

A NOTE ON SOME BOUNDARY VALUE PROBLEMS FOR HIGHER ORDER DIFFERENTIAL INCLUSIONS*

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Abstract

We establish Filippov existence theorems for solutions of certain boundary value problems associated to some higher order differential inclusions.

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1 Introduction

This paper is concerned with differential inclusions of the form

$$\mathcal{D}x \in F(t, x), \tag{1.1}$$

where \mathcal{D} is a differential operator and $F(.,.) : [0, 1] \times \mathbf{R} \rightarrow \mathcal{P}(\mathbf{R})$ is a set-valued map.

In the last years we observe a remarkable amount of interest in the study of existence of solutions of several boundary value problems associated to

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problem (1.1). Most of these existence results are obtained using fixed point techniques and are based on an integral form of the right inverse to the operator \mathcal{D} . This means that for every f the unique solution y of the equation $\mathcal{D}y = f$ can be written in the form $y = \mathcal{R}f$, when the operator \mathcal{R} has nonnegative Green's function.

For a first order differential inclusion defined by a lipschitzian set-valued map with nonconvex values, Filippov's theorem ([7]) consists in proving the existence of a solution starting from a given almost solution. Moreover, the result provides an estimate between the starting almost solution and the solution of the differential inclusion.

The aim of this note is to show that Filippov's ideas can be suitably adapted in order to obtain the existence of solutions for the following problems

$$x^{(n)} - \lambda x \in F(t, x), \quad a.e. (I) \quad (1.2)$$

with boundary conditions of the form

$$x^{(i)}(0) - x^{(i)}(T) = \mu_i, \quad i = 0, 1, \dots, n-1, \quad (1.3)$$

and

$$(p(t)x'(t))' \in F(t, x(t)) \quad a.e. (I), \quad (1.4)$$

with boundary conditions of the form

$$\alpha x(0) - \beta \lim_{t \rightarrow 0^+} p(t)x'(t) = 0, \quad \gamma x(T) + \delta \lim_{t \rightarrow T^-} p(t)x'(t) = 0, \quad (1.5)$$

where $I = [0, T]$, $\lambda \in \mathbf{R}$, $\mu_i \in \mathbf{R}$, $i = 0, 1, \dots, n-1$, $F : I \times \mathbf{R} \rightarrow \mathcal{P}(\mathbf{R})$ is a set-valued map, $p(\cdot) : I \rightarrow (0, \infty)$ is a continuous function and $\alpha, \beta, \gamma, \delta$ are nonnegative reals with $\alpha\delta + \beta\gamma + \gamma\alpha \int_0^T \frac{dt}{p(t)} \neq 0$.

Existence results obtained using fixed point techniques for problem (1.2)-(1.3) may be found in [2,3] and for problem (1.4)-(1.5) may be found in [4,5,9,10]. The results in the present paper are improvements of previous existence theorems from our papers [3] respectively, [4].

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and in Section 3 we prove our main results.

2 Preliminaries

Let (X, d) be a metric space. We recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}, \quad d^*(A, B) = \sup\{d(a, B); a \in A\},$$

where $d(x, B) = \inf_{y \in B} d(x, y)$.

In what follows $C(I, \mathbf{R})$ is the Banach space of all continuous functions from I to \mathbf{R} with the norm $\|x(\cdot)\|_C = \sup_{t \in I} |x(t)|$, $AC^i(I, \mathbf{R})$ is the space of i -times differentiable functions $x : I \rightarrow \mathbf{R}$ whose i -th derivative $x^{(i)}(\cdot)$ is absolutely continuous, $AC_p^1(I, \mathbf{R})$ is the space of continuous functions $x : I \rightarrow \mathbf{R}$ such that $p(\cdot)x'(\cdot)$ is absolutely continuous and $L^1(I, \mathbf{R})$ is the Banach space of integrable functions $u(\cdot) : I \rightarrow \mathbf{R}$ endowed with the norm $\|u(\cdot)\|_1 = \int_0^T |u(t)|dt$.

A function $x(\cdot) \in AC^{n-1}(I, \mathbf{R})$ is called a solution of problem (1.2)-(1.3) if there exists a function $v(\cdot) \in L^1(I, \mathbf{R})$ with $v(t) \in F(t, x(t))$, a.e. (I) such that $x^{(n)}(t) - \lambda x(t) = v(t)$, a.e. (I) and $x(\cdot)$ satisfies conditions (1.3).

We consider the Green function $G(\cdot, \cdot) : I \times I \rightarrow \mathbf{R}$ associated to the periodic boundary problem

$$x^{(n)} - \lambda x = 0, \quad x^{(i)}(0) - x^{(i)}(T) = 0, \quad i = 0, 1, \dots, n - 1.$$

For the properties of $G(\cdot, \cdot)$ we refer to [2].

