Annals of the Academy of Romanian Scientists
Series on Mathematics and its Applications
ISSN 2066 - 6594 Volume 3, Number 2 / 2011

MULTIGRID METHODS WITH CONSTRAINT LEVEL DECOMPOSITION FOR VARIATIONAL INEQUALITIES*

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Abstract

In this paper we introduce four multigrid algorithms for the constrained minimization of non-quadratic functionals. These algorithms are combinations of additive or multiplicative iterations on levels with additive or multiplicative ones over the levels. The convex set is decomposed as a sum of convex level subsets, and consequently, the algorithms have an optimal computing complexity. The methods are described as multigrid V-cycles, but the results hold for other iteration types, the W-cycle iterations, for instance. We estimate the global convergence rates of the proposed algorithms as functions of the number of levels, and compare them with the convergence rates of other existing multigrid methods. Even if the general convergence theory holds for convex sets which can be decomposed as a sum of convex level subsets, our algorithms are applied to the one-obstacle problems because, for these problems, we are able to construct optimal decompositions. But, in this case, the convergence rates of the methods introduced in this paper are better than those of the methods we know in the literature.

AMS subject classification: 65N55, 65N30, 65J15

Keywords: domain decomposition methods, variational inequalities, multigrid and multilevel methods

*Accepted for publication in revised form on March 3, 2011.

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1 Introduction

The multigrid or multilevel methods for the constrained minimization of functionals have been studied almost exclusively for the complementarity problems. Such a method has been proposed by Mandel in [22], [23] and [11]. Related methods have been introduced by Brandt and Cryer in [8] and Hackbush and Mittelmann in [14]. The method has been studied later by Kornhuber in [16] and extended to variational inequalities of the second kind in [17] and [18]. A variant of this method using truncated nodal basis functions has been introduced by Hoppe and Kornhuber in [15] and analyzed by Kornhuber and Yserentant in [20]. Also, versions of this method have been applied to Signorini's problem in elasticity by Kornhuber and Krause in [19] and Wohlmuth and Krause in [27]. Evidently, the above list of citations is not exhaustive and, for further information, we recommend the review article [13] written by Gräser and Kornhuber. For the two-level method, global convergence rates have been established by Badea, Tai and Wang in [7], and for its additive variant by Badea in [3]. A global convergence rate has been also estimated by Tai in [24] for a subset decomposition method.

In [2], a projected multilevel method has been introduced for the constrained minimization of non quadratic functionals. The convex set may be a little more general than of one- or two-obstacle type. The drawback of this method is its sub-optimal computing complexity because the convex set, which is defined on the finest mesh, is used in the smoothing steps on the coarse levels. Multigrid methods with optimal computing complexity have been introduced in [4] (see also, [5]) for the two-obstacle problems. In these algorithms, the convex level sets are recursively constructed for each smoothing step of the iterations. In the present paper, we introduce four multilevel algorithms in which the convex set is decomposed as a sum of convex level subsets. These algorithms, like those introduced in [4], have an optimal computing complexity, and are combinations of additive or multiplicative iterations on levels with additive or multiplicative ones over the levels. Even if the general convergence theory holds for convex sets which can be decomposed as a sum of convex level subsets, these algorithms are applied for the constrained minimization problems of the one-obstacle type. To our knowledge, optimal decompositions as sums of convex level sets for more general convex sets (two-obstacle convex sets, for instance) is an open problem. The methods are described as multigrid V-cycles, but the results hold for W-cycle iterations, for instance.

Regarding the convergence study of the classical multigrid method, an estimate of the asymptotic convergence rate of $1-1/(1+CJ^3)$, J being the number of levels, has been proved by Kornhuber in [16] for the complementarity problems in the bidimensional space. For these problems, the same estimate, but for the global convergence rate, is obtained for the methods in [4] which are of the multiplicative type over the levels. The methods in that paper which are of the additive type over the levels have a global convergence rate of $1-1/(1+CJ^4)$. The global convergence rates of the methods introduced in this paper are better than those of the methods in [4]. We found, for the complementarity problems in $\mathbf{R}^2$, that the convergence rate of the methods which are of the multiplicative type over the levels is of $1-1/(1+CJ^2)$, and of $1-1/(1+CJ^3)$ for the methods of additive type over the levels.

The paper is organized as follows. In Section 2, we state four algorithms in a general framework of reflexive Banach spaces, and prove their convergence under some assumptions. In Section 3, we show that these algorithms can be viewed as multilevel methods for the constrained minimization of non quadratic functionals, if we associate finite element spaces to the level meshes and consider decompositions of the domain at each level. We prove that the assumptions made in the previous section hold for convex sets of one-obstacle type. If the decompositions of the domain are made using the supports of the nodal basis functions we get, in Section 4, the multigrid methods. This particular choice of the domain decompositions allows us to obtain better estimates for the convergence rate of the methods.

2 Abstract convergence results

We consider a reflexive Banach space V and $V_1, \ldots, V_J$, are some closed subspaces of V, where $V_J = V$. Let $K \subset V$ be a nonempty closed convex set, and we assume that there exist some closed convex sets $K_j \subset V_j$, $j = 1, \ldots, J$ such that

$$K = K_1 + \ldots + K_J \tag{2.1}$$

The algorithms we introduce will be combinations of additive or multiplicative algorithms over levels with additive or multiplicative algorithms on each level. To this end, we assume that at each level $1 \le j \le J$ we have I_j closed subspaces of V_j, V_{ji}, $i = 1, \ldots, I_j$, and we shall write $I = \max_{j \in J} I_j$. Also, for a fixed $\sigma > 1$, we assume that there exists a constant C_1 such that

$$||\sum_{j=1}^{J}\sum_{i=1}^{I_j} w_{ji}|| \le C_1(\sum_{j=1}^{J}\sum_{i=1}^{I_j} ||w_{ji}||^{\sigma})^{\frac{1}{\sigma}} \tag{2.2}$$

for any $w_{ji} \in V_{ji}$, $j = J, \dots, 1$, $i = 1, \dots, I_j$. Evidently, we can take, for instance,

$$C_1 = (IJ)^{\frac{\sigma-1}{\sigma}} \tag{2.3}$$

but sharper estimations can be available in certain cases. In the case when we use multiplicative algorithms on the levels $1 \le j \le J$, we make the following

Assumption 2.1. *We assume that there exist two positive constants C_2 and C_3, and that any $w \in K$ can be written as $w = \sum_{j=1}^{J} w_j$, with $w_j \in K_j$, $j = 1, \dots, J$, such that*

- for any $v \in K$,

- and any $w_{ji} \in V_{ji}$ satisfying $w_j + \sum_{k=1}^{i} w_{jk} \in K_j$, $j = 1, \dots, J$, $i = 1, \dots, I_j$,

there exist $v_{ji} \in V_{ji}$, $j = 1, \dots, J$, $i = 1, \dots, I_j$, which satisfy

$$w_j + \sum_{k=1}^{i-1} w_{jk} + v_{ji} \in K_j \text{ for } j = 1, \dots, J,\ i = 1, \dots, I_j,$$

$$v - w = \sum_{j=1}^{J}\sum_{i=1}^{I_j} v_{ji} \text{ and } \sum_{j=1}^{J}\sum_{i=1}^{I_j} ||v_{ji}||^{\sigma} \le C_2^{\sigma}||v - w||^{\sigma} + C_3^{\sigma}\sum_{j=1}^{J}\sum_{i=1}^{I_j} ||w_{ji}||^{\sigma}.$$

If we use additive algorithms on the levels $1 \le j \le J$, we assume

Assumption 2.2. *We assume that there exists a constant $C_2 > 0$, and that any $w \in K$ can be written as $w = \sum_{j=1}^{J} w_j$, with $w_j \in K_j$, $j = 1, \dots, J$, such that for any $v \in K$,*

there exist $v_{ji} \in V_{ji}$, $j = 1, \dots, J$, $i = 1, \dots, I_j$, which satisfy

$$w_j + v_{ji} \in K_j \text{ for } j = 1, \dots, J,\ i = 1, \dots, I_j,$$

$$v - w = \sum_{j=1}^{J}\sum_{i=1}^{m} v_{ji} \text{ and } \sum_{j=1}^{J}\sum_{i=1}^{I_j} ||v_{ji}||^{\sigma} \le C_2^{\sigma}||v - w||^{\sigma}.$$

REMARK 2.1. *In the proofs, for the writing uniformity, we shall consider in Assumption 2.2 a constant* $C_3 = 0$ *and inequality* $\sum_{j=1}^{J}\sum_{i=1}^{I_j}||v_{ji}||^\sigma \leq C_2^\sigma||v-w||^\sigma$ *will be written like in Assumption 2.1,* $\sum_{j=1}^{J}\sum_{i=1}^{I_j}||v_{ji}||^\sigma \leq C_2^\sigma||v-w||^\sigma + C_3^\sigma\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}||^\sigma$*, for any* $w_{ji} \in V_{ji}$.

Now, we consider a Gâteaux differentiable functional $F : V \to \mathbf{R}$, which is assumed to be coercive on K, in the sense that $\frac{F(v)}{||v||} \to \infty$, as $||v|| \to \infty$, $v \in K$, if K is not bounded. Also, we assume that there exist two real numbers $p,\ q > 1$ such that $\frac{p}{p-q+1} \leq \sigma \leq p$ and that, for any real number $M > 0$ there exist $\alpha_M,\ \beta_M > 0$ for which

$$\begin{aligned} &\alpha_M||v-u||^p \leq < F'(v) - F'(u), v-u > \ \text{ and} \\ &||F'(v) - F'(u)||_{V'} \leq \beta_M||v-u||^{q-1} \end{aligned} \tag{2.4}$$

for any $u, v \in V$ with $||u||, ||v|| \leq M$. Above, we have denoted by F' the Gâteaux derivative of F, and we have marked that the constants α_M and β_M may depend on M. It is evident that if (2.4) holds, then for any $u, v \in V$, $||u||, ||v|| \leq M$, we have

$$\alpha_M||v-u||^p \leq < F'(v) - F'(u), v-u > \leq \beta_M||v-u||^q.$$

Following the way in [12], we can prove that for any $u, v \in V$, $||u||, ||v|| \leq M$, we have

$$\begin{aligned} &< F'(u), v-u > + \tfrac{\alpha_M}{p}||v-u||^p \leq F(v) - F(u) \leq \\ &< F'(u), v-u > + \tfrac{\beta_M}{q}||v-u||^q. \end{aligned} \tag{2.5}$$

Also, using the same techniques, we can prove that if F satisfies (2.4), then $1 < q \leq 2 \leq p$. We point out that since F is Gâteaux differentiable and satisfies (2.4), then F is a convex functional (see Proposition 5.5 in [10], pag. 25).

