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THE CLASSICAL MAXIMUM PRINCIPLE. SOME OF ITS EXTENSIONS AND APPLICATIONS.*

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Abstract

The intention of this paper is to survey some extensions (the P function method) and applications of the classical maximum principle for elliptic operators.

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1 Introduction

The intention of this paper is to survey some extensions (the P function method) and applications of the classical maximum principle for elliptic operators.

The maximum principle is one of the most useful and best known tools employed in the study of partial differential equations. The maximum principle enables us to obtain information about the uniqueness, approximation,

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boundedness and symmetry of the solution, the bounds for the first eigenvalue, for quantities of physical interest (maximum stress, the torsional stiffness, electrostatic capacity, charge density etc), the necessary conditions of solvability for some boundary value problems, etc.

The first chapter specializes the maximum principle for partial differential equations to the one variable case. We present the one dimensional classical maximum principle and a new extension.
In chapter two, we present the classical maximum principle of Hopf for elliptic operators and some possible extensions (the P function method (in honour of L. Payne, see [43]) and give a number of applications.

The maximum principle occurs in so many places and in such varied forms that is impossible to treat all topics. We treat here only the classical maximum principle and one of its extensions, namely the P function method for the elliptic case.

2 The one dimensional case

The one dimensional maximum principle represents a generalization of the following simple result: *Let the smooth function u satisfy the inequality $u'' \geq 0$ in $\Omega = (\alpha, \beta)$. Then the maximum of u in Ω occurs on $\partial\Omega = \{\alpha, \beta\}$ (on the boundary of Ω)*, i.e.,

$$\max_{\overline{\Omega}} u = \max\{u(\alpha), u(\beta)\}.$$

Theorem 1. *(one dimensional weak maximum principle) Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a nonconstant function satisfying $\mathrm{L}u \equiv u'' + b(x)u' \geq 0$ in Ω, with b bounded in closed subintervals of Ω. Then,*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Drawing the graph of a function u satisfying $u'' \geq 0 \quad (u'' \neq 0)$ reveals us the interesting fact that at a point on $\partial\Omega$ (where u attains its maximum), the slope of u is nonzero. More precisely, $du/dn > 0$ at such a point. Here d/dn denotes the outward derivative on $\partial\Omega$, i.e.,

$$\frac{du}{dn}(\alpha) = -u'(\alpha), \ \frac{du}{dn}(\beta) = u'(\beta).$$

The next theorem is an extension of this result:

Theorem 2. *(one dimensional strong maximum principle) Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a nonconstant function satisfying* $\mathrm{L}u \equiv u''+b(x)u'+c(x)u \geq 0$ *in Ω, with b and c bounded in closed subintervals of Ω and $c \leq 0$ in Ω. Then a nonnegative maximum can occur only on $\partial\Omega$, and $du/dn > 0$ there. If $c \equiv 0$ in Ω then, u takes its maximum on $\partial\Omega$ and $du/dn > 0$ there.*

The following simple counterexample shows that we have to impose some restrictions to c: The function $u(x) = e^{-x} \sin x$ satisfies

$$\mathrm{L}u \equiv u'' + 2u' + 3u \geq 0 \text{ in } \Omega = (0, \pi).$$

We see that the nonnegative u vanishes on $\partial\Omega$ and hence there can be no maximum principle. A result can still be proven if $c \geq 0$. The result is a version of Theorem 5 on page 9 in [65].

Theorem 3. *(one dimensional generalized maximum principle) Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a nonconstant function satisfying* $\mathrm{L}u \equiv u''+c(x)u \geq 0$ *in Ω. Suppose that*

$$\sup_{\Omega} c < \frac{\pi^2}{(\operatorname{diam}\Omega)^2}. \tag{1}$$

Then, the function u/w_ε cannot attain a nonnegative maximum in Ω unless it is a constant. $\operatorname{diam}\Omega$ *represents the diameter of Ω and*

$$w_\varepsilon = \cos \frac{\pi(2x - \operatorname{diam}\Omega)}{2(\operatorname{diam}\Omega + \varepsilon)} \cosh(\varepsilon x),$$

where $\varepsilon > 0$ is small.

