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MATHEMATICAL MODELLING OF TWO-SPECIES RELATIVISTIC FLUIDS*

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Abstract

An interface-capturing method is used to deduce equations governing fluid motion in a relativistic two-species flow. These kind of methods combine simple fluid flow equations, which are the balance law for particle number and energy-momentum tensor conservation equation for global fluid, the balance laws for particle number density of each species, with extra equations. Since equations of multi-species relativistic fluid are not closed assigning laws of the state of each species, closure equations are necessarily introduced. A model based on the axiom of existence of a temperature and an entropy for the global fluid, which verify an equation analogous to that holding in the case of a simple fluid, is formulated. Weak discontinuities compatible with such kind of mixture are also studied.

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1 Introduction

A very large variety of scientific and technological problems are of a two-species flow. Flows relevant in chemistry, petrolchemical industry, biology, geophysics, nuclear processes or propulsion technology, for example, are often considered as two-species flows.

There are several approaches to two-fluid flow processes [1, 2, 12, 23, 27, 36, 37, 39, 40, 41, 42, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62]. In one of these approaches, for example, the governing equations are directly formulated according to conservation principles and treating a two-fluid mixture as a set of interacting subregions of individual fluids. Another of these approaches derives the governing equations from structural continuum fluid models and the mathematical model is expressed in terms of balance equations by treating a two-fluid mixture as one or two averaged continua.

In recent years the dynamics of two-species relativistic fluids plays an important role in areas of astrophysics, high energy particle beams, high energy nuclear collisions and free-electron laser technology. So two-fluid flows have received increasing attention and they are still the subject of numerous investigations [3, 4, 5, 6, 7, 8, 9, 10, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 25, 26, 28, 29, 30, 31, 32, 33, 34, 35, 38, 43, 44, 45, 47, 48, 63]. For some of these relativistic flows the hyperbolic aspects of the phenomenon play a crucial role.

This is the motivation of our interest in a system of governing equations for a two-species fluid, based on the physical balance of particle number and energy-momentum tensor, taking into account the interface exchange. This modeling approach is based on a relativistic two-species flow model, in which a separate fluid is interacting with the other one by interfacial transfer.

In this paper, a capturing method, which is a relativistic extension of the method introduced by Wackers and Koren [61] for classical compressible two-fluid flow, is used.

In order to obtain a closed governing system, it is necessary to examine the following problem. If we consider a simple relativistic fluid, the conservation equations for the particle number and for the energy tensor are completed by the fluid state law that, for example, allows to express the pressure in terms of the particle number and the internal energy density. Whereas, the multi-species conservation equations can not be completed by giving state laws to each species. Therefore, it is necessary to insert further closure equations.

The purpose of this paper, following Lagoutiére [40], Dellacherie and Rency [23], is to consider same closure laws based on thermodynamic considerations ensuring the hyperbolicity of the system and consists in bringing to the case of two species and two pressures of the investigation done in paper [30], in which we consider the case of a single species and two-phases with single pressure.

Moreover, the weak discontinuities, propagating in this relativistic mixture, are examined.

Finally, a special case in which each fluid-species is supposed to satisfy the equation of state of a perfect gas is considered.

In what follows, the space-time is a four dimensional manifold V_4, whose normal hyperbolic metric ds^2, with signature $+,-,-,-$, is expressed in local coordinates in the usual form $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$; the metric tensor is assumed to be of class C^1 and piecewise C^2; the 4-velocity is defined as $u^\mu = dx^\mu/ds$, which implies its unitary character $u^\mu u_\mu = 1$; ∇_μ is the covariant differentiation operator with respect to the given metric; the units are such that the velocity of light is unitary, *i.e.* $c = 1$.