The next result is well known.

Lemma 2.1. ([2]) *If $v(\cdot) : [0, T] \rightarrow \mathbf{R}$ is an integrable function then the problem*

$$\begin{aligned} x^{(n)}(t) - \lambda x(t) &= v(t) \quad \text{a.e. } (I) \\ x^{(i)}(0) - x^{(i)}(T) &= \mu_i, \quad i = 0, 1, \dots, n - 1. \end{aligned}$$

has a unique solution $x(\cdot) \in AC^{n-1}(I, \mathbf{R})$ given by

$$x(t) = P_\mu(t) + \int_0^T G(t, s)v(s)ds,$$

where

$$P_\mu(t) = \sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t, 0)\mu_{n-1-i}. \tag{2.1}$$

A function $x(\cdot) \in AC_p^1(I, \mathbf{R})$ is called a solution of problem (1.4)-(1.5) if there exists a function $v(\cdot) \in L^1(I, \mathbf{R})$ with $v(t) \in F(t, x(t))$, a.e. (I) such that $(p(t)x'(t))' = v(t)$, a.e. (I) and conditions (1.5) are satisfied.

Lemma 2.2. ([9]) *If $v(\cdot) : [0, T] \rightarrow \mathbf{R}$ is an integrable function then the problem*

$$(p(t)x'(t))' = v(t) \quad \text{a.e. } (I),$$

$$\alpha x(0) - \beta \lim_{t \rightarrow 0^+} p(t)x'(t) = 0, \quad \gamma x(T) + \delta \lim_{t \rightarrow T^-} p(t)x'(t) = 0$$

has a unique solution $x(\cdot) \in AC_p^1(I, \mathbf{R})$ given by

$$x(t) = \int_0^T G_1(t, s)v(s)ds,$$

where

$$G_1(t, s) := \frac{1}{\rho} \begin{cases} (\beta + \alpha \int_0^s \frac{du}{p(u)})(\delta + \gamma \int_t^T \frac{du}{p(u)}) & \text{if } 0 \leq s < t \leq T \\ (\beta + \alpha \int_0^t \frac{du}{p(u)})(\delta + \gamma \int_s^T \frac{du}{p(u)}) & \text{if } 0 \leq t < s \leq T \end{cases}$$

and $\rho := \alpha\delta + \beta\gamma + \gamma\alpha \int_0^T \frac{dt}{p(t)} \neq 0$.

Finally, we recall a selection result which is a version of the celebrated Kuratowski and Ryll-Nardzewski selection theorem ([8]).

Lemma 2.3. ([1]) *Consider X a separable Banach space, B is the closed unit ball in X , $H : I \rightarrow \mathcal{P}(X)$ is a set-valued map with nonempty closed values and $g : I \rightarrow X, L : I \rightarrow \mathbf{R}_+$ are measurable functions. If*

$$H(t) \cap (g(t) + L(t)B) \neq \emptyset \quad \text{a.e.}(I),$$

then the set-valued map $t \rightarrow H(t) \cap (g(t) + L(t)B)$ has a measurable selection.

In the sequel we assume the following conditions on F .

Hypothesis 2.4. (i) $F(\cdot, \cdot) : I \times \mathbf{R} \rightarrow \mathcal{P}(\mathbf{R})$ has nonempty closed values and for every $x \in \mathbf{R}$ $F(\cdot, x)$ is measurable.

(ii) There exists $L(\cdot) \in L^1(I, \mathbf{R})$ such that for almost all $t \in I, F(t, \cdot)$ is $L(t)$ -Lipschitz in the sense that

$$d_H(F(t, x), F(t, y)) \leq L(t)|x - y| \quad \forall x, y \in \mathbf{R}.$$

3 The main results

We are now ready to prove the main result of this paper.

Denote $L_0 := \int_0^T L(s)ds$ and $M_0 := \sup_{t,s \in I} |G(t, s)|$.

Theorem 3.1. *Assume that Hypothesis 2.4 is satisfied and $M_0L_0 < 1$. Let $y(\cdot) \in AC^{n-1}(I, \mathbf{R})$ be such that there exists $q(\cdot) \in L^1(I, \mathbf{R})$ with $d(y^{(n)}(t) - \lambda y(t), F(t, y(t))) \leq q(t)$, a.e. (I). Denote $\tilde{\mu}_i = y^{(i)}(0) - y^{(i)}(T)$, $i = 0, 1, \dots, n - 1$.*

Then there exists $x(\cdot) : I \rightarrow \mathbf{R}$ a solution of (1.2)-(1.3) satisfying for all $t \in I$

$$|x(t) - y(t)| \leq \frac{1}{1 - M_0L_0} \sup_{t \in I} |P_\mu(t) - P_{\tilde{\mu}}(t)| + \frac{M_0}{1 - M_0L_0} \int_0^T q(t)dt,$$

where $P_\mu(t)$ is defined in (2.1).