In certain cases, the second equation in (2.4) can be refined, and we assume that there exist some constants $0 < \beta_{jk} \leq 1$, $\beta_{jk} = \beta_{kj}$, $j,\ k = J, \ldots, 1$, such that

$$\langle F'(v+v_{ji}) - F'(v), v_{kl}\rangle \leq \beta_M\beta_{jk}||v_{ji}||^{q-1}||v_{kl}|| \tag{2.6}$$

for any $v \in V$, $v_{ji} \in V_{ji}$, $v_{kl} \in V_{kl}$ with $||v||$, $||v+v_{ji}||$, $||v_{kl}|| \leq M$, $i = 1, \ldots, I_j$ and $l = 1, \ldots, I_k$. Evidently, in view of (2.4), the above inequality holds for

$$\beta_{jk} = 1,\ j,\ k = J, \ldots, 1 \tag{2.7}$$

We consider the variational inequality

$$u \in K: \ < F'(u), v - u > \geq 0, \text{ for any } v \in K, \tag{2.8}$$

and since the functional F is convex and differentiable, it is equivalent with the minimization problem

$$u \in K: F(u) \leq F(v), \text{ for any } v \in K. \tag{2.9}$$

We can use, for instance, Theorem 8.5 in [21], pag. 251, to prove that problem (2.9) has a unique solution if F has the above properties. In view of (2.5), for a given $M > 0$ such that the solution $u \in K$ of (2.9) satisfies $||u|| \leq M$, we have

$$\frac{\alpha_M}{p}||v - u||^p \leq F(v) - F(u) \text{ for any } v \in K, \ ||v|| \leq M. \tag{2.10}$$

To solve problem (2.8), we propose four algorithms which are either of additive or multiplicative type from a level to another one, in combination with additive or multiplicative iterations on the levels. We first define the algorithm which is of the multiplicative type over the levels as well as on each level.

ALGORITHM 2.1. *We start the algorithm with a $u^0 \in K$ and decompose it as in Assumption 2.1 with $w = u^0$, $u^0 = u_1^0 + \ldots + u_J^0$, $u_j^0 \in K_j$, $j = 1, \ldots, J$. At iteration $n+1$, $n \geq 0$, assuming that we have $u^n \in K$, we decompose it as in Assumption 2.1 with $w = u^n$, $u^n = u_1^n + \ldots + u_J^n$, $u_j^n \in K_j$, $j = 1, \ldots, J$. Then, for $j \in J, \ldots, 1$,*

- we successively calculate, the corrections $w_j^{n+1} \in V_j$, $u_j^n + w_j^{n+1} \in K_j$, by the multiplicative algorithm: we first write $w_j^n = 0$, and for $i = 1, \ldots, I_j$, successively calculate $w_{ji}^{n+1} \in V_{ji}$, $u_j^n + w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1} \in K_j$, the solution of the inequality

$$\langle F'\left(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}\right), v_{ji} - w_{ji}^{n+1}\rangle \geq 0 \tag{2.11}$$

for any $v_{ji} \in V_{ji}$, $u_j^n + w_j^{n+\frac{i-1}{I_j}} + v_{ji} \in K_j$, and write $w_j^{n+\frac{i}{I_j}} = w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}$,

- then, we write, $u^{n+\frac{J-j+1}{J}} = u^{n+\frac{J-j}{J}} + w_j^{n+1}$.

The algorithm which is of multiplicative type over the levels and of the additive type on levels is written as,

ALGORITHM 2.2. *We start the algorithm with an $u^0 \in K$ and decompose it as in Assumption 2.2 with $w = u^0$, $u^0 = u_1^0 + \ldots + u_J^0$, $u_j^0 \in K_j$, $j = 1, \ldots, J$. At iteration $n+1$, $n \geq 0$, assuming that we have $u^n \in K$, we decompose it as in Assumption 2.2 with $w = u^n$, $u^n = u_1^n + \ldots + u_J^n$, $u_j^n \in K_j$, $j = 1, \ldots, J$. Then, for $j = J, \ldots, 1$,*

- we successively calculate, the corrections $w_j^{n+1} \in V_j$, $u_j^n + w_j^{n+1} \in K_j$, by the additive algorithm: we simultaneously calculate $w_{ji}^{n+1} \in V_{ji}$, $u_j^n + w_{ji}^{n+1} \in K_j$, the solution of the inequality

$$\langle F'\left(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_{ji}^{n+1}\right), v_{ji} - w_{ji}^{n+1}\rangle \geq 0 \tag{2.12}$$

for any $v_{ji} \in V_{ji}$, $u_j^n + v_{ji} \in K_j$, and write $w_j^{n+1} = \frac{r}{I} \sum_{i=1}^{I_j} w_{ji}^{n+1}$, with a fixed $0 < r \leq 1$.

- then, we write, $u^{n+\frac{J-j+1}{J}} = u^{n+\frac{J-j}{J}} + w_j^{n+1}$.

Now, the additive algorithm over levels and which is of multiplicative type on each level reads,

ALGORITHM 2.3. *We start the algorithm with an $u^0 \in K$ and decompose it as in Assumption 2.1 with $w = u^0$, $u^0 = u_1^0 + \ldots + u_J^0$, $u_j^0 \in K_j$, $j = 1, \ldots, J$. At iteration $n+1$, $n \geq 0$, assuming that we have $u^n \in K$, we decompose it as in Assumption 2.1 with $w = u^n$, $u^n = u_1^n + \ldots + u_J^n$, $u_j^n \in K_j$, $j = 1, \ldots, J$. Then we simultaneously calculate, for $j = 1, \ldots, J$, the corrections $w_j^{n+1} \in V_j$, $u_j^n + w_j^{n+1} \in K_j$, by the multiplicative algorithm:*

– we first write $w_j^n = 0$, and for $i = 1, \ldots, I_j$, successively calculate $w_{ji}^{n+1} \in V_{ji}$, $u_j^n + w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1} \in K_j$, the solution of the inequality

$$\langle F'\left(u^n + w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}\right), v_{ji} - w_{ji}^{n+1}\rangle \geq 0 \tag{2.13}$$

for any $v_{ji} \in V_{ji}$, $u_j^n + w_j^{n+\frac{i-1}{I_j}} + v_{ji} \in K_j$, and write $w_j^{n+\frac{i}{I_j}} = w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}$, Then, we write $u^{n+1} = u^n + \frac{s}{J} \sum_{j=1}^{J} w_j^{n+1}$, with a fixed $0 < s \leq 1$.

Finally, the algorithm which is of additive type over the levels as well as on each level is written as,

ALGORITHM 2.4. *We start the algorithm with an $u^0 \in K$ and decompose it as in Assumption 2.2 with $w = u^0$, $u^0 = u_1^0 + \ldots + u_J^0$, $u_j^0 \in K_j$, $j = 1, \ldots, J$. At iteration $n+1$, $n \geq 0$, assuming that we have $u^n \in K$, we decompose it as in Assumption 2.2 with $w = u^n$, $u^n = u_1^n + \ldots + u_J^n$, $u_j^n \in K_j$, $j = 1, \ldots, J$. Then we simultaneously calculate, for $j = 1, \ldots, J$, the corrections $w_j^{n+1} \in V_j$, $u_j^n + w_j^{n+1} \in K_j$, by the additive algorithms:*

– we simultaneously calculate $w_{ji}^{n+1} \in V_{ji}$, $u_j^n + w_{ji}^{n+1} \in K_j$, the solution of the inequality

$$\langle F'\left(u^n + w_{ji}^{n+1}\right), v_{ji} - w_{ji}^{n+1}\rangle \geq 0 \tag{2.14}$$

for any $v_{ji} \in V_{ji}$, $u_j^n + v_{ji} \in K_j$, and write $w_j^{n+1} = \frac{r}{I}\sum_{i=1}^{I_j} w_{ji}^{n+1}$, with a fixed $0 < r \leq 1$.

Then, we write $u^{n+1} = u^n + \frac{s}{J}\sum_{j=1}^{J} w_j^{n+1}$, with a fixed $0 < s \leq 1$.

Like inequality (2.8), inequalities (2.11)–(2.14) are equivalent with minimization problems (see [6]).

The convergence result is given by

Theorem 2.1. *We consider that V is a reflexive Banach, V_j, $j = 1, \ldots, J$, are closed subspaces of V, and V_{ji}, $i = 1, \ldots, I_j$, are some closed subspaces of V_j, $j = 1, \ldots, J$. Let K be a non empty closed convex subset of V decomposed as in (2.1) where K_j are closed convex subsets of V_j, $j = 1, \ldots, J$, and F be a Gâteaux differentiable functional on V which is supposed to be coercive if K is not bounded, and satisfies (2.4). Also, we assume that Assumption 2.1 holds for Algorithms 2.1 and 2.3, and Assumption 2.2 holds for Algorithms 2.2 and 2.4. On these conditions, if u is the solution of problem (2.8) and u^n, $n \geq 0$, are its approximations obtained from the above described algorithms, then there exists $M > 0$ such that $||u||$, $||u^n|| \leq M$, $n \geq 0$, and the following error estimations hold:*

(i) if $p = q = 2$ we have

$$F(u^n) - F(u) \leq \left(\frac{\tilde{C}_1}{\tilde{C}_1 + 1}\right)^n \left[F(u^0) - F(u)\right], \tag{2.15}$$

$$||u^n - u||^2 \leq \frac{2}{\alpha_M}\left(\frac{\tilde{C}_1}{\tilde{C}_1 + 1}\right)^n \left[F(u^0) - F(u)\right], \tag{2.16}$$

where $\tilde{C}_1$ is given in (2.30), and
(ii) if $p > q$ we have

$$F(u^n) - F(u) \leq \frac{F(u^0) - F(u)}{\left[1 + n\tilde{C}_2 \left(F(u^0) - F(u)\right)^{\frac{p-q}{q-1}}\right]^{\frac{q-1}{p-q}}}, \tag{2.17}$$

$$||u - u^n||^p \leq \frac{p}{\alpha_M} \frac{F(u^0) - F(u)}{\left[1 + n\tilde{C}_2 \left(F(u^0) - F(u)\right)^{\frac{p-q}{q-1}}\right]^{\frac{q-1}{p-q}}}, \tag{2.18}$$

where $\tilde{C}_2$ is given in (2.34).