2 Simple relativistic fluid

The standard equations for a simple relativistic fluid [11, 46] are the particle number conservation

$$\nabla_\alpha(ru^\alpha) = 0 \, , \tag{1}$$

and the total energy-momentum conservation

$$\nabla_\alpha T^{\alpha\beta} = 0 \, , \tag{2}$$

where u^α is the 4-velocity, r is the particle number density and the stress-energy tensor is given by

$$T^{\alpha\beta} = rfu^\alpha u^\beta - pg^{\alpha\beta} \, ; \tag{3}$$

here f is the relativistic specific enthalpy

$$f = 1 + h = 1 + \varepsilon + \frac{p}{r} = \frac{\rho + p}{r} \, , \tag{4}$$

where $h = \varepsilon + p/r$ is the "classical" specific enthalpy, ε the specific internal energy, p the pressure and $\rho = r(1+\varepsilon)$ the energy density.

Moreover, the spatial projection and the projection along u^α of equation (2) give, respectively,

$$r f u^\alpha \nabla_\alpha u^\beta - \gamma^{\alpha\beta} \partial_\alpha p = 0\,, \tag{5}$$

$$u^\alpha \partial_\alpha \rho + (\rho + p)\vartheta = 0\,, \tag{6}$$

where

$$\vartheta = \nabla_\alpha u^\alpha\,, \tag{7}$$

and $\gamma^{\alpha\beta} = g^{\alpha\beta} - u^\alpha u^\beta$ is the projection tensor onto the 3-space orthogonal to u^α, *i.e.* the rest space of an observer moving with 4-velocity u^α.

The five equations system (1), (5) and (6) in the six unknown variables u^α, r, ε, p is completed by an equation of state. For example, pressure p can be expressed in terms of particle number density r and specific internal energy ε:

$$p = p(r, \varepsilon)\,. \tag{8}$$

Moreover, we state the general hypothesis that there exist two functions $T(r, \varepsilon)$ and $S(r, \varepsilon)$ such that

$$T dS = d\varepsilon + p d\frac{1}{r}\,. \tag{9}$$

More precisely, T is the temperature and S is the entropy of the fluid. This last equation, well-known as the Gibbs' equation, resumes the first and the second principle of thermodynamics for a system subject to a reversible transformation.

Using equations (6) and (9), it is possible to deduce that

$$\nabla_\alpha (r u^\alpha) = 0 \;\Leftrightarrow\; u^\alpha \partial_\alpha S = 0\,. \tag{10}$$

3 The central hypothesis for a fluid mixture

Let us consider a two-species fluid mixture, flowing with a unique velocity. Each fluid species has its own particle number density, r_k, its specific internal energy, ε_k, and its pressure, p_k, that can be expressed in terms of r_k and ε_k:

$$p_k = p_k(r_k, \varepsilon_k)\,, \qquad (k = 1, 2)\,. \tag{11}$$

Also, let us suppose that each species k admits a thermodynamic temperature, $T_k = T_k(r_k, \varepsilon_k)$, and an entropy density (strictly convex), $S_k = S_k(r_k, \varepsilon_k)$, which satisfy the Gibbs' relation:

$$T_k dS_k = d\varepsilon_k + p_k d\frac{1}{r_k}\,, \qquad (k = 1, 2)\,. \tag{12}$$

Now, we introduce another field variable, the mass fraction Y of fluid 1, which is defined by

$$Y = \frac{r_1}{r}\,, \tag{13}$$

where

$$r = r_1 + r_2 \tag{14}$$

is the particle number density for the global fluid.

Let ε be the specific internal energy of the fluid mixture. Since it is an extensive variable, we have

$$\varepsilon = Y_1\varepsilon_1 + Y_2\varepsilon_2\,, \tag{15}$$

with

$$Y_1 = Y\,, Y_2 = 1 - Y\,, \tag{16}$$

and we suppose that the equations (5) and (6), for a simple relativistic fluid flow, are also valid for the two-species fluid model.

Using the partial densities r_k $(k = 1, 2)$, the balance laws for particle number density of each species write as

$$\nabla_\alpha(r_k u^\alpha) = 0\,, \qquad (k = 1, 2)\,. \tag{17}$$

Let us observe that, together with (14), equations (17) yields the balance equation for the bulk particle number density (1).