The proof follows from Theorem 5, page 9 in [65] and Lemma 2.1. [11]. Although our result is stated only for a particular operator L ($b \equiv 0$), is it more precise than the result stated for general operators $\mathrm{L}u \equiv u'' + b(x)u' + c(x)u$ (see Theorem 5, page 9 in [65]). The authors do not indicate when a maximum principle is valid. They state that a maximum principle is valid for " any sufficiently short interval Ω ".

The proofs of these theorems as well as their applications (uniqueness of the solution of the boundary value problem, approximation in boundary value problems, the classical Sturm- Liouville theory, existence for nonlinear equations via monotone methods) can be found in the excellent book of Protter and Weinberger [65].

Certain solutions of equations of higher order exhibit a maximum principle:

Theorem 4. *Let $1 \leq k \leq n-1$, $n \geq 2$ and $u \in C^n(\overline{\Omega})$ be a nonconstant function satisfying* $\mathrm{L}u \equiv u^n \geq 0$ *in* Ω. *Suppose that*

$$(-1)^{n-k}u^{(i)}(\alpha) \geq 0,\ i = 1, \dots, k-1\ (\textit{if such } i \textit{ exist}),$$

$$(-1)^{n-k+j}u^{(j)}(\beta) \geq 0,\ j = 1, \dots, n-k-1\ (\textit{if such } j \textit{ exist}).$$

Then, in the case $n-k$ even u attains its minimum value and in case $n-k$ odd u attains its maximum either at α or β.

The nontrivial proof is given in [76]. For n=4, k=2, Theorem 4 generalizes the maximum principle in [6]: *Let u satisfy the inequality $u^{(4)} \leq 0$ in Ω. If $u'(\alpha) \leq 0$, $u'(\beta) \geq 0$, then u attains its maximum at α or β.*
A maximum principle for general fourth order operators appears in [41].

3 The n dimensional case

In this section, we treat the n dimensional variants of results presented in section 1, some possible extensions for nonlinear equations and for equations of higher order as well as their applications.

We consider the linear operator (summation convention is assumed, i.e., summation from 1 to n is understood on repeated indices)

$$\mathrm{L}u = a^{ij}(x)u_{ij} + b^i(x)u_i + c(x)u,\ a^{ij}(x) = a^{ji}(x),$$

where $x = (x_1, ..., x_n) \in \Omega$, Ω is a bounded domain (unless otherwise stated) of $\mathbb{R}^n$, $n \geq 1$ and $u_i = \frac{\partial u}{\partial x_i}, u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$.
The operator L is called elliptic at a point $x \in \Omega$ if the matrix $[a^{ij}(x)]$ is positive, i.e., if $\lambda(x)$ and $\Lambda(x)$ denote respectively the minimum and maximum eigenvalues of $[a^{ij}(x)]$, then

$$0 < \lambda(x)|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda(x)|\xi|^2,$$

for all $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n - \{0\}$. If $\lambda \geq 0$, then L is called elliptic in Ω. If Λ/λ is bounded in Ω, we shall call L uniformly elliptic in Ω.

Theorem 5. *(weak maximum principle) ([25]). Let* L *be elliptic in* Ω*. Suppose that* $|b^i|/\lambda < +\infty$ *in* Ω*,* $i = 1, ..., n$*. If* $\mathrm{L}u \geq 0$ *in* Ω*,* $c = 0$ *in* Ω *and* $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$*, then the maximum of* u *in* $\overline{\Omega}$ *is achieved on* $\partial\Omega$*, that is:*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u. \tag{2}$$

Remarks: 1). Theorem 5 holds under the weaker hypothesis: the matrix $[a^{ij}]$ is nonnegative and the ratio $|b^k|/a^{kk}$ is locally bounded for some $k \in \{1, ..., n\}$.

2). The maximum principle for subharmonic functions goes back to Gauss (1838) ([17]). The first proof of a maximum principle for operators more general than the Laplace operator was proved in two dimensions by Paraf in 1892 ([42]).

