t)) + \xi(t) \le 0, \\[2mm] (u_\nu(t) - g)\big(\sigma_\nu(t) + p(u_\nu(t)) + \xi(t)\big) = 0, \\[2mm] 0 \le \xi(t) \le \displaystyle\int_0^t b(t-s)\, u_\nu^+(s)\, ds, \\[2mm] \xi(t) = 0 \ \text{ if } \ u_\nu(t) < 0, \\[2mm] \xi(t) = \displaystyle\int_0^t b(t-s)\, u_\nu^+(s)\, ds \ \text{ if } \ u_\nu(t) > 0 \end{array}\right\} \quad \text{on} \quad \Gamma_3, \tag{5}$$

$$\boldsymbol{\sigma}_\tau(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_3, \tag{6}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0,\ \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \quad \text{in} \quad \Omega. \tag{7}$$

Equation (1) represents the viscoplastic constitutive law of the material in which $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the linearized stress tensor, $\mathcal{E}$ is the elasticity tensor and $\mathcal{G}$ is a given constitutive function. Equation (2) is the equilibrium equation in which Div denotes the divergence operator for tensor valued functions. Conditions (3) and (4) are the displacement and traction boundary conditions, respectively, and condition (5) represents the contact condition with normal compliance, unilateral constraint and memory term, in which σ_ν denotes the normal stress, u_ν is the normal displacement, $g \geq 0$ and p, b are given functions. In the case when b vanishes, this condition was used in [1, 3], for instance. Condition (6) shows that the tangential stress on the contact surface, denoted $\boldsymbol{\sigma}_\tau$, vanishes. We use it here since we assume that the contact process is frictionless. Finally, (7) represents the initial conditions in which $\boldsymbol{u}_0$ and $\boldsymbol{\sigma}_0$ denote the initial displacement and the initial stress field, respectively.

Quasistatic frictionless and frictional contact problems for viscoplastic materials with a constitutive law of the form (1) have been studied in various papers, see [2] for a survey. There, various models of contact were stated and their variational analysis, including existence and uniqueness results, was provided. The novelty of the current paper arises on the contact condition (5); it describes a deformable foundation which becomes rigid when the penetration reaches the critical bound g and which develops memory effects. Considering such condition leads to a new and nonstandard mathematical model which, in a variational formulation, is governed by a history-dependent variational inequality for the displacement field.

The rest of the paper is structured as follows. In Section 2 we list the assumptions on the data and introduce the variational formulation of the problem. Then, in Section 3 we state our main result, Theorem 1, and provide a sketch of the proof.

2 Variational formulation

In the study of problem $\mathcal{P}$ we use the standard notation for Sobolev and Lebesgue spaces associated to Ω and Γ. Also, we denote by "$\cdot$" and $\|\cdot\|$ the inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively. For each Banach space X we use the notation $C(\mathbb{R}_+;X)$ for the space of continuously functions defined on $\mathbb{R}_+$ with values on X and, for a subset $K \subset X$, we still use the symbol $C(\mathbb{R}_+;K)$ for the set of continuous functions defined on $\mathbb{R}_+$ with values on

K. We also consider the spaces

$$V = \{\, \boldsymbol{v} \in H^1(\Omega)^d : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1 \,\},\ Q = \{\, \boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega)^d : \tau_{ij} = \tau_{ji} \,\}.$$

These are Hilbert spaces together with the inner products $(\cdot,\cdot)_V$, $(\cdot,\cdot)_Q$,

$$(\boldsymbol{u},\boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \qquad (\boldsymbol{\sigma},\boldsymbol{\tau})_Q = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, dx,$$

and the associated norms $\|\cdot\|_V$, $\|\cdot\|_Q$, respectively. For an element $\boldsymbol{v} \in V$ we still write $\boldsymbol{v}$ for the trace of V and we denote by v_ν the normal component of $\boldsymbol{v}$ on Γ given by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$.

We assume that the elasticity tensor $\mathcal{E}$, the nonlinear constitutive function $\mathcal{G}$ and the normal compliance function p satisfy the following conditions.

$$\left\{\begin{array}{l}
\text{(a) } \mathcal{E} = (\mathcal{E}_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d.\\
\text{(b) } \mathcal{E}_{ijkl} = \mathcal{E}_{klij} = \mathcal{E}_{jikl} \in L^\infty(\Omega),\ 1 \le i,j,k,l \le d.\\
\text{(c) There exists } m_{\mathcal{E}} > 0 \text{ such that}\\
\qquad \mathcal{E}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \ge m_{\mathcal{E}} \|\boldsymbol{\tau}\|^2\ \ \forall\, \boldsymbol{\tau} \in \mathbb{S}^d,\ \text{a.e. in } \Omega.
\end{array}\right. \tag{8}$$