Equation $(17)_1$ can also be written as

$$\nabla_\alpha(Y r u^\alpha) = 0\,, \tag{18}$$

which, taking into account (1), gives the following equation

$$u^\alpha \partial_\alpha Y = 0\,. \tag{19}$$

Thus, searching for regular solutions, the mathematical study of the model can be performed in terms of a set of 10 independent field variables,

u^α, r, ε_1, ε_2, p, p_1, p_2 and Y. The governing system (5), (6), (11), (13) and (19) is a set of 8 equations in 10 unknown variables. Thus, two further equations are needed in order to close the system.

According to Lagoutiére [40], a criterion for choosing this closure relations is to suppose that there exists a priori the temperature T of the mixture, function of all the thermodynamic variables of the problem, such that

$$TDS = D\varepsilon + pD\frac{1}{r}\,, \tag{20}$$

where p and S, given by

$$S = YS_1 + (1-Y)S_2\,, \tag{21}$$

are the pressure and the entropy of the whole fluid and $D = u^\alpha\partial_\alpha$. This hypothesis is called "central hypothesis".

Multiplying (12) by Y_k and summing over k, using (19) and the mixture law (13), the following equation is obtained

$$Y_1T_1DS_1 + Y_2T_2DS_2 = D\varepsilon + (p_1+p_2)D\frac{1}{r}\,, \tag{22}$$

and, for the central hypothesis (20), equation (22) gives the compatibility conditions

$$Y_1T_1DS_1 + Y_2T_2DS_2 - TDS = (p_1+p_2-p)D\frac{1}{r}\,. \tag{23}$$

Now, we assume an additional hypothesis: the closure relation must be verify the vanishing of both sides of (23). So, it gets

$$Y_1T_1DS_1 + Y_2T_2DS_2 - TDS = 0\,, \tag{24}$$
$$p_1 + p_2 - p = 0\,. \tag{25}$$

It is noted that (25) implies that the pressure is closed

$$p = p_1 + p_2\,, \tag{26}$$

that is the well-known Dalton's law.

The last closure relation must be satisfy eq. (24). Hence, it is possible impose one of the following closures

$$\begin{aligned}
&\frac{DS_1}{S_1} = \frac{DS_2}{S_2}\,,\\
&T_1 DS_1 = T_2 DS_2\,,\\
&DS_1 = DS_2\,,\\
&Y_1 DS_1 = Y_2 DS_2\,,\\
&T_1 = T_2\,.
\end{aligned} \tag{27}$$

Each closure relation defined above, by virtue of equation (24), allows to define a temperature that verifies (20); respectively, we have

$$\begin{aligned}
&T = \frac{1}{S}(Y_1 S_1 T_1 + Y_2 S_2 T_2)\,,\\
&\frac{1}{T} = \frac{Y_1}{T_1} + \frac{Y_2}{T_2}\,,\\
&T = Y_1 T_1 + Y_2 T_2\,,\\
&T = \frac{1}{2}(T_1 + T_2)\,,\\
&T = T_1 = T_2\,.
\end{aligned} \tag{28}$$

Now, we consider system given by the following equations

$$\begin{cases}
\nabla_\alpha (r u^\alpha) = 0\,,\\
r f u^\alpha \nabla_\alpha u^\beta - \gamma^{\alpha\beta}\partial_\alpha p = 0\,,\\
u^\alpha \partial_\alpha \varepsilon + p u^\alpha \partial_\alpha \dfrac{1}{r}\,,\\
u^\alpha \partial_\alpha Y = 0\,,\\
\varepsilon = Y_1 \varepsilon_1 + Y_2 \varepsilon_2\,,\\
Y_1 + Y_2 = 1 \;\Leftrightarrow\; r = r_1 + r_2\,,\\
r_k = Y_k r\,,\\
p_k = p_k(r_k, \varepsilon_k)\,, \qquad (k = 1, 2)\,,
\end{cases} \tag{29}$$

which we need add the two closure relations $p = p_1 + p_2$ and one of (27).