Proof. Step 1. We first prove the boundedness of the approximations u^n of u as well as of the corrections w_{ji}^{n+1} obtained from the above algorithms. If K is not bounded, using the coercivity and convexity of F, we get that there exists a $M > 0$, such that $||u||$, $||u^n||$, $||w_{ji}^{n+1}|| \leq M$, $n \geq 0$, $j = J, \ldots, 1$, $i = 1, \ldots, I_j$, for the four Algorithms 2.1–2.4. The proof is similar with that given in [1], [3] or [4], and can be found in [6].

Step 2. Now, we study the boundedness of $\sum_{j=1}^{J} \sum_{i=1}^{I_j} ||w_{ji}^{n+1}||^p$. For Algorithm 2.1, in view of (2.5) and (2.11), we have

$$\frac{\alpha_M}{p} ||w_{ji}^{n+1}||^p \leq F\left(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_j^{n+\frac{i-1}{I_j}}\right) - F\left(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_j^{n+\frac{i}{I_j}}\right)$$

ie.,

$$\frac{\alpha_M}{p} \sum_{i=1}^{I_j} ||w_{ji}^{n+1}||^p \leq F\left(u^n + \sum_{k=j+1}^{J} w_k^{n+1}\right) - F\left(u^n + \sum_{k=j}^{J} w_k^{n+1}\right) \tag{2.19}$$

Also, for Algorithm 2.2, from (2.12), we get

$$\frac{\alpha_M}{p} ||w_{ji}^{n+1}||^p \leq F\left(u^n + \sum_{k=j+1}^{J} w_k^{n+1}\right) - F\left(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_{ji}^{n+1}\right)$$

But,

$$F\left(u^n+\sum_{k=j+1}^{J} w_k^{n+1}+w_j^{n+1}\right)=F\left(u^n+\sum_{k=j+1}^{J} w_k^{n+1}+\frac{r}{I}\sum_{i=1}^{I_j} w_{ji}^{n+1}\right)\leq$$
$$(1-\frac{rI_j}{I})F\left(u^n+\sum_{k=j+1}^{J} w_k^{n+1}\right)+\frac{r}{I}\sum_{i=1}^{I_j} F\left(u^n+\sum_{k=j+1}^{J} w_k^{n+1}+w_{ji}^{n+1}\right)$$

From the above two equations, we get

$$\begin{aligned}&\frac{r}{I}\frac{\alpha_M}{p}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^p\leq F\left(u^n+\sum_{k=j+1}^{J} w_k^{n+1}\right)-\\&F\left(u^n+\sum_{k=j}^{J} w_k^{n+1}\right)\end{aligned}\tag{2.20}$$

By a similar proof, for Algorithm 2.3, using (2.13), we get

$$\frac{\alpha_M}{p}\sum_{i=1}^{I_J}||w_{ji}^{n+1}||^p\leq F\left(u^n\right)-F\left(u^n+w_j^{n+1}\right)\tag{2.21}$$

and, in view of (2.14), for Algorithm 2.4, we have,

$$\frac{r}{I}\frac{\alpha_M}{p}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^p\leq F\left(u^n\right)-F\left(u^n+w_j^{n+1}\right)\tag{2.22}$$

Now, let us write

$$t=\begin{cases}1 & \text{for Algorithm 2.1}\\ \frac{r}{I} & \text{for Algorithm 2.2}\\ \frac{s}{J} & \text{for Algorithm 2.3}\\ \frac{s}{J}\frac{r}{I} & \text{for Algorithm 2.4}\end{cases}\tag{2.23}$$

For Algorithms 2.1 and 2.2, in view of (2.19) and (2.20), we can write

$$t\frac{\alpha_M}{p}\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^p\leq F\left(u^n\right)-F\left(u^{n+1}\right)\tag{2.24}$$

With t in (2.23), the same equation holds for Algorithms 2.3 and 2.4. Indeed,

$$F\left(u^{n+1}\right) = F\left(u^n + \frac{s}{J}\sum_{j=1}^{J} w_j^{n+1}\right) \leq (1-s)F(u^n) + \frac{s}{J}\sum_{j=1}^{J} F\left(u^n + w_j^{n+1}\right)$$

and (2.24) follows from (2.21) and (2.22).

Step 3. We now estimate $F(u^{n+1}) - F(u)$. For a given $j \in J$, we write $\overline{w}_j^{n+1} = \sum_{i=1}^{I_j} w_{ji}^{n+1}$. Evidently, for Algorithm 2.1, we have

$$F(u^{n+1}) = F\left(u^n + \sum_{j=1}^{J} \overline{w}_j^{n+1}\right)$$

For Algorithm 2.2, we get,

$$F(u^{n+1}) = F\left(u^n + \frac{r}{I}\sum_{j=1}^{J} \overline{w}_j^{n+1}\right) \leq (1-\frac{r}{I})F\left(u^n\right) + \frac{r}{I}F\left(u^n + \sum_{j=1}^{J} \overline{w}_j^{n+1}\right)$$

It is clear that for Algorithm 2.3, we have

$$F(u^{n+1}) \leq (1-\frac{s}{J})F(u^n) + \frac{s}{J}F\left(u^n + \sum_{j=1}^{J} \overline{w}_j^{n+1}\right)$$

Finally, for Algorithm 2.4, we get,

$$F(u^{n+1}) = F\left(u^n + \frac{s}{J}\frac{r}{I}\sum_{j=1}^{J} \overline{w}_j^{n+1}\right) \leq$$
$$(1-\frac{s}{J}\frac{r}{I})F(u^n) + \frac{s}{J}\frac{r}{I}F\left(u^n + \sum_{j=1}^{J} \overline{w}_j^{n+1}\right)$$

From the above four equations we conclude that

$$F(u^{n+1}) \leq (1-t)F(u^n) + tF\left(u^n + \sum_{j=1}^{J} \overline{w}_j^{n+1}\right) \tag{2.25}$$

where t is given in (2.23). With $v = u$ and $w = u^n$, we consider a decomposition $\sum_{j=1}^{J}\sum_{i=1}^{I_j} v_{ji}^n$ of $u - u^n$ as in Assumption 2.1, in the case of Algorithms

2.1 and 2.3, or as in Assumption 2.2, in the case of Algorithms 2.2 and 2.4. In Assumption 2.1, we take $w_{ji} = w_{ji}^{n+1}$, $j = J, \ldots, 1$, $i = 1, \ldots, I_j$, which are obtained from Algorithms 2.1 or 2.3. In view of (2.5), we have

$$\begin{aligned}
&F\left(u^n + \sum_{j=1}^{J} \overline{w}_j^{n+1}\right) - F(u) + \frac{\alpha_M}{p} ||u^n + \sum_{j=1}^{J} \overline{w}_j^{n+1} - u||^p \leq \\
&\langle F'\left(u^n + \sum_{j=1}^{J} \overline{w}_j^{n+1}\right), u^n + \sum_{j=1}^{J} \overline{w}_j^{n+1} - u\rangle = \\
&-\sum_{k=1}^{J} \sum_{i=1}^{I_k} \langle F'\left(u^n + \sum_{j=1}^{J} \overline{w}_j^{n+1}\right), v_{ki}^n - w_{ki}^{n+1}\rangle
\end{aligned} \tag{2.26}$$

For Algorithm 2.1, in view of (2.11) and (2.6), we have,

$$\begin{aligned}
&-\langle F'\left(u^n + \sum_{j=1}^{J} \overline{w}_j^{n+1}\right), v_{ki}^n - w_{ki}^{n+1}\rangle \leq \\
&\langle F'\left(u^n + \sum_{l=k+1}^{J} \overline{w}_l^{n+1} + w_k^{n+\frac{i-1}{I_k}} + w_{ki}^{n+1}\right) - \\
&F'\left(u^n + \sum_{j=1}^{J} \overline{w}_j^{n+1}\right), v_{ki}^n - w_{ki}^{n+1}\rangle \leq \\
&\beta_M \sum_{j=1}^{J} \beta_{kj} \sum_{l=1}^{I_j} ||w_{jl}^{n+1}||^{q-1} ||v_{ki}^n - w_{ki}^{n+1}||
\end{aligned}$$

Above, we have added and subtracted the missing terms between $F'\left(u^n + \sum_{l=k+1}^{J} \overline{w}_l^{n+1} + w_k^{n+\frac{i-1}{I_j}} + w_{ki}^{n+1}\right)$ and $F'\left(u^n + \sum_{j=1}^{J} \overline{w}_j^{n+1}\right)$. Also, for Algorithm 2.2, from (2.12), we have

$$-\langle F'\left(u^n+\sum_{j=1}^{J}\overline{w}_j^{n+1}\right), v_{ki}^n-w_{ki}^{n+1}\rangle \le$$
$$\langle F'\left(u^n+\frac{r}{I}\sum_{j=k+1}^{J}\overline{w}_j^{n+1}+w_{ki}^{n+1}\right)-F'\left(u^n+\sum_{j=1}^{J}\overline{w}_j^{n+1}\right), v_{ki}^n-w_{ki}^{n+1}\rangle \le$$
$$2\beta_M\sum_{j=1}^{J}\beta_{kj}\sum_{l=1}^{I_j}||w_{jl}^{n+1}||^{q-1}||v_{ki}^n-w_{ki}^{n+1}||$$