Theorem 6. *(the strong maximum principle of E. Hopf) ([30]). Let* L *be uniformly elliptic,* $c = 0$ *and* $\mathrm{L}u \geq 0$ *in* Ω *(not necessarily bounded), where* $u \in C^2(\Omega)$*. Then, if* u *attains its maximum in the interior of* Ω*, then* u *is constant. If* $c \leq 0$ *and* c/λ *is bounded then* u *cannot attain a nonnegative maximum in the interior of* Ω*, unless* u *is constant.*

The proof is a consequence of the following useful result known as Hopf's lemma [30]:

Lemma 1. *Suppose that* L *is uniformly elliptic in* Ω*,* $c = 0$ *in* Ω *and* $\mathrm{L}u \geq 0$ *in* Ω*. Let* $x_0 \in \partial\Omega$ *be such that*
i) u *is continuous at* x_0*,*
ii) $u(x_0) > u(x)$ *for all* $x \in \Omega$*,*
iii) $\partial\Omega$ *satisfies an interior sphere condition at* x_0 *(i.e., there exists a ball* $B \subset \Omega$ *with* $x_0 \in \partial B$*).*
Then the outer normal derivative of u *at* x_0*, if it exists, satisfies the strict inequality*

$$\frac{\partial u}{\partial n}(x_0) > 0. \tag{3}$$

If $c \leq 0$ *and* c/λ *is bounded in* Ω*, then the same conclusion holds provided* $u(x_0) \geq 0$*, and if* $u(x_0) = 0$ *then, the same conclusion holds irrespective of the sign of* c*.*

We now restrict ourselves to the case $b^i \equiv 0$ and prove Danet, [11]:

Theorem 7. *(generalized maximum principle) Let* $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *satisfy the inequality* $\mathrm{L}u \equiv \Delta u + c(x)u \geq 0$, *where* $c \geq 0$ *in* Ω. *Suppose that*

$$\sup_{\Omega} c < C_1 = \frac{4n+4}{(\operatorname{diam}\Omega)^2}. \tag{4}$$

Then, the function u/w_1 *cannot attain a nonnegative maximum in* Ω, *unless it is a constant.*

Similarly, if Ω *lies in a slab of width* d *and*

$$\sup_{\Omega} c < C_2 = \frac{\pi^2}{d^2}, \tag{5}$$

we obtain a similar result for u/w_2. *Here*

$$w_1(x) = 1 - (\sup_{\Omega} c/2n)(x_1^2 + \cdots + x_n^2)$$

and

$$w_2 = \cos \frac{\pi(2x_i - d)}{2(d+\varepsilon)} \prod_{j=1}^{n} \cosh(\varepsilon x_j),$$

for some $i \in \{1, \ldots, n\}$, *where* $\varepsilon > 0$ *is small.*

Comments

1. A broad class of domains satisfy $\Omega \subset B_{\mathrm{diam}\Omega/2}$. For these domains C_1 may be replaced by $C_3 = 8n/(\mathrm{diam}\Omega)^2$.

2. We may improve the constant C_3 (i.e., choose a larger constant) if $\Omega = \{x \in \mathbb{R}^n \mid 0 < R <| x |< R + \varepsilon\}$, where $\varepsilon > 0$ is sufficiently small. A maximum principle holds if

$$\sup_{\Omega} c < C_4 = \frac{2(n-1)}{(\varepsilon + \delta)\mathrm{diam}\Omega}. \tag{6}$$

For sufficiently small ε we have $C_4 > C_3$.

3. A similar result was given in [65], Theorem 10, p.73. for general operators. The authors proved that if

$$\sup_{\Omega} \gamma < \frac{4}{d^2 e^2}, \tag{7}$$

then a similar maximum principle is valid. Here $\mathrm{L}u \equiv \Delta u + c(x)u$, $c \geq 0$ in Ω and Ω is supposed to lie in a strip of width d. Of course, Theorem 7 (valid only for the case $b^i \equiv 0, c > 0$) is sharper that their result, but does not hold for general operators.

4. We have to impose some restrictions to c. Otherwise, as the following example shows, the maximum principle (Theorem 7) is false. The function $u(x,y) = \sin x \sin y$ satisfies $u = 0$ on $\partial\Omega$ and is solution of the equation $\Delta u + 2u = 0$ in $\Omega = (0,\pi) \times (0,\pi)$. Of course, (4) does not hold.

The maximum principles that we have presented above are valid only for the class $C^2(\Omega) \cap C^0(\overline{\Omega})$, i.e., the results are valid for classical solutions. We may consider operators L of the divergence form

$$\mathrm{L}u \equiv (a^{ij}(x)_j u + b^i(x)u)_i + c^i(x)u_i + d(x)u,$$

whose coefficients $a^{ij},\ b^i,\ c^i,\ d,\ i,j = 1,2,...,n$ are assumed to be measurable functions on a domain $\Omega \subset \mathbb{R}^n$.