For each regular solution, the system (29), together with relations (26) and (27), is equivalent to system which is obtained replacing expression of energy $(29)_3$ with relation

$$u^\alpha \partial_\alpha S = 0\,. \tag{30}$$

In fact, by virtue of (20) and (21), we have

$$TDS = D\varepsilon + pD\frac{1}{r} = 0\,. \tag{31}$$

Moreover, for every regular solution of the above system of evolution, we deduce

$$(27)_1 \text{ or } (27)_2 \text{ or } (27)_3 \text{ or } (27)_4 \text{ or } (27)_5 \;\Leftrightarrow\; DS_k = 0\,. \tag{32}$$

Ultimately, the complete system of governing differential equations may be written in terms of variables u^α, r_1, r_2, S_1, S_2, Y as

$$\begin{cases} rfu^\alpha \nabla_\alpha u^\beta - \gamma^{\alpha\beta}\partial_\alpha p = 0\,, \\ \nabla_\alpha (r_1 u^\alpha) = 0\,, \\ \nabla_\alpha (r_2 u^\alpha) = 0\,, \\ u^\alpha \partial_\alpha S_1 = 0\,, \\ u^\alpha \partial_\alpha S_2 = 0\,, \\ u^\alpha \partial_\alpha Y = 0\,, \end{cases} \tag{33}$$

where

$$p = p_1(r_1, S_1) + p_2(r_2, S_2)\,. \tag{34}$$

4 Weak discontinuities

In a domain Ω of space-time V_4, let Σ be a regular hypersurface, not generated by the flow lines, being $\varphi(x^\alpha) = 0$ its local equation. We set $L_\alpha = \partial_\alpha \varphi$. As it will be clear below, the hypersurface Σ is space-like, *i.e.* $L_\alpha L^\alpha < 0$. In the following, N_α will denote the normalized vector

$$N_\alpha = \frac{L_\alpha}{\sqrt{-L^\beta L_\beta}}\,, \qquad N^\alpha N_\alpha = -1\,. \tag{35}$$

We are interested in a particular class of solutions of system (33) namely, weak discontinuity waves Σ across which the field variables u^α, r_1, r_2, S_1, S_2 and Y are continuous, but, conversely, jump discontinuities may occur in their normal derivatives (at least one of the partial derivative suffers a jump across Σ). In this case, if Q denotes any of these fields, then there exists [11, 46] the distribution δQ, with support Σ, such that

$$\bar{\delta}[\nabla_\alpha Q] = N_\alpha \delta Q\,, \tag{36}$$

where $\bar{\delta}$ is the Dirac measure defined by φ with Σ as support, square brackets denote the discontinuity, δ being an operator of infinitesimal discontinuity; δ behaves like a derivative insofar as algebraic manipulations are concerned.

By virtue of (36), from system (33) we obtain the following linear homogeneous system in the distribution δu^α, δr_1, δr_2, δS_1, δS_2 and δY:

$$\begin{cases} rfL\delta u^\beta - \gamma^{\alpha\beta} N_\alpha \left[\left(\dfrac{\partial p_1}{\partial r_1}\right)_{S_1} \delta r_1 + \left(\dfrac{\partial p_2}{\partial r_2}\right)_{S_2} \delta r_2 \right. \\ \qquad\qquad \left. + \left(\dfrac{\partial p_1}{\partial S_1}\right)_{r_1} \delta S_1 + \left(\dfrac{\partial p_2}{\partial S_2}\right)_{r_2} \delta S_2 \right] = 0\,, \\ L\delta r_1 + r_1 N_\alpha \delta u^\alpha = 0\,, \\ L\delta r_2 + r_2 N_\alpha \delta u^\alpha = 0\,, \\ L\delta S_1 = 0\,, \\ L\delta S_2 = 0\,, \\ L\delta Y = 0\,, \end{cases} \tag{37}$$

where $L = u^\alpha N_\alpha$.