Here, we have added and subtracted the missing terms between $F'(u^n)$ and $F'\left(u^n+\frac{r}{I}\sum_{j=k+1}^{J}\overline{w}_j^{n+1}+w_{ki}^{n+1}\right)$, between $F'(u^n)$ and $F'\left(u^n+\sum_{j=1}^{J}\overline{w}_j^{n+1}\right)$, and used the fact that $\frac{r}{I}\le 1$. Similarly, we get the above inequality from (2.13) for Algorithm 2.3, and from (2.14) for Algorithm 2.4. Consequently, in view of (2.26), we can write for all the four algorithms,

$$F(u^n+\sum_{j=1}^{J}\overline{w}_j^{n+1})-F(u)+\frac{\alpha_M}{p}||u^n+\sum_{j=1}^{J}\overline{w}_j^{n+1}-u||^p \le$$
$$2\beta_M\sum_{j=1}^{J}\sum_{k=1}^{J}\beta_{kj}\sum_{l=1}^{I_j}||w_{jl}^{n+1}||^{q-1}\sum_{i=1}^{I_k}||v_{ki}^n-w_{ki}^{n+1}|| \le$$
$$2\beta_M I^{\frac{\sigma-1}{\sigma}+\frac{p-q+1}{p}}\sum_{j=1}^{J}\left(\sum_{k=1}^{J}\beta_{kj}(\sum_{i=1}^{I_k}||v_{ki}^n-w_{ki}^{n+1}||^\sigma)^{\frac{1}{\sigma}}\right)\left(\sum_{l=1}^{I_j}||w_{jl}^{n+1}||^p\right)^{\frac{q-1}{p}} \le$$
$$2\beta_M I^{\frac{\sigma-1}{\sigma}+\frac{p-q+1}{p}}\left[\sum_{j=1}^{J}\left(\sum_{k=1}^{J}\beta_{kj}(\sum_{i=1}^{I_k}||v_{ki}^n-w_{ki}^{n+1}||^\sigma)^{\frac{1}{\sigma}}\right)^\sigma\right]^{\frac{1}{\sigma}}\cdot$$
$$\left(\sum_{j=1}^{J}(\sum_{l=1}^{I_j}||w_{jl}^{n+1}||^p)^{\frac{q-1}{p}\frac{\sigma}{\sigma-1}}\right)^{\frac{\sigma-1}{\sigma}} \le 2\beta_M I^{\frac{\sigma-1}{\sigma}+\frac{p-q+1}{p}}J^{\frac{\sigma-1}{\sigma}-\frac{q-1}{p}}(\max_{k=1,\cdots,J}\sum_{j=1}^{J}\beta_{kj})\cdot$$
$$(\sum_{j=1}^{J}\sum_{i=1}^{I_j}||v_{ji}^n-w_{ji}^{n+1}||^\sigma)^{\frac{1}{\sigma}}(\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^p)^{\frac{q-1}{p}}$$

Above, we have used the inequality (see Corollary 4.1 in [25])

$$||Ax||_{l^\sigma} \le (\max_i \sum_j |A_{ij}|)||x||_{l^\sigma} \tag{2.27}$$

where $A = (A_{ij})_{ij}$ is a symmetric matrix. In view of (2.2), Assumptions 2.1 and 2.2 and Remark 2.1, we have

$$\begin{aligned}
&(\sum_{j=1}^{J}\sum_{i=1}^{I_j}||v_{ji}^n - w_{ji}^{n+1}||^\sigma)^{\frac{1}{\sigma}} \le (\sum_{j=1}^{J}\sum_{i=1}^{I_j}||v_{ji}||^\sigma)^{\frac{1}{\sigma}} + (\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^\sigma)^{\frac{1}{\sigma}} \le \\
&(C_2^\sigma||u-u^n||^\sigma + C_3^\sigma\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^\sigma)^{\frac{1}{\sigma}} + (\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^\sigma)^{\frac{1}{\sigma}} \le \\
&C_2||u-u^n|| + (1+C_3)(\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^\sigma)^{\frac{1}{\sigma}} \le \\
&C_2||u-u^n - \sum_{j=1}^{J}\overline{w}_j^{n+1}|| + (1+C_1C_2+C_3)(IJ)^{\frac{p-\sigma}{p\sigma}}(\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^p)^{\frac{1}{p}}
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&F(u^n + \sum_{j=1}^{J}\overline{w}_j^{n+1}) - F(u) + \frac{\alpha_M}{p}||u^n + \sum_{j=1}^{J}\overline{w}_j^{n+1} - u||^p \le \\
&2\beta_M I^{\frac{\sigma-1}{\sigma}+\frac{p-q+1}{p}} J^{\frac{\sigma-1}{\sigma}-\frac{q-1}{p}}(\max_{k=1,\cdots,J}\sum_{j=1}^{J}\beta_{kj})\cdot \\
&\left[C_2||u-u^n-\sum_{j=1}^{J}\overline{w}_j^{n+1}||(\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^p)^{\frac{q-1}{p}} + \right. \\
&\left.(1+C_1C_2+C_3)(IJ)^{\frac{p-\sigma}{p\sigma}}(\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^p)^{\frac{q}{p}}\right]
\end{aligned}$$

But, for any $\varepsilon > 0$, $p > 1$ and $x, y \ge 0$, we have $xy \le \varepsilon x^p + \frac{1}{\varepsilon^{\frac{1}{p-1}}} y^{\frac{p}{p-1}}$.

Consequently, we have

$$F(u^n+\sum_{j=1}^{J}\overline{w}_j^{n+1})-F(u)+\frac{\alpha_M}{p}||u^n+\sum_{j=1}^{J}\overline{w}_j^{n+1}-u||^p\leq$$
$$2\beta_M I^{\frac{\sigma-1}{\sigma}+\frac{p-q+1}{p}}J^{\frac{\sigma-1}{\sigma}-\frac{q-1}{p}}(\max_{k=1,\cdots,J}\sum_{j=1}^{J}\beta_{kj})\cdot$$
$$\left[C_2\varepsilon||u-u^n-\sum_{j=1}^{J}\overline{w}_j^{n+1}||^p+C_2\frac{1}{\varepsilon^{\frac{1}{p-1}}}(\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^p)^{\frac{q-1}{p-1}}+\right.$$
$$\left.(1+C_1C_2+C_3)(IJ)^{\frac{p-\sigma}{p\sigma}}(\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^p)^{\frac{q}{p}}\right]$$

for any $\varepsilon>0$. With

$$\varepsilon=\frac{\alpha_M}{p}\frac{1}{2C_2\beta_M I^{\frac{\sigma-1}{\sigma}+\frac{p-q+1}{p}}J^{\frac{\sigma-1}{\sigma}-\frac{q-1}{p}}(\max_{k=1,\cdots,J}\sum_{j=1}^{J}\beta_{kj})}\tag{2.28}$$

the above equation becomes,

$$F(u^n+\sum_{j=1}^{J}\overline{w}_j^{n+1})-F(u)\leq\frac{\frac{\alpha_M}{p}}{C_2\varepsilon}\cdot$$
$$\left[\frac{C_2}{\varepsilon^{\frac{1}{p-1}}}(\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^p)^{\frac{q-1}{p-1}}+(1+C_1C_2+C_3)(IJ)^{\frac{p-\sigma}{p\sigma}}(\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^p)^{\frac{q}{p}}\right]$$

From this equation and (2.24)

$$F(u^n+\sum_{j=1}^{J}\overline{w}_j^{n+1})-F(u)\leq\frac{\frac{\alpha_M}{p}}{C_2\varepsilon}\cdot\left[\frac{C_2}{\varepsilon^{\frac{1}{p-1}}(t\frac{\alpha_M}{p})^{\frac{q-1}{p-1}}}(F(u^n)-F(u^{n+1}))^{\frac{q-1}{p-1}}+\right.$$
$$\left.\frac{(1+C_1C_2+C_3)(IJ)^{\frac{p-\sigma}{p\sigma}}}{(t\frac{\alpha_M}{p})^{\frac{q}{p}}}(F(u^n)-F(u^{n+1}))^{\frac{q}{p}}\right]$$

with t in (2.23) and ε in (2.28). In view of the above equation and (2.25), we have

$$F(u^{n+1}) - F(u) \le \tfrac{1-t}{t}(F(u^n) - F(u^{n+1})) + \frac{\frac{\alpha_M}{p}}{C_2\varepsilon} \cdot \left[\frac{C_2}{\varepsilon^{\frac{1}{p-1}}(t\frac{\alpha_M}{p})^{\frac{q-1}{p-1}}}(F(u^n) - F(u^{n+1}))^{\frac{q-1}{p-1}} + \frac{(1 + C_1C_2 + C_3)(IJ)^{\frac{p-\sigma}{p\sigma}}}{(t\frac{\alpha_M}{p})^{\frac{q}{p}}}(F(u^n) - F(u^{n+1}))^{\frac{q}{p}} \right] \tag{2.29}$$

Step 4. We prove error estimations (2.15)–(2.18). First, using (2.10), we see that error estimations in (2.16) and (2.18) can be obtained from (2.15) and (2.17), respectively. Now, if $p = q = 2$, then $\sigma = 2$, and from the above equation, we easily get equation (2.15), where

$$\tilde{C}_1 = \frac{1-t}{t} + \frac{1}{C_2 t\varepsilon}\left[\frac{C_2}{\varepsilon} + 1 + C_1C_2 + C_3\right] \text{ with } \varepsilon = \frac{\frac{\alpha_M}{2}}{2C_2\beta_M I(\max\limits_{k=1,\cdots,J}\sum\limits_{j=1}^{J}\beta_{kj})} \tag{2.30}$$

Finally, if $p > q$, from (2.29), we have

$$F(u^{n+1}) - F(u) \le \tilde{C}_3(F(u^n) - F(u^{n+1}))^{\frac{q-1}{p-1}} \tag{2.31}$$

where

$$\tilde{C}_3 = \frac{1-t}{t}(F(u^0) - F(u))^{\frac{p-q}{p-1}} + \frac{\frac{\alpha_M}{p}}{C_2\varepsilon}\left[\frac{C_2}{\varepsilon^{\frac{1}{p-1}}(t\frac{\alpha_M}{p})^{\frac{q-1}{p-1}}} + \frac{(1 + C_1C_2 + C_3)(IJ)^{\frac{p-\sigma}{p\sigma}}}{(t\frac{\alpha_M}{p})^{\frac{q}{p}}}(F(u^0) - F(u))^{\frac{p-q}{p(p-1)}}\right] \tag{2.32}$$

with ε in (2.28). From (2.31), we get

$$F(u^{n+1}) - F(u) + \frac{1}{\tilde{C}_3^{\frac{p-1}{q-1}}}(F(u^{n+1}) - F(u))^{\frac{p-1}{q-1}} \le F(u^n) - F(u),$$

and we know (see Lemma 3.2 in [25]) that for any $r > 1$ and $c > 0$, if $x \in (0, x_0]$ and $y > 0$ satisfy $y + cy^r \le x$, then $y \le (\frac{c(r-1)}{crx_0^{r-1}+1} + x^{1-r})^{\frac{1}{1-r}}$.