The divergence form has the advantage that the operator L may be defined for a significant broader class of functions than the class $C^2(\Omega)$.

Assume that u is weakly differentiable and that $a^{ij}D_j u + b^i u$ and that $c_i D_i u + du,\ i = 1,2,...,n$ are locally integrable. Then u satisfies in a weak sense $\mathrm{L}u = 0\,(\geq 0,\ \leq 0)$ in Ω if :

$$\mathcal{L}(u,\varphi) = \int_\Omega \left[(a^{ij}u_j + b^i u)\varphi_i - (c^i u_i + du)\varphi\right] dx = 0\,(\leq 0,\ \geq 0),$$

for all non-negative $\varphi \geq \in C_0^1(\Omega)$.
We shall assume that L is strictly elliptic in Ω and that L has bounded coefficients, i.e. there exists some constants Λ and $\nu \geq 0$ such that:

$$\sum_{i,j} |a^{ij}|^2 \leq \Lambda,\ \lambda^{-2} \sum_i \left(|b^i|^2 + |c^i|^2\right) + \lambda^{-1}|d| \leq \nu^2. \tag{8}$$

is valid in Ω.

We state now the weak maximum principle for weak solutions.

Theorem 8. *([4], [25]) Let $u \in W^{1,2}(\Omega) \cap C^0(\overline{\Omega})$ satisfy $\mathrm{L}u \geq 0$ in Ω. If*

$$\int_\Omega (d\varphi - b^i\varphi_i)dx \leq 0,\ \forall\ \varphi \geq 0,\ \varphi \in C_0^1(\Omega) \tag{9}$$

then,

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+.$$

Here $u^+ = \max\{u, 0\}$.

Extensions and application of this result are presented in the book of Gilbarg and Trudinger [25].

We now deal with a possible extension of the maximum principle, namely the P function method. The method consists in determining a function $\mathrm{P} = \mathrm{P}(x, u, \nabla u, \dots)$, satisfying a maximum principle, i.e.,

$$\max_{\overline{\Omega}} \mathrm{P} = \max_{\partial\Omega} \mathrm{P},$$

where u is a solution of the studied equation (boundary value problem). This powerful method has many applications of interest and represents the core of the paper.

I. **The second order case**

1. The St.-Venant problem. (the torsion problem)

First, we examine one of the simplest cases, the problem of the torsional rigidity of a beam

$$\begin{cases} \Delta u = -2 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{10}$$

Theorem 9. *The function* $\mathrm{P}_1 = |\nabla u|^2 + 4u$ *takes its maximum value either at a critical point of* u *or at some point on the boundary, unless* P_1 *is a constant. If* Ω *is convex and smooth (*$\partial\Omega \in C^{2+\varepsilon}$*), then* P_1 *cannot take its maximum value on* $\partial\Omega$*. Moreover, if* Ω *degenerates to an infinite strip, then* $\mathrm{P}_1 \equiv$ *const. Similarly the function* $\mathrm{P}_2 = |\nabla u|^2$ *attains its maximum on* $\partial\Omega$*.*

The proof is due to L.E.Payne, [43] and follows from the differential inequality

$$\Delta \mathrm{P}_1 + \frac{1}{|\nabla u|^2}\{4\nabla \mathrm{P}_1 \cdot \nabla u + \frac{1}{2}|\nabla \mathrm{P}_1|^2\} \geq 0 \text{ in } \Omega,$$

and the maximum principle.

Theorem 10. *([83])*

The function $\mathrm{P}_3 = |\nabla u|^2 + (4/n)u$ *takes its maximum value at some point on the boundary, unless* P_3 *is a constant. Moreover,* P_3 *is identically constant in* Ω *if and only if* Ω *is a* n *dimensional ball.*

Remarks. 1. The simplest P function is $\mathrm{P} = u$ (the classical maximum principle).
2. There are no general methods to determine P functions. Sometimes we can check the one dimensional case in order to get an idea of what types of P functions we have to look in the n dimensional case. For example considering the one dimensional equation

$$u'' + 2 = 0 \text{ in } \Omega = (0, \alpha)$$

and multiplying it by u' and then integrating it we get that

$$\mathrm{P} = (u')^2 + 4u \equiv \text{const. in } \Omega.$$

This function is the one dimensional version of P_1.