Moreover, from the unitary character of u^α we get the relation

$$u_\alpha \delta u^\alpha = 0\,. \tag{38}$$

Now, we focus on the normal speeds of propagation of the various waves with respect to an observer moving with the mixture velocity u^α. The normal speed λ_Σ of propagation of the wave front Σ, described by a time-like world line having tangent vector field u^α, that is with respect to the time direction u^α, is given by [11, 46]

$$\lambda_\Sigma^2 = \frac{L^2}{\ell^2}\,, \quad \ell^2 = 1 + L^2\,. \tag{39}$$

The local causality condition, *i.e.* the requirement that the characteristic hypersurface Σ has to be time-like or null (or, equivalently, that the normal N_α has to be space-like or null, that is $g^{\alpha\beta}N_\alpha N_\beta \leq 0$), is equivalent to the condition $0 \leq \lambda_\Sigma^2 \leq 1$.

From the above equations (37), we obtain as first the solution $L = 0$, which represents a wave moving with the mixture.

For the corresponding discontinuities, we find

$$\begin{aligned} &N_\alpha \delta u^\alpha = 0\,,\\ &\delta p = \left[\left(\frac{\partial p_1}{\partial r_1}\right)_{S_1}\delta r_1 + \left(\frac{\partial p_2}{\partial r_2}\right)_{S_2}\delta r_2 + \left(\frac{\partial p_1}{\partial S_1}\right)_{r_1}\delta S_1\right.\\ &\qquad\qquad\qquad\left. + \left(\frac{\partial p_2}{\partial S_2}\right)_{r_2}\delta S_2\right] = 0\,. \end{aligned} \tag{40}$$

From system (37), we see that the coefficients characterizing the discontinuities have 6 degrees of freedom and this correspond to 6 independent eigenvectors relevant to $L = 0$ in the space of the field variables.

From now on we suppose $L \neq 0$. Equations $(37)_4$, $(37)_5$ and $(37)_6$ give, respectively, $\delta S_1 = \delta S_2 = \delta Y = 0$, whereas equation $(37)_1$, multiplied by N_β, gives us:

$$rfLN_\beta \delta u^\beta + \ell^2\left[\left(\frac{\partial p_1}{\partial r_1}\right)_{S_1}\delta r_1 + \left(\frac{\partial p_2}{\partial r_2}\right)_{S_2}\delta r_2\right] = 0\,. \tag{41}$$

Writing

$$p_k = p_k(r_k, S_k) = p_k[\rho_k(r_k, S_k), S_k] \tag{42}$$

and taking into account that

$$\begin{cases} \left(\dfrac{\partial p_k}{\partial r_k}\right)_{S_k} = \left(\dfrac{\partial p_k}{\partial \rho_k}\right)_{S_k}\left(\dfrac{\partial \rho_k}{\partial r_k}\right)_{S_k}\,,\\[2ex] \left(\dfrac{\partial \rho_k}{\partial r_k}\right)_{S_k} = f_k\,, \end{cases} \tag{43}$$

equation (41) gives

$$rfLN_\alpha\delta u^\alpha + \ell^2(f_1\lambda_1^2\delta r_1 + f_2\lambda_2^2\delta r_2) = 0\,, \tag{44}$$

where we denote

$$\lambda_1^2 = \left(\frac{\partial p_1}{\partial \rho_1}\right)_{S_1}\,, \qquad \lambda_2^2 = \left(\frac{\partial p_2}{\partial \rho_2}\right)_{S_2}\,. \tag{45}$$

Consequently, (44), $(37)_2$ and $(37)_3$ represent a linear homogeneous system in the 3 scalar distributions $N_\alpha\delta u^\alpha$, δr_1 and δr_2, which may be different from zero only if the determinant of the coefficient vanishes.