Proof. The set-valued map $t \rightarrow F(t, y(t))$ is measurable with closed values and

$$F(t, y(t)) \cap \{y^{(n)}(t) - \lambda y(t) + q(t)[-1, 1]\} \neq \emptyset \quad \text{a.e. (I)}.$$

From Lemma 2.3 it follows that there exists a measurable selection $f_1(t) \in F(t, y(t))$ a.e. (I) such that

$$|f_1(t) - y^{(n)}(t) + \lambda y(t)| \leq q(t) \quad \text{a.e. (I)} \tag{3.2}$$

Define $x_1(t) = P_\mu(t) + \int_0^T G(t, s)f_1(s)ds$ and one has

$$|x_1(t) - y(t)| \leq \sup_{t \in I} |P_\mu(t) - P_{\tilde{\mu}}(t)| + M_0\|q\|_1.$$

We claim that it is enough to construct the sequences $x_n(\cdot) \in C(I, \mathbf{R})$, $f_n(\cdot) \in L^1(I, \mathbf{R})$, $n \geq 1$ with the following properties

$$x_n(t) = P_\mu(t) + \int_0^T G(t, s)f_n(s)ds, \quad t \in I, \tag{3.3}$$

$$f_n(t) \in F(t, x_{n-1}(t)) \quad \text{a.e. (I)}, \quad n \geq 1, \tag{3.4}$$

$$|f_{n+1}(t) - f_n(t)| \leq L(t)|x_n(t) - x_{n-1}(t)| \quad \text{a.e. (I)}, \quad n \geq 1. \tag{3.5}$$

If this construction is realized then from (3.2)-(3.5) we have for almost all $t \in I$

$$\begin{aligned}
 |x_{n+1}(t) - x_n(t)| &\leq \int_0^T |G(t, t_1)| \cdot |f_{n+1}(t_1) - f_n(t_1)| dt_1 \leq \\
 M_0 \int_0^T L(t_1) |x_n(t_1) - x_{n-1}(t_1)| dt_1 &\leq M_0 \int_0^T L(t_1) \int_0^T |G(t_1, t_2)| \cdot \\
 |f_n(t_2) - f_{n-1}(t_2)| dt_2 &\leq M_0^2 \int_0^T L(t_1) \int_0^T L(t_2) |x_{n-1}(t_2) - x_{n-2}(t_2)| dt_2 dt_1 \\
 &\leq M_0^n \int_0^T L(t_1) \int_0^T L(t_2) \dots \int_0^T L(t_n) |x_1(t_n) - y(t_n)| dt_n \dots dt_1 \leq \\
 &\leq (M_0 L_0)^n (\sup_{t \in I} |P_\mu(t) - P_{\bar{\mu}}(t)| + M_0 \|q\|_1).
 \end{aligned}$$

Therefore $\{x_n(\cdot)\}$ is a Cauchy sequence in the Banach space $C(I, \mathbf{R})$, hence converging uniformly to some $x(\cdot) \in C(I, \mathbf{R})$. Therefore, by (3.5), for almost all $t \in I$, the sequence $\{f_n(t)\}$ is Cauchy in \mathbf{R} . Let $f(\cdot)$ be the pointwise limit of $f_n(\cdot)$.

Moreover, one has

$$\begin{aligned}
 |x_n(t) - y(t)| &\leq |x_1(t) - y(t)| + \sum_{i=1}^{n-1} |x_{i+1}(t) - x_i(t)| \leq \sup_{t \in I} |P_\mu(t) - P_{\bar{\mu}}(t)| + M_0 \|q\|_1 + \sum_{i=1}^{n-1} (\sup_{t \in I} |P_\mu(t) - P_{\bar{\mu}}(t)| + M_0 \|q\|_1) (M_0 L_0)^i \leq \\
 &\frac{\sup_{t \in I} |P_\mu(t) - P_{\bar{\mu}}(t)| + M_0 \|q\|_1}{1 - M_0 L_0}.
 \end{aligned} \tag{3.6}$$

On the other hand, from (3.2), (3.5) and (3.6) we obtain for almost all $t \in I$

$$\begin{aligned}
 |f_n(t) - y^{(n)}(t) + \lambda y(t)| &\leq \sum_{i=1}^{n-1} |f_{i+1}(t) - f_i(t)| + |f_1(t) - y^{(n)}(t) + \lambda y(t)| \leq L(t) \frac{\sup_{t \in I} |P_\mu(t) - P_{\bar{\mu}}(t)| + M_0 \|q\|_1}{1 - M_0 L_0} + q(t).
 \end{aligned}$$

Hence the sequence $f_n(\cdot)$ is integrably bounded and therefore $f(\cdot) \in L^1(I, \mathbf{R})$.