Applications

a). Upper bound for the stress function u, if Ω is convex.

Let M be the unique critical point of u and Q a point on $\partial\Omega$, nearest to P. Let r measure the distance from M along the ray connecting M and Q. Hence

$$-\frac{du}{dr} \leq |\nabla u| \text{ in } \Omega. \tag{11}$$

From Theorem 9 we have $|\nabla u|^2 \leq 4(u_M - u(x)) \quad \text{in } \Omega$, where $u_M = \sup_\Omega u$. Using (11) we get

$$\int_0^{u_M} \frac{du}{2\sqrt{u_M - u}} \leq \int_Q^M dr = |MQ|.$$

Hence

$$\sqrt{u_M} \leq |MQ| \leq \rho,$$

where ρ is the radius of the largest ball contained in Ω.

Note that the following bound was also obtained using similar methods ([15])

$$u_M \leq \frac{\alpha}{\beta}\left[\frac{1}{\cos(\rho\sqrt{\beta})} - 1\right],$$

where $\alpha \geq 1 + \sqrt{2}$ and $0 < \beta < \pi^2/4\rho^2$.

A lower bound for u_M was given in [54] (in the case Ω convex and bidimensional). Further isoperimetric inequalities as well as bounds in terms of the stress function for the curvature of the level curves u=const are presented in [54].

b). Upper bound for the maximum stress.

An important quantity is the maximum stress $\sigma = \max_{\partial\Omega} |\nabla u|$. Since P_2 and P_3 attain their maximum value on the boundary of Ω and using standard calculations (see [78]) we get,

$$|\nabla u|^2 \leq \sigma \leq \frac{2}{nK(P)} \leq \frac{2}{nK_{min}}, \tag{12}$$

where $K_{min} = \min_{\partial\Omega} K$, K represents the average curvature of $\partial\Omega$ (the curvature if $n = 2$) and P is a point on the boundary where P_2 assumes its maximum.

c). Upper bound for the average curvature of $\partial\Omega$.

Integrating (12) over Ω we obtain

$$K(P) \leq \frac{|\partial\Omega|}{n|\Omega|}, \tag{13}$$

where P is defined above, $|\partial\Omega|$ stands for the $n-1$ dimensional measure and $|\Omega|$ stands for the n dimensional measure.

Equation (13) tells us that at a point of maximum stress, the boundary must be sufficiently flat.

d). Upper bound for the torsional rigidity.

The torsional rigidity of Ω is $T = 2\int_\Omega u dx = \int_\Omega |\nabla u|^2 dx$. We have the following bound:

$$T \leq \frac{4}{3}|\Omega| u_M \leq \frac{4}{3}|\Omega|\rho^2.$$

e). An overdetermined St. - Venant problem.

We consider the problem (10) overdedermined by the boundary condition

$$K|\nabla u|^3 = \text{const.} > 0 \text{ on } \partial\Omega, \tag{14}$$

where Ω is a simply connected domain in $\mathbb{R}^2$ and K the curvature of $\partial\Omega$.

Makar-Limanov ([34]) introduced the function

$$P_4 = u_{ij}u_iu_j - |\nabla u|^2\Delta u + u((\Delta u)^2 - u_{ij}u_{ij}),$$

(u is a solution of St. - Venant problem (10)) and showed that it satisfies a maximum principle. A consequence is the convexity of level lines $\{u = \text{const.}\}$. Moreover, we have $\mathrm{P} = K|\nabla u|^3 \geq 0$ on $\partial\Omega$. and P_4 is constant in Ω if and only if Ω is an ellipse. The next theorem (Henrot and Philippin, [32]) tells us that ellipses are the only domains for which condition (14) holds.

Theorem 11. *The over determined problem (10), (14) is solvable only if Ω is an ellipse.*

The proof follows from the implication:

$$\mathrm{P}_4 = \text{const. on}\, \partial\Omega \Rightarrow \mathrm{P}_4 = \text{const. in}\, \overline{\Omega}.$$

Standard methods of investigation for overdetermined problems may not work (Serrin's moving plane method). In this case, we can take advantage of the P function method.