Therefore, we obtain the equation

$$\mathcal{H} = fL^2 - \omega\ell^2 = 0\,, \tag{46}$$

where

$$\omega = \sum_{k=1}^{2} Y_k f_k \left(\frac{\partial p_k}{\partial \rho_k}\right)_{S_k} = Y_1 f_1 \lambda_1^2 + Y_2 f_2 \lambda_2^2\,. \tag{47}$$

Equation (46) corresponds to two hydrodynamical waves propagating in such a two fluid system with speeds of propagation, λ_Σ, given by

$$rf\lambda_\Sigma^2 = r_1 f_1 \lambda_1^2 + r_2 f_2 \lambda_2^2\,, \tag{48}$$

where λ_1 and λ_2 represent the speeds of propagation of hydrodynamical waves in each species.

Now, we assume that each species satisfies the equation of state of perfect gases:

$$p_k = (\gamma_k - 1) r_k \varepsilon_k\,, \qquad k = 1, 2\,, \tag{49}$$

where

$$\gamma_k = \frac{c_{p_k}}{c_{V_k}}\,, \qquad k = 1, 2\,, \tag{50}$$

is the ratio between specific heats at constant pressure, c_{p_k}, and volume, c_{V_k}, of the k-th species.

So, we have

$$\lambda_k^2 = \frac{\gamma_k p_k}{r_k f_k}\,. \tag{51}$$

Therefore, from equation (48), the following expression for the velocity of propagation is obtained

$$\lambda_\Sigma^2 = \frac{1}{rf}(\gamma_1 p_1 + \gamma_2 p_2)\,, \tag{52}$$

which coincides with expression (27) found in [31].

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References

[1] R. Abgrall, R. Saurel. Discrete equations for physical and numerical compressible multiphase mixture. *J. Comput. Phys.* 186: 361-396, 2003.

[2] G. Allaire, S. Clerc, S. Kokh. A five-equation model for the simulation of interfaces between compressible fluids. *J. Comput. Phys.* 181: 577-616, 2002.

[3] N. Anderson. Gravitational waves from instabilities in relativistic stars. *Class. Quantum Grav.* 20: R105, 2003.

[4] N. Anderson, G. L. Comer. On the dynamics of superfluid neutron star cores. *Mon. Not. R. Astron. Soc.* 328: 1129-1143, 2001.

[5] N. Anderson, G. L. Comer. A flux-conservative formalism for convective and dissipative multi-fluid systems, with application to Newtonian superfluid neutron stars. *Class. Quantum Grav.* 23: 5505-5529, 2006.

[6] N. Anderson, G. L. Comer. Relativistic fluid dynamics: physics for many different scales. *Living Reviews in Relativity* 10: 1, 2007.

[7] N. Anderson, G. L. Comer, D. Langlois. Oscillations of general relativistic superfluid neutron stars. *Phys. Rev. D* 66: 104002, 2002.

[8] N. Anderson, G. L. Comer, K. Glampedakis. How viscous is a superfluid neutron star core? *Nucl. Phys. A* 763: 212-229, 2005.

[9] N. Anderson, T. Sidery, G. L. Comer. Mutual friction in superfluid neutron stars. *Mon. Not. R. Astron. Soc.* 368: 162-170, 2006.

[10] N. Anderson, T. Sidery, G. L. Comer. Waves and instabilities in dissipative rotating superfluid neutron stars. *Mon. Not. R. Astron. Soc.* 385: 335-348, 2008.

[11] A. M. Anile. *Relativistic fluids and magneto-fluids.* Cambridge University Press, Cambridge, 1989.

[12] M. Baer, J. Nunziato. A two-phase mixture theory for the deflagration-to-detonation transition (DDT) in reactive granular materials. *Int. J. Multiphase Flow* 12: 861-889, 1986.

[13] S. S. Bayin. Anisotropic fluid spheres in general relativity. *Phys. Rev. D* 26: 1262-1274, 1982.

[14] B. Carter, N. Chamel. Covariant analysis of Newtonian multifluid models for neutron stars: I Milne-Cartan structure and variational formulation. *Int. J. Mod. Phys. D* 13: 291-325, 2004.

[15] B. Carter, N. Chamel. Covariant analysis of Newtonian multifluid models for neutron stars: II Stress - energy tensors and virial theorems. *Int. J. Mod. Phys. D* 14: 717-748, 2005.