Consequently, we have $F(u^{n+1}) - F(u) \le [\tilde{C}_2 + (F(u^n) - F(u))^{\frac{q-p}{q-1}}]^{\frac{q-1}{q-p}}$, from which,

$$F(u^{n+1}) - F(u) \le [(n+1)\tilde{C}_2 + (F(u^0) - F(u))^{\frac{q-p}{q-1}}]^{\frac{q-1}{q-p}}, \tag{2.33}$$

where

$$\tilde{C}_2 = \frac{p-q}{(p-1)(F(u^0) - F(u))^{\frac{p-q}{q-1}} + (q-1)\tilde{C}_3^{\frac{p-1}{q-1}}}. \tag{2.34}$$

Equation (2.33) is another form of equation (2.17). □

3 Multilevel Schwarz methods

We consider a family of regular meshes $\mathcal{T}_{h_j}$ of mesh sizes h_j, $j = 1, \ldots, J$ over the domain $\Omega \subset \mathbf{R}^d$. We write $\Omega_j = \cup_{\tau \in \mathcal{T}_{h_j}} \tau$ and assume that $\mathcal{T}_{h_{j+1}}$ is a refinement of $\mathcal{T}_{h_j}$ on Ω_j, $j = 1, \ldots, J-1$, and $\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega_J = \Omega$. Also, we assume that, if a node of $\mathcal{T}_{h_j}$ lies on $\partial\Omega_j$, then it lies on $\partial\Omega_{j+1}$, too, that is, it lies on $\partial\Omega$. Besides, we suppose that $\mathrm{dist}(x_{j+1}, \Omega_j) \le Ch_j$, for any node x_{j+1} of $\mathcal{T}_{h_{j+1}}$, $j = 1, \ldots, J-1$. In this section, C denotes a generic positive constant independent of the mesh sizes, the number of meshes, as well as of the overlapping parameters and the number of subdomains in the domain decompositions which will be considered later. Since the mesh $\mathcal{T}_{h_{j+1}}$ is a refinement of $\mathcal{T}_{h_j}$, we have $h_{j+1} \le h_j$, and assume that there exists a constant γ, independent of the number of meshes or their sizes, such that

$$1 < \gamma \le \frac{h_j}{h_{j+1}} \le C\gamma, \quad j = 1, \ldots, J-1. \tag{3.1}$$

At each level $j = 1, \ldots, J$, we consider an overlapping decomposition $\{\Omega_j^i\}_{1 \le i \le I_j}$ of Ω_j, and assume that the mesh partition $\mathcal{T}_{h_j}$ of Ω_j supplies a mesh partition for each Ω_j^i, $1 \le i \le I_j$. Also, we assume that the overlapping size for the domain decomposition at the level $1 \le j \le J$ is δ_j. In addition, we suppose that if ω_{j+1}^i is a connected component of Ω_{j+1}^i, $j = 1, \ldots, J-1$, $i = 1, \ldots, I_j$, then

$$\mathrm{diam}(\omega_{j+1}^i) \le Ch_j \tag{3.2}$$

Since $h_{j+1} \le \delta_{j+1}$, from (3.1), we also have

$$\frac{h_j}{\delta_{j+1}} \le C\gamma, \quad j = 1, \ldots, J-1. \tag{3.3}$$

Finally, we assume that $I_1 = 1$.

At each level $j = 1, \dots, J$, we introduce the linear finite element spaces,

$$V_{h_j} = \{v \in C(\bar{\Omega}_j) : v|_\tau \in P_1(\tau),\ \tau \in \mathcal{T}_{h_j},\ v = 0 \text{ on } \partial\Omega_j\}, \tag{3.4}$$

and, for $i = 1, \dots, I_j$, we write

$$V_{h_j}^i = \{v \in V_{h_j} : v = 0 \text{ in } \Omega_j \backslash \Omega_j^i\}. \tag{3.5}$$

The functions in V_{h_j} $j = 1, \dots, J-1$, will be extended with zero outside Ω_j and the spaces will be considered as subspaces of $W^{1,\sigma}$, $1 \le \sigma \le \infty$. We denote by $||\cdot||_{0,\sigma}$ the norm in L^σ, and by $||\cdot||_{1,\sigma}$ and $|\cdot|_{1,\sigma}$ the norm and seminorm in $W^{1,\sigma}$, respectively.

We consider the obstacle problem

$$u \in K : \ < F'(u), v - u > \ge 0, \text{ for any } v \in K, \tag{3.6}$$

where

$$K = \{v \in V_{h_J} : \varphi \le v\}, \tag{3.7}$$

with $\varphi \in V_{h_J}$. We shall prove that Assumptions 2.1 and 2.2 hold for this type of convex set, and explicitly write the constants C_2 and C_3 as functions of the mesh and overlapping parameters. We can then conclude from Theorem 2.1 that if the functional F has the asked properties, then Algorithms 2.1–2.4 are globally convergent.

We first introduce the operators $I_{h_j} : V_{h_{j+1}} \to V_{h_j}$, $j = 1, \dots, J-1$, defined as follows. Let us denote by x_{ji} a node of $\mathcal{T}_{h_j}$, by ϕ_{ji} the linear nodal basis function associated with x_{ji} and $\mathcal{T}_{h_j}$, and by ω_{ji} the support of ϕ_{ji}. Given a $v \in V_{h_{j+1}}$, we write $I_{ji}v = \min_{x \in \omega_{ji}} v(x)$. Finally, we define $I_{h_j}v := \sum_{x_{ji} \text{node of } \mathcal{T}_{h_j}} (I_{ji}v)\phi_{ji}(x)$.

Remark 3.1. 1) In [24], similar operators, $I_{h_j} : V_{h_J} \to V_{h_j}$, are defined. For a $v \in V_{h_J}$, we write as above, $I_{ji}v = \min_{x \in \omega_{ji}} v(x)$ and $I_{h_j}v := \sum_{x_{ji} \text{node of } \mathcal{T}_{h_j}} (I_{ji}v)\phi_{ji}(x)$. These operators have the disadvantage that $I_{h_j}v$ can not be computed from $I_{h_{j+1}}v$. For this reason, in the case of the multigrid method, their definition is modified in [13] by taking $I_{ji}v = \min_{x \in \text{Int}\omega_{ji}} v(x)$ in the place of $I_{ji}v = \min_{x \in \omega_{ji}} v(x)$.

2) Since the finite element spaces are linear, for a $v \in V_{h_j}$, we can take $I_{ji}v = \min\limits_{x \in \omega_{ji},\, x \text{ node of } \mathcal{T}_{h_{j+1}}} v(x)$ in the place of $I_{ji}v = \min_{x \in \omega_{ji}} v(x)$ in our above definition.

3) In [2] some more general operators $I_{h_j} : V_{h_{j+1}} \to V_{h_j}$ have been introduced. They coincide with those above defined ones for $v \geq 0$.

For a $v \in V_{h_J}$, we recursively define

$$v^J = v \text{ and } v^j = I_{h_j} v^{j+1},\ j = J-1, \ldots, 1 \tag{3.8}$$

Writing

$$C_{d,\sigma}(H,h) = \begin{cases} 1 & \text{if } d = \sigma = 1 \\ & \text{or } 1 \leq d < \sigma < \infty \\ (\ln \frac{H}{h} + 1)^{\frac{d-1}{d}} & \text{if } 1 < d = \sigma < \infty \\ (\frac{H}{h})^{\frac{d-\sigma}{\sigma}} & \text{if } 1 \leq \sigma < d < \infty, \end{cases} \tag{3.9}$$

we have the following result

Lemma 3.1. *Let $v^j, w^j \in V_{h_j}$, $j = J, \ldots, 1$ defined as in (3.8) for some $v, w \in V_{h_J}$, respectively. Then, for $j = J, \ldots, 1$, we have*

$$|v^j - w^j|_{1,\sigma} \leq C C_{d,\sigma}(h_j, h_J) |v - w|_{1,\sigma} \tag{3.10}$$

Proof. Equation (3.10) is evident for $j = J$. For a $j = J-1, \ldots, 1$, let $\omega_j(x_j)$ be the support of the nodal basis function in V_{h_j} corresponding to the node x_j of $\mathcal{T}_{h_j}$. Then there exist two nodes of $\mathcal{T}_{h_j}$, $x_j^1, x_j^2 \in \omega_j(x_j)$, such that

$$|v^j - w^j|^\sigma_{1,\sigma,\omega_j(x_j)} \leq C h_j^{d-\sigma} |(v^j - w^j)(x_j^1) - (v^j - w^j)(x_j^2)|^\sigma \tag{3.11}$$

and let us assume that

$$|(v^j - w^j)(x_j^1) - (v^j - w^j)(x_j^1)| = (v^j - w^j)(x_j^1) - (v^j - w^j)(x_j^2) \tag{3.12}$$