$$\left\{\begin{array}{l}
\text{(a) } \mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d.\\
\text{(b) There exists } L_{\mathcal{G}} > 0 \text{ such that}\\
\qquad \|\mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma}_1,\boldsymbol{\varepsilon}_1) - \mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{G}}\left(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|\right)\\
\qquad\quad \forall\, \boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in \mathbb{S}^d,\ \text{a.e. } \boldsymbol{x} \in \Omega.\\
\text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma},\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega,\\
\qquad \text{for any } \boldsymbol{\sigma},\boldsymbol{\varepsilon} \in \mathbb{S}^d.\\
\text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x},\boldsymbol{0},\boldsymbol{0}) \text{ belongs to } Q.
\end{array}\right. \tag{9}$$

$$\left\{\begin{array}{l}
\text{(a) } p : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+.\\
\text{(b) There exists } L_p > 0 \text{ such that}\\
\qquad |p(\boldsymbol{x},r_1) - p(\boldsymbol{x},r_2)| \le L_p\,|r_1 - r_2|\ \forall\, r_1,\, r_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3.\\
\text{(c) } (p(\boldsymbol{x},r_1) - p(\boldsymbol{x},r_2))(r_1 - r_2) \ge 0\ \forall\, r_1,\, r_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3.\\
\text{(d) The mapping } \boldsymbol{x} \mapsto p(\boldsymbol{x},r) \text{ is measurable on } \Gamma_3,\\
\qquad \text{for any } r \in \mathbb{R}.\\
\text{(e) } p(\boldsymbol{x},r) = 0 \text{ for all } r \le 0,\ \text{a.e. } \boldsymbol{x} \in \Gamma_3.
\end{array}\right. \tag{10}$$

Moreover, the densities of body forces and surface tractions, the memory function and the initial data are such that

$$\boldsymbol{f}_0 \in C(\mathbb{R}_+; L^2(\Omega)^d), \qquad \boldsymbol{f}_2 \in C(\mathbb{R}_+; L^2(\Gamma_2)^d), \tag{11}$$

$$b \in C(\mathbb{R}_+; L^\infty(\Gamma_3)), \qquad b(t,\boldsymbol{x}) \ge 0 \quad \text{a.e. } \boldsymbol{x} \in \Gamma_3, \tag{12}$$

$$\boldsymbol{u}_0 \in V, \quad \boldsymbol{\sigma}_0 \in Q. \tag{13}$$

Consider now the subset $U \subset V$, the operators $P : V \to V$, $\mathcal{B} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; L^2(\Gamma_3))$ and the function $\boldsymbol{f} : \mathbb{R}_+ \to V$ defined by

$$U = \{\, \boldsymbol{v} \in V : v_\nu \le g \text{ on } \Gamma_3 \,\}, \tag{14}$$

$$(P\boldsymbol{u}, \boldsymbol{v})_V = \int_{\Gamma_3} p(u_\nu) v_\nu \, da \quad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V, \tag{15}$$

$$(\mathcal{B}\boldsymbol{u}(t), \xi)_{L^2(\Gamma_3)} = \Big(\int_0^t b(t-s)\, u_\nu^+(s)\, ds, \xi \Big)_{L^2(\Gamma_3)} \tag{16}$$
$$\forall\, \boldsymbol{u} \in C(\mathbb{R}_+; V),\ \xi \in L^2(\Gamma_3),\ t \in \mathbb{R}_+,$$

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}\, da \quad \forall\, \boldsymbol{v} \in V,\ t \in \mathbb{R}_+. \tag{17}$$

Then, the variational formulation of Problem $\mathcal{P}$ is the following.

Problem $\mathcal{P}_V$. *Find a displacement field $\boldsymbol{u} : \mathbb{R}_+ \to U$ and a stress field $\boldsymbol{\sigma} : \mathbb{R}_+ \to Q$ such that, for all $t \in \mathbb{R}_+$,*

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0), \tag{18}$$

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + (P\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \tag{19}$$
$$+(\mathcal{B}\boldsymbol{u}(t), v_\nu^+ - u_\nu^+(t))_{L^2(\Gamma_3)} \ge (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \forall\, \boldsymbol{v} \in U.$$

Note that (18) is a consequence of (1) and (7), while (19) can be easily obtained by using integrations by parts, (2)–(6) and notation (14)–(17).

3 Existence and uniqueness

The unique solvability of Problem $\mathcal{P}_V$ is given by the following result.

Theorem 1 *Assume that* (8)–(13) *hold. Then Problem $\mathcal{P}_V$ has a unique solution, which satisfies $\boldsymbol{u} \in C(\mathbb{R}_+; U)$ and $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q)$.*

Proof. The proof is carried out in several steps which we describe below.