[16] B. Carter, N. Chamel. Covariant analysis of Newtonian multi-fluid models for neutron stars: III Transvective, voscous and superfluid drag dissipation. *Int. J. Mod. Phys. D* 14: 749-774, 2005.

[17] B. Carter, D. Langlois. Equation of state for cool relativistic superfluids. *Class. Quantum Grav.* 51: 5855-5864, 1995.

[18] B. Carter, D. Langlois. Relativistic models for superconducting-superfluid mixtures. *Nucl. Phys. B* 531: 478-504, 1998.

[19] M. Cissoko. Wavefronts in a relativistic cosmic two-component fluid. *Gen. rel. Grav.* 30: 521-534, 1998.

[20] M. Cissoko. Wave fronts in a mixture of two relativistic perfect fluids flowing with two distinct four-velocities. *Phys. Rev. D* 63: 083516, 2001.

[21] G. L. Comer, D. Langlois. Hamiltonian formulation for multi-constituent relativistic perfect fluids. *Class. Quantum Grav.* 10: 2317, 1993.

[22] G. L. Comer, D. Langlois. Hamiltonian formulation for relativistic superfluids. *Class. Quantum Grav.* 11: 709, 1994.

[23] S. Dellacherie, N. Rency. Relations de fermeture pour le systm des équations d'Euler multi-espices. Rapport CEA-R-5999, 2001.

[24] K. Dunn. Two-fluid cosmological models in Gödel-type spacetimes. *Gen. Rel. Grav.* 21: 137-147, 1989.

[25] R. I. Epstein. Acoustic properties of neutron stars. *ApJ* 333: 880-894, 1988.

[26] J. J. Ferrando, J. A. Morales, M. Portilla. Two-perfect fluid interpretation of an energy tensor. *Gen. rel. Grav.* 22: 1021-1032, 1990.

[27] S. L. Gavrilyuk, R. Saurel. Rankine-Hugoniot relations for shocks in heterogeneous mixtures. *J. Fluid Mech.* 575: 495-507, 2007.

[28] S. Giambò. Mathematical Modeling of Two-Component Relativistic Fluids. In *"WASCOM 2009"15th Conference on Waves and Stability in Continuous Media.* Singapore World Scientific Publishing, 2010.

[29] S. Giambò, S. Giambò. A seven equation model for relativistic two fluid flows-I. *Romai J.* 5: 59-70, 2010.

[30] S. Giambò, S. Giambò. A seven equation model for relativistic two fluid flows-II. *Romai J.* 6: 95-105, 2010.

[31] S. Giambò, S. Giambò. A model for relativistic fluid mixture with different pressure laws. *EPL* 90: 54004, 2010.

[32] S. Giambò, S. Giambò, G. Muscianisi. On the equation of two-component relativistic gas flow. In *"WASCOM 2009" 15th Conference on Waves and Stability in Continuous Media.* Singapore World Scientific Publishing, 2010.

[33] K. Glampedakis, N. Anderson. Ekman layer damping of r modes revisited. *Mon. Not. R. Astron. Soc.* 371: 1311-1321, 2006.

[34] K. Glampedakis, N. Anderson, D. J. Jones. Stability of precessing superfluid neutron stars. *Phys. Rev. Lett.* 100: 081101, 2008.

[35] E. N. Glass, J. P. Krisch. Two-fluid atmosphere for relativistic stars. *Class. Quantum Grav.* 16: 1175-1184, 1999.

[36] S. Jiang, G. Ni. A second-order y-model BGK scheme for multimaterial compressible flows. *Appl. Numer. Math.* 57: 597-608, 2007.

[37] S. Karmi. Multicomponent flow calculations by a consistent primitive algorithm. *J. Comput. Phys.* 112: 31-43, 1993.

[38] J. P. Krisch, L. L. Smalley. Two fluid acoustic modes and inhomogeneous cosmologies. *Class. Quantum Grav.* 10: 2615-2623, 1993.

[39] F. Lagoutiére, B. Després. Étude mathématique de modeles de mélange. Rapport. CEA, 1999.