Now, we have

$$\begin{aligned} &(v^j - w^j)(x_j^1) - (v^j - w^j)(x_j^2) = \\ &(I_{h_j} v^{j+1} - I_{h_j} w^{j+1})(x_j^1) + (I_{h_j} w^{j+1} - I_{h_j} v^{j+1})(x_j^2) \end{aligned}$$

and let

$$I_{h_j} w^{j+1}(x_j^1) = w^{j+1}(x_{j+1}^1) \text{ and } I_{h_j} v^{j+1}(x_j^2) = v^{j+1}(x_{j+1}^2)$$

where $x_{j+1}^1 \in \omega_j(x_j^1)$ and $x_{j+1}^2 \in \omega_j(x_j^2)$ are two nodes of $\mathcal{T}_{h_{j+1}}$, $\omega_j(x_j^1)$ and $\omega_j(x_j^2)$ being the supports of the nodal basis functions in V_{h_j} corresponding to the nodes x_j^1 and x_j^2, respectively. Consequently, we get

$$\begin{aligned}&(v^j - w^j)(x_j^1) - (v^j - w^j)(x_j^2) \leq \\ &(v^{j+1} - w^{j+1})(x_{j+1}^1) - (v^{j+1} - w^{j+1})(x_{j+1}^2)\end{aligned}$$

Repeating the above reasoning, we get that, for $k = j, \dots, J-1$, there exist $x_1^{k+1} \in \omega_k(x_1^k)$ and $x_2^{k+1} \in \omega_k(x_2^k)$ are two nodes of $\mathcal{T}_{h_{k+1}}$, $\omega_k(x_1^k)$ and $\omega_k(x_2^k)$ being the supports of the nodal basis functions in V_{h_k} corresponding to the nodes x_1^k and x_2^k, respectively, such that

$$\begin{aligned}&(v^k - w^k)(x_1^k) - (v^k - w^k)(x_2^k) \leq \\ &(v^{k+1} - w^{k+1})(x_1^{k+1}) - (v^{k+1} - w^{k+1})(x_2^{k+1})\end{aligned} \tag{3.13}$$

From (3.11), (3.12) and (3.13), we get

$$|v^j - w^j|_{1,\sigma,\omega_j(x_j)}^\sigma \leq C h_j^{d-\sigma}[(v-w)(x_J^1) - (v-w)(x_J^2)]^\sigma \tag{3.14}$$

Since the radius of ω_k is less than h_k, and in view of (3.1), it follows that $\text{dist}(x_j, x_J^1), \text{dist}(x_j, x_J^1) \leq h_j + (h_j + \ldots + h_{J-1}) \leq (1 + 1 + \frac{1}{\gamma} + \ldots + \frac{1}{\gamma^{J-1-j}}) h_j \leq \frac{2\gamma-1}{\gamma-1} h_j$. Therefore, if we write

$$\tilde{\omega}_j(x_j) = \bigcup_{\tau \in \mathcal{T}_{h_j},\ \text{dist}(x_j,\tau) \leq \frac{\gamma}{\gamma-1} h_j} \tau,$$

then $x_J^1, x_J^2 \in \tilde{\omega}_j(x_j)$. Subtracting and adding $(v-w)(x)$, $x \in \tilde{\omega}_j(x_j)$, in the right hand side of (3.14), integrating over $\tilde{\omega}_j(x_j)$, in view of Lemma 4.1 in [2], we have

$$\begin{aligned}&(\tfrac{2\gamma-1}{\gamma-1} h_j)^d |v^j - w^j|_{1,\sigma,\omega_j(x_j)}^\sigma \leq C h_j^{d-\sigma} \Big[||(v-w)(x_j^1) - (v-w)(x)||_{0,\sigma,\tilde{\omega}_j(x_j)}^\sigma + \\ &||(v-w)(x_j^2) - (v-w)(x)||_{0,\sigma,\tilde{\omega}_j(x_j)}^\sigma \Big] \leq \\ &C h_j^{d-\sigma} (2\tfrac{2\gamma-1}{\gamma-1} h_j)^\sigma C_{d,\sigma}(2\tfrac{2\gamma-1}{\gamma-1} h_j, h_J)^\sigma |v-w|_{1,\sigma,\tilde{\omega}_j(x_j)}^\sigma,\end{aligned}$$

ie.,

$$|v^j - w^j|_{1,\sigma,\omega_j(x_j)} \leq C C_{d,\sigma}(h_j, h_J) |v-w|_{1,\sigma,\tilde{\omega}_j(x_j)}$$

Finally, since the mesh $\mathcal{T}_{h_j}$ is regular and γ is independent of J and of the mesh parameters, then $\omega_j(x_j)$ and $\tilde{\omega}_j(x_j)$ contain a bounded number of simplexes of $\mathcal{T}_{h_j}$, which is also independent of J and of the mesh parameters. Consequently, we get (3.10). □

Another result we shall utilize is given by the following lemma.

Lemma 3.2. *For any $v, w \in V_{h_{j+1}}$, $j = J-1, \dots, 1$, we have*

$$||v - w - I_{h_j}v + I_{h_j}w||_{0,\sigma} \le Ch_jC_{d,\sigma}(h_j, h_{j+1})|v - w|_{1,\sigma} \tag{3.15}$$

Proof. As in the proof of the previous lemma, we denote by $\omega_j(x_j)$ the support of the nodal basis function ϕ_j in V_{h_j} corresponding to the node x_j of $\mathcal{T}_{h_j}$. For a $\tau \in \mathcal{T}_{h_j}$, we have

$$\begin{aligned} &||v - w - I_{h_j}v + I_{h_j}w||_{0,\sigma,\tau} = \\ &|| \sum_{x_j \text{ node of } \tau} [v - w - (I_{h_j}v - I_{h_j}w)(x_j)]\phi_j||_{0,\sigma,\tau} \le \\ &\sum_{x_j \text{ node of } \tau} ||v - w - (I_{h_j}v - I_{h_j}w)(x_j)||_{0,\sigma,\tau} \end{aligned}$$

From the definition of I_{h_j}, there exist two nodes of $\mathcal{T}_{h_{j+1}}$, $x^1_{j+1}, x^2_{j+1} \in \omega_j(x_j)$, such that $(I_{h_j}v)(x_j) = v(x^1_{j+1})$ and $(I_{h_j}w)(x_j) = w(x^2_{j+1})$. Therefore,

$$\begin{aligned} &||v - w - I_{h_j}v + I_{h_j}w||_{0,\sigma,\tau} \le \\ &\sum_{x_j \text{ node of } \tau} ||v - w - v(x^1_{j+1}) + w(x^2_{j+1})||_{0,\sigma,\omega_j(x_j)} \end{aligned}$$

Now, let $\omega_j(x_j)^+ = \{x \in \omega_j(x_j) : v - w - v(x^1_{j+1}) + w(x^2_{j+1}) \ge 0\}$ and $\omega_j(x_j)^- = \{x \in \omega_j(x_j) : v - w - v(x^1_{j+1}) + w(x^2_{j+1}) \le 0\}$. From the above equation, the definition of I_{h_j} and Lemma 4.1 in [2], we get

$$\begin{aligned} &||v - w - I_{h_j}v + I_{h_j}w||_{0,\sigma,\tau} \le \\ &\sum_{x_j \text{ node of } \tau} \Big[||v - w - v(x^1_{j+1}) + w(x^1_{j+1})||^\sigma_{0,\sigma,\omega_j(x_j)^+} + \\ &||w - v - w(x^2_{j+1}) + v(x^2_{j+1})||^\sigma_{0,\sigma,\omega_j(x_j)^-}\Big]^{1/\sigma} \le \\ &Ch_jC_{d,\sigma}(h_j, h_{j+1}) \sum_{x_j \text{ node of } \tau} \Big[|v - w|^\sigma_{1,\sigma,\omega_j(x_j)^+} + |w - v|^\sigma_{1,\sigma,\omega_j(x_j)^-}\Big]^{1/\sigma} = \\ &Ch_jC_{d,\sigma}(h_j, h_{j+1}) \sum_{x_j \text{ node of } \tau} |v - w|_{1,\sigma,\omega_j(x_j)} \end{aligned}$$

Since the mesh $\mathcal{T}_{h_j}$ is regular, $\omega_j(x_j)$ contains a bounded number of simplexes of $\mathcal{T}_{h_j}$, which is independent of J and of the mesh parameters. Consequently, inequality (3.15) can be obtained from the above equation. $\square$

Now, we consider a decomposition of $\varphi = \varphi_J + \ldots + \varphi_1$ with $\varphi_j \in V_{h_j}$, $j = J, \ldots, 1$, and define

$$K_j = \{v \in V_{h_j} : \varphi_j \leq v\},\ j = J, \ldots, 1 \tag{3.16}$$

In this way, we get a decomposition of K as in (2.1). For a $v \in K$, with the notation in (3.8), we write

$$\begin{aligned} v_j &= \varphi_j + (v-\varphi)^j - (v-\varphi)^{j-1},\ j = J, \ldots, 2 \\ v_1 &= \varphi_1 + (v-\varphi)^1 \end{aligned} \tag{3.17}$$

Evidently,

$$v_j \in K_j,\ j = J, \ldots, 1, \text{ and } v = v_J + \ldots + v_1 \tag{3.18}$$

We have the following

Lemma 3.3. *If $v_j, w_j \in K_j$, $j = J, \ldots, 1$, are defined as in (3.17) for some $v, w \in K$, respectively, then*

$$|v_j - w_j|_{1,\sigma} \leq C C_{d,\sigma}(h_{j-1}, h_J)|v - w|_{1,\sigma} \tag{3.19}$$

and

$$||v_j - w_j||_{0,\sigma} \leq C h_{j-1} C_{d,\sigma}(h_j, h_J)|v - w|_{1,\sigma} \tag{3.20}$$

where we take $h_0 = h_1$ for $j = 1$.