Using Lebesgue's dominated convergence theorem and taking the limit in (3.3), (3.4) we deduce that $x(\cdot)$ is a solution of (1.1). Finally, passing to the limit in (3.6) we obtained the desired estimate on $x(\cdot)$.

It remains to construct the sequences $x_n(\cdot), f_n(\cdot)$ with the properties in (3.3)-(3.5). The construction will be done by induction.

Since the first step is already realized, assume that for some $N \geq 1$ we already constructed $x_n(\cdot) \in C(I, \mathbf{R})$ and $f_n(\cdot) \in L^1(I, \mathbf{R}), n = 1, 2, \dots, N$ satisfying (3.3), (3.5) for $n = 1, 2, \dots, N$ and (3.4) for $n = 1, 2, \dots, N - 1$. The set-valued map $t \rightarrow F(t, x_N(t))$ is measurable. Moreover, the map $t \rightarrow L(t)|x_N(t) - x_{N-1}(t)|$ is measurable. By the Lipschitzianity of $F(t, \cdot)$ we have that for almost all $t \in I$

$$F(t, x_N(t)) \cap \{f_N(t) + L(t)|x_N(t) - x_{N-1}(t)|[-1, 1]\} \neq \emptyset.$$

From Lemma 2.3 we obtain that there exists a measurable selection $f_{N+1}(\cdot)$ of $F(\cdot, x_N(\cdot))$ such that

$$|f_{N+1}(t) - f_N(t)| \leq L(t)|x_N(t) - x_{N-1}(t)| \quad \text{a.e. } (I).$$

We define $x_{N+1}(\cdot)$ as in (3.3) with $n = N + 1$. Thus $f_{N+1}(\cdot)$ satisfies (3.4) and (3.5) and the proof is complete.

Remark 3.2. In [3], using Covitz-Nadler set-valued contraction principle ([6]) one obtains another Filippov type existence result for problem (1.2)-(1.3). More exactly, according to Theorem 3.1 in [3], for any $\varepsilon > 0$ there exists $x_\varepsilon(\cdot)$ a solution of problem (1.2)-(1.3) satisfying for all $t \in I$

$$|x_\varepsilon(t) - y(t)| \leq \frac{1}{1 - M_0 L_0} \sup_{t \in I} |P_\mu(t) - P_{\bar{\mu}}(t)| + \frac{M_0}{1 - M_0 L_0} \int_0^T q(t) dt + \varepsilon \quad (3.7)$$

Obviously, the estimate in (3.1) is better than the one in (3.7). Moreover, in [3] it is required that the set-valued map $F(\cdot, \cdot)$ satisfy an additional hypothesis, namely $d(0, F(t, 0)) \leq L(t)$ a.e. (I) .

We are concerned now with the boundary value problem (1.4)-(1.5).

Set $M_1 := \max_{t,s \in I} |G_1(t, s)|$.

Theorem 3.3. Assume that Hypothesis 2.4 is satisfied and $M_1 L_0 < 1$. Let $y(\cdot) \in AC_p^1(I, \mathbf{R})$ be such that there exists $q(\cdot) \in L^1(I, \mathbf{R})$ with $d((p(t)y(t))', F(t, y(t))) \leq q(t)$, a.e. (I) , $\alpha y(0) - \beta \lim_{t \rightarrow 0^+} p(t)y'(t) = 0$, $\gamma y(T) + \delta \lim_{t \rightarrow T^-} p(t)y'(t) = 0$.

Then there exists $x(\cdot) : I \rightarrow \mathbf{R}$ a solution of (1.1)-(1.2) satisfying for all $t \in I$

$$|x(t) - y(t)| \leq \frac{M_1}{1 - M_1 L_0} \int_0^T q(t) dt. \quad (3.8)$$

The proof of Theorem 3.3 is similar to the one of Theorem 3.1.

Remark 3.4. In [4], using Covitz-Nadler set-valued contraction principle one obtains another Filippov type existence result for problem (1.4)-(1.5). More precisely, according to Theorem 3.1 in [4], for any $\varepsilon > 0$ there exists $x_\varepsilon(\cdot)$ a solution of problem (1.4)-(1.5) satisfying for all $t \in I$

$$|x_\varepsilon(t) - y(t)| \leq \frac{M_1}{1 - M_1 L_0} \int_0^T q(t) dt + \varepsilon. \quad (3.9)$$

Obviously, the estimate in (3.8) is better than the one in (3.9).

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