[40] F. Lagoutiére. *Modélisation mathématique et résolution numérique de problémes de fluides compressibles á plusieurs constituants.* Université de Paris VI, 2000.

[41] B. Larrouturou. How to preserve mass fractions positivity when computing compressible multi-component flows. *J. Comput. Phys.* 95: 59-84, 1991.

[42] B. Larrouturou, L. Fezoui. On the equations of multi-component perfect or real inviscid flow. In *Nonlinear Hyperbolic Problems.* Springer-Verlag, 1988.

[43] U. Lee, S. Yoshida. r-modes of neutron stars with superfluid cores. *ApJ.* 586: 403, 2003.

[44] P. S. Letelier. Anisotropic fluids with two-perfect fluid components. *Phys. Rev. D* 22: 807-813, 1980.

[45] P. S. Letelier, P. S. C. Alencar. Anisotropic fluids with multifluid components. *Phys. Rev. D* 34: 343-351, 1986.

[46] A. Lichnerowicz. *Relativistic fluid dynamics.* Cremonese, Roma, 1971.

[47] L.-M. Lin, G. L. Comer, N. Anderson. Oscillations of general relativistic multifluid/multilayer compact stars. *Phys. Rev. D* 78: 083008, 2008.

[48] G. Mendell. Superfluid hydrodynamics in rotating neutron stars. I - Nondissipative equations. II - Dissipative effects. *ApJ.* 380:515-540, 1991.

[49] A. Murrone, H. Guillard. A five equation reduced model for compressible two phase flow problem. *J. Comput. Phys.* 202: 664-698, 2005.

[50] R. R. Nourgaliev, T. N. Dinh, T. G. Theofanous. Adaptive characteristics-based matching for compressible multifluid dynamics. *J. Comput. Phys.* 213: 500-529, 2006.

[51] R. Prix. Variational description of multifluid hydrodynamics: Uncharged fluids. *Phys. Rev. D* 69: 043001, 2004.

[52] S. Qamar, G. Warnecke. Simulation of multicomponent flows using high order central schemes. *Appl. Numer. Math.* 50: 183-201, 2004.

[53] S. Qamar, G. Warnecke. Solving population balance equation for two-component aggregation by a finite volume scheme. *Chem. Eng. Sci.* 62: 513-532, 2007.

[54] V. Ransom, D. Hicks. Hyperbolic two-pressure models for two-phase flow. *J. Comput. Phys.* 53: 124-151, 1984.

[55] E. Romenski, E. F. Toro. Compressible two-phase flow models: two-pressure models and numerical methods. *Comp. Fluid Dyn. Journ.* 13: 403-416, 2004.

[56] E. Romenski, E. F. Toro. Hyperbolicity and one-dimensional waves in compressible two-phase flow models. *Shock Waves* 13: 473-487, 2004.

[57] R. Saurel, R. Abgrall. A simple method for compressible multifluid flows. *SIAM J. Comput. Sci.* 21: 115-145, 1999.

[58] R. Saurel, R. Abgrall. A multiphase Godunov method for compressible multifluid on multiphase flows. *J. Comput. Phys.* 150: 425-467, 1999.

[59] K. M. Shyue. A wave-propagation based volume tracking method for compressible multicomponent flow in two space dimensions. *J. Comput. Phys.* 215: 219-244, 2006.

[60] H. B. Stewart, B. Wendroff. Two-phase flow: models and methods. *J. Comput. Phys.* 56: 363-409, 1984.

[61] J. Wackers, B. Koren. Five-equation model for compressible two-fluid flow. Report MAS-EO414, 2004.

[62] D. Zeidan, E. Romenski, A. Slaouti, E. F. Toro. Numerical study of wave propagation in compressible two-phase flow. *Int. J. Numer. Math. Fluids* 54, 393-417, 2007.

[63] W. Zimdahl. Reacting fluids in the expanding univese: a new mechanism for entropy production. *Mon. Not. R. Astron. Soc.* 288: 665-673, 1997.