Proof. For $j = J, \ldots, 2$, in view of (3.10), we have

$$\begin{aligned} &|v_j - w_j|_{1,\sigma} = |(v-\varphi)^j - (v-\varphi)^{j-1} - (w-\varphi)^j + (w-\varphi)^{j-1}|_{1,\sigma} \leq \\ &C[C_{d,\sigma}(h_j, h_J) + C_{d,\sigma}(h_{j-1}, h_J)]|v - w|_{1,\sigma} \end{aligned}$$

ie., (3.19) holds for $j = J, \ldots, 2$. Also, by a similar proof, we get that (3.19) for $j = 1$. Now, using (3.15) and (3.10), for $j = J, \ldots, 2$, we get

$$\begin{aligned} &||v_j - w_j||_{0,\sigma} = \\ &||(v-\varphi)^j - I_{h_{j-1}}(v-\varphi)^j - (w-\varphi)^j + I_{h_{j-1}}(w-\varphi)^j||_{0,\sigma} \leq \\ &C h_{j-1} C_{d,\sigma}(h_{j-1}, h_j)|(v-\varphi)^j - (w-\varphi)^j|_{1,\sigma} \leq \\ &C h_{j-1} C_{d,\sigma}(h_{j-1}, h_j) C_{d,\sigma}(h_j, h_J)|v - w|_{1,\sigma} \end{aligned}$$

and therefore, (3.20) holds for $j = J, \ldots, 2$. For $j = 1$, from the classical Friedrichs-Poincaré inequality and (3.10), we have

$$\begin{aligned} &||v_1 - w_1||_{0,\sigma} = ||(v-\varphi)^1 - (w-\varphi)^1||_{0,\sigma} \leq \\ &C h_1 |(v-\varphi)^1 - (w-\varphi)^1|_{1,\sigma} \leq C h_1 C_{d,\sigma}(h_1, h_J)|v - w|_{1,\sigma}, \end{aligned}$$

ie., we obtained (3.20) for $j = 1$. □

To prove that Assumption 2.1 holds, we associate to the decomposition $\{\Omega_j^i\}_{1\le i\le I_j}$ of Ω_j, some functions $\theta_j^i \in C(\bar{\Omega}_j)$, $\theta_j^i|_\tau \in P_1(\tau)$ for any $\tau \in \mathcal{T}_{h_j}$, $i = 1, \cdots, I_j$, such that

$$\begin{aligned}&0 \le \theta_j^i \le 1 \text{ on } \Omega_j,\\ &\theta_j^i = 0 \text{ on } \cup_{l=i+1}^{I_j} \Omega_j^l \backslash \Omega_j^i,\ \theta_j^i = 1 \text{ on } \Omega_j^i \backslash \cup_{l=i+1}^{I_j} \Omega_j^l\end{aligned} \tag{3.21}$$

Also, for Assumption 2.2, we associate a unity partition to each domain decomposition $\{\Omega_j^i\}_{1\le i\le I_j}$, $j = J, \ldots, 1$,

$$0 \le \theta_j^i \le 1 \text{ and } \sum_{i=1}^{I_j} \theta_j^i = 1 \text{ on } \Omega_j \tag{3.22}$$

with $\theta_j^i \in C(\bar{\Omega}_j)$, $\theta_j^i|_\tau \in P_1(\tau)$ for any $\tau \in \mathcal{T}_{h_j}$, $i = 1, \cdots, I_j$. Such functions θ_j^i with the above properties exist (see [2] or [26] p. 59, for instance). Moreover, since the overlapping size of the domain decomposition on a level $j = J, \ldots, 1$ is δ_j, the above functions θ_j^i can be chosen to satisfy

$$|\partial_{x_k} \theta_j^i| \le C/\delta_j, \text{ a.e. in } \Omega_j, \text{ for any } k = 1, \ldots, d \tag{3.23}$$

Finally, we recall some interpolation properties. For a $v \in V_{h_j}$ and a continuous functions θ which is of polynomial form on the elements of $\tau \in \mathcal{T}_{h_j}$, we have (see [9] and [28]),

$$||\theta v - L_{h_j}(\theta v)||_{0,\sigma} \le C h_j |\theta v|_{1,\sigma} \text{ and } |L_{h_j}(\theta v)|_{1,\sigma} \le C|\theta v|_{1,\sigma}$$

where L_{h_j} is the P_1-Lagrangian interpolation operator which uses the function values at the nodes of the mesh $\mathcal{T}_{h_j}$. Therefore, we have

$$||L_{h_j}(\theta v)||_{1,\sigma} \le C||\theta v||_{1,\sigma} \tag{3.24}$$

Now, we can prove

Proposition 3.1. *Assumption 2.1 holds with the constants C_2 and C_3 are given in (3.29) for the convex sets K_j, $j = J, \ldots, 1$, defined in (3.16).*

Proof. Let us consider $v,\ w \in K$ and let $v_j, w_j \in K_j$, $j = J, \ldots, 1$, be their decompositions defined as in (3.17), respectively. Also, let $w_{ji} \in V_{h_j}^i$

such that $w_j + w_{j1} + \ldots + w_{ji} \in K_j$, $j = J, \ldots, 1$, $i = 1, \ldots, I_j$. Now, for $j = J, \ldots, 2$, we define

$$v_{j1} = L_{h_j}(\theta_j^1(v_j - w_j) + (1 - \theta_j^1)w_{j1})$$
$$v_{ji} = L_{h_j}(\theta_j^i((v_j - w_j) - \sum_{l=1}^{i-1} v_{jl}) + (1 - \theta_j^i)w_{ji}), \quad i = 2, \ldots, I_j$$

with θ_j^i in (3.21). Like in Proposition 3.1 in [2], where we take $v = v_j$ and $w = w_j$, we can prove that

$$\begin{aligned} & v_{ji} \in V_{h_j}^i, \quad w_j + w_{j1} + \ldots + w_{ji-1} + v_{ji} \in K_j, \ i = 1, \ldots, I_j \\ & v_j - w_j = \sum_{i=1}^{I_j} v_{ji} \end{aligned} \tag{3.25}$$

We point out that here, the condition $w_j + w_{j1} + \ldots + w_{ji-1} + v_{ji} \in K_j$ can be proved by verifying that it is satisfied only at the nodes of $\mathcal{T}_{h_j}$. At the level $j = 1$, we do not have a domain decomposition, $I_1 = 1$, and we take

$$v_{11} = v_1 - w_1.$$

From this equation, (3.18) and (3.25), we get that the first two conditions of Assumption 2.1 are satisfied.

We estimate now the constants C_2 and C_3. Using Lemma 3.3 and the same techniques as in [2] or [4] (see [6] for details), we can write

$$||v_{ji}||_{1,\sigma}^\sigma \leq C I^\sigma \left\{ C_{d,\sigma}(h_{j-1}, h_J)^\sigma |u - w|_{1,\sigma}^\sigma + \sum_{k=1}^{I_j} |w_{jk}|_{1,\sigma}^\sigma \right\} \tag{3.26}$$

for any $j = J, \ldots, 2$ and $i = 1, \ldots, I_j$. At the level $j = 1$, from Lemma 3.3, we have

$$||v_{11}||_{1,\sigma}^\sigma \leq C C_{d,\sigma}(h_1, h_J)^\sigma |v - w|_{1,\sigma}^\sigma \tag{3.27}$$

From (3.26) and (3.27), we get

$$\begin{aligned} \sum_{j=1}^{J} \sum_{i=1}^{I_j} ||v_{ji}||_{1,\sigma}^\sigma \leq C I^{\sigma+1} \Bigg\{ & \sum_{j=2}^{J} \sum_{i=1}^{I_j} |w_{ji}|_{1,\sigma}^\sigma + \\ & \left[\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^\sigma \right] |u - w|_{1,\sigma}^\sigma \Bigg\} \end{aligned} \tag{3.28}$$

Consequently, from (3.28), we get that the constants C_2 and C_3 can be written as

$$C_2 = CI^{\frac{\sigma+1}{\sigma}} \left[\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^\sigma \right]^{\frac{1}{\sigma}} \text{ and } C_3 = CI^{\frac{\sigma+1}{\sigma}} \tag{3.29}$$

□

Concerning Assumption 2.2 we have

Proposition 3.2. *Assumption 2.2 holds with the constants C_2 and C_3 are given in (3.32) for the convex sets K_j, $j = J, \dots, 1$, defined in (3.16).*

Proof. Let us consider v, $w \in K$, and let $v_j, w_j \in K_j$, $j = J, \dots, 1$, be their decompositions defined as in (3.17), respectively. Now, we define

$$\begin{aligned} & v_{ji} = L_{h_j}(\theta_j^i(v_j - w_j)),\ i = 1, \dots, I_j,\ \text{for } j = J, \dots, 2, \\ & \text{and } v_{11} = v_1 - w_1 \end{aligned} \tag{3.30}$$

with θ_j^i in (3.22). In view of (3.18) and (3.30), we get that the first two conditions of Assumption 2.2 hold.

We estimate now the constants C_2 and C_3. For $j = J, \dots, 2$, from (3.23) and (3.24), we get

$$||v_{ji}||_{1,\sigma}^\sigma \le C(|v_j - w_j|_{1,\sigma}^\sigma + (1 + \frac{1}{\delta_j})^\sigma ||v_j - w_j||_{0,\sigma}^\sigma)$$

Using this equation, the proof is similar with that of the previous proposition. For $j = J, \dots, 2$, in view of (3.19) and (3.20), we have

$$||v_{ji}||_{1,\sigma}^\sigma \le CC_{d,\sigma}(h_{j-1}, h_J)^\sigma |v - w|_{1,\sigma}^\sigma$$

and we use (3.27) for the estimation of $||v_{11}||_{1,\sigma}$. From these equations, we get

$$\sum_{j=1}^{J} \sum_{i=1}^{I_j} ||v_{ji}||_{1,\sigma}^\sigma \le CI \left[\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^\sigma \right] |v - w|_{1,\sigma}^\sigma \tag{3.31}$$

Consequently, the constants C_2 and C_3, can be written as

$$C_2 = CI^{\frac{1}{\sigma}} [\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^\sigma]^{\frac{1}{\sigma}} \text{ and } C_3 = 0 \tag{3.32}$$

□

The constants C_1 and β_{jk}, $j,\ k = J, \dots, 1$, can be taken as in (2.3) and (2.7), but better choices are available in the case of the multigrid methods in the next section. As we see form the above estimations, the convergence rates given in Theorem 2.1 depend on the functional F, the maximum number of the subdomains on each level, I, and the number J of levels. The number of subdomains on levels can be associated with the number of colors needed to mark the subdomains such that the subdomains with the same color do not intersect with each other. Since this number of colors depends in general on the dimension of the Euclidean space where the domain lies, we can conclude that our convergence rate essentially depends on the number J of levels.