(i) We use the Banach fixed point argument to prove that for each function $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ there exists a unique function $\mathcal{S}\boldsymbol{u} \in C(\mathbb{R}_+; Q)$ such that

$$\mathcal{S}\boldsymbol{u}(t) = \int_0^t \mathcal{G}(\mathcal{S}\boldsymbol{u}(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) \quad \forall\, t \in \mathbb{R}_+.$$

(ii) Next, we note that $(\boldsymbol{u},\boldsymbol{\sigma})$ is a solution of Problem $\mathcal{P}_V$ iff

$$\boldsymbol{\sigma}(t)=\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))+\mathcal{S}\boldsymbol{u}(t)\quad \forall\, t\in\mathbb{R}_+, \tag{20}$$

$$\begin{aligned}&(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q+(\mathcal{S}\boldsymbol{u}(t),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q\\ &\quad+(\mathcal{B}\boldsymbol{u}(t),v_\nu^+-u_\nu^+(t))_{L^2(\Gamma_3)}+(P\boldsymbol{u}(t),\boldsymbol{v}-\boldsymbol{u}(t))_V\\ &\quad\geq(\boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{u}(t))_V\quad \forall\,\boldsymbol{v}\in U,\ \forall\, t\in\mathbb{R}_+.\end{aligned} \tag{21}$$

(iii) Let $A:V\to V$ and $\varphi:Q\times L^2(\Gamma_3)\times V\to\mathbb{R}$ be defined by equalities

$$\begin{aligned}&(A\boldsymbol{u},\boldsymbol{v})_V=(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q+(P\boldsymbol{u},\boldsymbol{v})_V,\\ &\varphi(x,\boldsymbol{v})=(\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q+(\xi,v_\nu^+)_{L^2(\Gamma_3)}\end{aligned}$$

for all $\boldsymbol{u},\ \boldsymbol{v}\in V$, $x=(\boldsymbol{\sigma},\xi)\in Q\times L^2(\Gamma_3)$. We prove that $A:V\to V$ is a strongly monotone and Lipschitz continuous operator and there exists $\beta\geq 0$ such that

$$\begin{aligned}&\varphi(x_1,\boldsymbol{u}_2)-\varphi(x_1,\boldsymbol{u}_1)+\varphi(x_2,\boldsymbol{u}_1)-\varphi(x_2,\boldsymbol{u}_2)\\ &\quad\leq\beta\,\|x_1-x_2\|_{Q\times L^2(\Gamma_3)}\,\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V\quad\forall\, x_1,\,x_2\in Q\times L^2(\Gamma_3),\ \boldsymbol{u}_1,\,\boldsymbol{u}_2\in V.\end{aligned}$$

Moreover, we prove that for every $n\in\mathbb{N}$ there exists $s_n>0$ such that

$$\begin{aligned}&\|\mathcal{S}\boldsymbol{u}_1(t)-\mathcal{S}\boldsymbol{u}_2(t)\|_Q+\|\mathcal{B}\boldsymbol{u}_1(t)-\mathcal{B}\boldsymbol{u}_2(t)\|_{L^2(\Gamma_3)}\\ &\quad\leq s_n\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V\,ds\quad\forall\,\boldsymbol{u}_1,\,\boldsymbol{u}_2\in C(\mathbb{R}_+;V),\,\forall\, t\in[0,n].\end{aligned}$$

These properties allow to use Theorem 2 in [3]. In this way we prove the existence of a unique function $\boldsymbol{u}\in C(\mathbb{R}_+;U)$ which satisfies the history-dependent variational inequality (21), for all $t\in\mathbb{R}_+$.

(iv) Let $\boldsymbol{\sigma}$ be the function given by (20); then, the couple $(\boldsymbol{u},\boldsymbol{\sigma})$ satisfies (20)–(21) for all $t\in\mathbb{R}_+$ and, moreover, it has the regularity $\boldsymbol{u}\in C(\mathbb{R}_+;U)$, $\boldsymbol{\sigma}\in C(\mathbb{R}_+;Q)$. This concludes the existence part in Theorem 1. The uniqueness part follows from the uniqueness of the solution of the inequality (21), guaranteed by Theorem 2 in [3]. □

Acknowledgement

The work of the first two authors was supported within the Sectoral Operational Programme for Human Resources Development 2007-2013, co-financed by the European Social Fund, under the projects POSDRU/107/1.5/S/76841 and POSDRU/88/1.5/S/60185, respectively, entitled *Modern Doctoral Studies: Internationalization and Interdisciplinarity*, at University Babeş-Bolyai, Cluj-Napoca, Romania.

References

[1] J. Jarušek and M. Sofonea. On the solvability of dynamic elastic-visco-plastic contact problems. *Zeitschrift für Angewandte Matematik und Mechanik* (*ZAMM*). 88:3–22, 2008.

[2] M. Shillor, M. Sofonea, J.J. Telega. *Models and Analysis of Quasistatic Contact*. Lect. Notes Phys. Springer, Berlin Heidelberg, 2004.

[3] M. Sofonea and A. Matei. History-dependent quasivariational inequalities arising in Contact Mechanics. *European Journal of Applied Mathematics*. 22:471–491, 2011.