We first estimate the constants C_1–C_3 as functions of J. To this end, in the remainder of this section, C will be a generic constant which does not depend on J. Writing $S_{d,\sigma}(J) = \left[\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^\sigma\right]^{\frac{1}{\sigma}}$ from (3.1) and (3.9), we can consider

$$S_{d,\sigma}(J) = \begin{cases} (J-1)^{\frac{1}{\sigma}} & \text{if } d = \sigma = 1 \\ & \text{or } 1 \le d < \sigma < \infty \\ CJ & \text{if } 1 < d = \sigma < \infty \\ C^J & \text{if } 1 \le \sigma < d < \infty \end{cases} \tag{3.33}$$

in our estimations. In this general framework, we take C_1, and β_{jk}, $j,\ k = J, \dots, 1$, as in (2.3) and (2.7),

$$C_1 = CJ^{\frac{\sigma-1}{\sigma}} \text{ and } \max_{k=1,\cdots,J} \sum_{j=1}^{J} \beta_{kj} = J \tag{3.34}$$

Also, from (3.29) and (3.32), we get

$$C_2 = CS_{d,\sigma}(J) \text{ and } C_3 = \begin{cases} C & \text{for Algorithms 2.1 and 2.3} \\ 0 & \text{for Algorithms 2.2 and 2.4} \end{cases} \tag{3.35}$$

As a consequence of Theorem 2.1 and Propositions 3.1 and 3.2 we have

Corollary 3.1. *Let us consider the finite element spaces V_{h_j} defined in (3.4) which are associated with the levels $j = 1, \dots, J$, and their subspaces $V_{h_j}^i$, $i = 1, \dots, I_j$, given in (3.5), which are associated with the level domain decompositions. Also, let K be the closed convex subset of $V = V_J$ given in (3.7), which is decomposed as a sum of the level closed convex sets $K_j \subset V_{h_j}$,*

$j = J, \dots, 1$, defined in (3.16). If F is a Gâteaux differentiable functional on V which is supposed to be coercive and to satisfy (2.4), then the approximation sequences u^n, $n \geq 0$ obtained from Algorithms 2.1–2.4 converge to the solution u of the one-obstacle problem (3.6) and the error estimations in Theorem 2.1 hold. The constants $\tilde{C}_1$ and $\tilde{C}_2$ in these error estimations depend on the number of levels J through the constants C_1–C_3 given in (3.34) and (3.35).

Remark 3.2. 1) The results of this section have referred to problems in $W^{1,\sigma}$ with Dirichlet boundary conditions, and the functions corresponding to the coarse levels have been extended with zero outside the domains Ω_j, $j = J-1, \dots, 1$. Let us assume that the problem has mixed boundary conditions: $\partial\Omega_J = \Gamma_d \cup \Gamma_n$, with Dirichlet conditions on Γ_d and Neumann conditions on Γ_n. In this case, if a node of $\mathcal{T}_{h_j}$, $j = J-1, \dots, 1$, lies in $\mathrm{Int}(\Gamma_n)$, we have to assume that all the sides of the elements $\tau \in \mathcal{T}_{h_j}$ having that node are included in Γ_n.

2) Similar convergence results with those ones presented in this section can be obtained for problems in $(W^{1,s})^d$.

4 Multigrid methods

In the above multilevel methods a mesh is the refinement of that one on the previous level, but the domain decompositions are almost independent from one level to another. We obtain similar multigrid methods by decomposing the level domains by the supports of the nodal basis functions. Consequently, the subspaces $V_{h_j}^i$, $i = 1, \dots, I_j$, are one-dimensional spaces generated by the nodal basis functions associated with the nodes of $\mathcal{T}_{h_j}$, $j = J, \dots, 1$. In this case Algorithms 2.1–2.4 are V-cycle multigrid iterations in which the smoothing steps are performed by a combination of multiplicative methods with additive ones. Evidently, similar results can be given for the W-cycle multigrid iterations.

In this section, we show that the estimations given in (2.3) and (2.7) for the constants C_1 and β_{jk}, j, $k = J, \dots, 1$ can be improved in the case of the multigrid methods. Finally, we summarize the previous results by writing the convergence rates of the four algorithms as functions of the number J of the levels, for the varied values of the constants p, q, σ and d.

Concerning the constants β_{jk}, $j, k = J, \dots, 1$, in (2.6), we can prove (see [4] or [5], for instance) that, in the case of the multigrid methods, there exist such constants such that

$$\max_{k=1,\ldots,J} \sum_{j=1}^{J} \beta_{kj} = C$$

where C is a constant independent of the meshes and their number. Also, the constant C_1 in (2.2) is estimated in Lemma 4.1 in [4] or [5],

$$C_1 = (n!)^{\frac{1}{\sigma}} C^{\frac{n-1}{n}} \left(I \frac{\gamma^{\frac{d}{n}}}{\gamma^{\frac{d}{n}} - 1} \right)^{\frac{n-1}{\sigma}}$$

where $n \in \mathbf{N}$, $n-1 < \sigma \leq n$, and C is a constant independent of the meshes and their number.

Now, we shall write the convergence rate of the multigrid Algorithms 2.1–2.4 in function of the number J of levels. To this end, we write the error estimations in Theorem 2.1 of the four algorithms using the above estimations of C_1 and $\max_{k=J,\ldots,1} \sum_{j=1}^{J} \beta_{kj}$, and C_2 and C_3 given in (3.35). In order to be more conclusive, we limit ourselves to a typical example where

$$F(v) = \frac{1}{\sigma} ||v||_{1,\sigma}^{\sigma} - L(v), \quad v \in W^{1,\sigma}(\Omega) \tag{4.1}$$

where L is a linear and continuous functional on $W^{1,\sigma}(\Omega)$, $\sigma > 1$. In this case (see [1], for instance),

$$p = 2,\ q = \sigma \text{ if } \sigma < 2; \quad p = 2,\ q = 2 \text{ if } \sigma = 2; \quad p = \sigma,\ q = 2 \text{ if } \sigma > 2$$

Evidently, we can use the same procedure for other problems, too.

For $\sigma = 2$ and $p = q = 2$, in view of (2.30), (2.23) and (3.35), we get

$$\tilde{C}_1(J) = \begin{cases} CS_{d,2}(J)^2 & \text{for Algorithms 2.1 and 2.2} \\ CJS_{d,2}(J)^2 & \text{for Algorithms 2.3 and 2.4} \end{cases} \tag{4.2}$$

and, from Theorem 2.1, we have

$$||u^n - u||_{1,2}^2 \leq \tilde{C}_0 \left(1 - \frac{1}{1 + \tilde{C}_1(J)} \right)^n \tag{4.3}$$

where $\tilde{C}_0$ is a constant independent of J.

For $1 < q = \sigma < 2$ and $p = 2$, in view of (2.32), (2.23) and (3.35), we get

$$\tilde{C}_3(J) = \begin{cases} CJ^{\frac{(\sigma-1)(2-\sigma)}{\sigma}} S_{d,\sigma}(J)^2 & \text{for Algorithms 2.1 and 2.2} \\ CJ^{\frac{2(\sigma-1)}{\sigma}} S_{d,\sigma}(J)^2 & \text{for Algorithms 2.3 and 2.4} \end{cases} \tag{4.4}$$

From Theorem 2.1, we get that

$$||u^n - u||^2_{1,\sigma} \le \tilde{C}_0 \frac{1}{\left(1 + n\tilde{C}_2(J)\right)^{\frac{\sigma-1}{2-\sigma}}} \tag{4.5}$$

where, in view of (2.34), we can take

$$\tilde{C}_2(J) = \frac{1}{1 + \tilde{C}_3(J)^{\frac{1}{\sigma-1}}} \tag{4.6}$$

For $p = \sigma > 2$ and $q = 2$, we get

$$\tilde{C}_3(J) = \begin{cases} CJ^{\frac{\sigma-2}{\sigma-1}} S_{d,\sigma}(J)^{\frac{\sigma}{\sigma-1}} & \text{for Algorithms 2.1 and 2.2} \\ CJS_{d,\sigma}(J)^{\frac{\sigma}{\sigma-1}} & \text{for Algorithms 2.3 and 2.4} \end{cases} \tag{4.7}$$

Finally, in this case, we have

$$||u^n - u||^\sigma_{1,\sigma} \le \tilde{C}_0 \frac{1}{\left(1 + n\tilde{C}_2(J)\right)^{\frac{1}{\sigma-2}}} \tag{4.8}$$

where

$$\tilde{C}_2(J) = \frac{1}{1 + \tilde{C}_3(J)^{\sigma-1}} \tag{4.9}$$

We make now some remarks on the above error estimations of the four algorithms. First, we point out that the above convergence results give global rate estimations. As we have expected, the multiplicative (over the levels) Algorithms 2.1 and 2.2 converge better than their additive variants, Algorithms 2.3 and 2.4. For the complementarity problems, we can compare the convergence rates of the four multigrid algorithms with the similar ones in the literature. In this case, $p = q = \sigma = d = 2$ in the above example, from (4.3) and (4.2), we get that the convergence rate of Algorithms 2.1 and 2.2 is of $1 - \frac{1}{1+CJ^2}$, and that of Algorithms 2.3 and 2.4 is of $1 - \frac{1}{1+CJ^3}$. These convergence rates are better, with a factor J, than those of the similar algorithms introduced in [4], which are of $1 - \frac{1}{1+CJ^3}$ and $1 - \frac{1}{1+CJ^4}$, respectively.

For the truncated monotone multigrid method, an asymptotic convergence rate of $1-\frac{1}{1+CJ^4}$, and under some conditions, of $1-\frac{1}{1+CJ^3}$, is found in [16] and [13]. An estimate of $1-\frac{1}{1+CJ^3}$ is also obtained in [16] for the asymptotic convergence rate of the standard monotone multigrid methods. In [13], it is mentioned that this asymptotic rate may be of $1-\frac{1}{1+CJ^2}$, or even of $1-\frac{1}{1+CJ}$, under some conditions.

Acknowledgment. This work has been supported by the CNCSIS under the contract ID-PCE nr. 566/2009.

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