2. The membrane problem.

We are concerned now with eigenvalues of elastically supported membrane problem:

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in}\, \Omega \subset \mathbb{R}^2 \\ \partial u/\partial n + \alpha u = 0 & \text{on}\, \partial\Omega, \end{cases} \tag{15}$$

where $\partial/\partial n$ is the outward normal derivative operator, α is a positive constant and is Ω simply connected, smooth and convex .

If α is large, Payne and Schaefer [53] derived a lower bound for the first eigenvalue λ_1

$$\lambda_1 > \rho^{-2}(\tan^{-1}(\alpha/\sqrt{\Lambda_1}))^2, \tag{16}$$

using that the P function $\mathrm{P}_5 = |\nabla u_1|^2 + \lambda_1 u_1^2$ takes its maximum either on $\partial\Omega$ or at an interior point at which $\nabla u = 0$. Here u_1 represents the first eigenfunction and ρ the radius of the largest inscribed disc. We see that the bound (16) involves Λ_1, the first eigenvalue for the problem,

$$\begin{cases} \Delta v + \Lambda v = 0 & \text{in}\, \Omega \\ v = 0 & \text{on}\, \partial\Omega. \end{cases} \tag{17}$$

If necessary, we can use upper bounds for Λ_1. A known bound for convex regions was given by Hersch [28]

$$\Lambda_1 \geq \frac{\pi^2}{4\rho^2}. \tag{18}$$

On the other hand if α is small we have

$$\lambda_1 > \rho^{-2}(\tan^{-1}(\alpha A/L)^{1/2})^2, \tag{19}$$

where L is the perimeter of Ω and A its area.

Bounds for eigenvalue of (15) have been previously obtained by Sperb [77], [78], [79], Payne and Weinberger [55].

Bounds for the first positive eigenvalue in the free membrane problem

$$\begin{cases} \Delta u + \mu u = 0 & \text{in } \Omega \subset \mathbb{R}^2 \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

are discussed in the book of Sperb [78].

3). A classical problem of electrostatics.

We consider the exterior Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega^* \equiv \mathbb{R}^3 - \overline{\Omega} \\ u = 1 & \text{on } \partial\Omega \\ u = O(\frac{1}{r}) \text{ as } r \to \infty. \end{cases} \tag{20}$$

u is the electrostatic potential of the conductor and r measures the distance from some origin inside Ω.

The following useful result was proven by Payne and Philippin [49].

Theorem 12. *Let H and h be harmonic functions in Ω, where $H \in C^1(\Omega)$, $h \in C^0(\Omega)$ and let $f(h)$ be a positive C^2 function. Assume that f satisfies*

$$[f^{n-2/2(n-1)}]'' \leq 0, \text{ if } n \geq 3,$$

$$[\log f]'' \leq 0, \text{ if } n = 2.$$

Then the function

$$\mathrm{P}_6 = \frac{\nabla H \cdot \nabla H}{f(h)},$$

assumes its maximum on $\partial\Omega$.

Theorem 12 tells that the function

$$\mathrm{P}_6 = \frac{\nabla u \cdot \nabla u}{u^4}, x \in \Omega^*$$

satisfies

$$\mathrm{P}_6 \leq \max_{\partial\Omega} \mathrm{P}_6, \tag{21}$$

with equality if Ω is a sphere. Moreover

$$C^{-2} \leq \max_{\partial\Omega} \mathrm{P}_6, \tag{22}$$

with equality if Ω is a sphere where, C is the capacity $C = \int_{\Omega^*} |\nabla u|^2 dx$.

At the point $P_0 \in \partial\Omega$, where P_6 assumes its maximum it follows from Hopf's lemma (lemma 1) that either Ω is a sphere or

$$\frac{\partial \mathrm{P}_6}{\partial n} = 2\frac{\partial u}{\partial n}\frac{\partial^2 u}{\partial n^2} - 4\Big(\frac{\partial u}{\partial n}\Big)^2 > 0.$$

For a smooth hypersurface S in $\mathbb{R}^n$ we have on S the relation (see [78], p.62)

$$\Delta u = \Delta_s u + (n-1)K\frac{\partial u}{\partial n} + \frac{\partial^2 u}{\partial n^2}, \tag{23}$$

where $\Delta_s u$ is the Laplacian in the induced metric of S and K the mean curvature (the curvature if $n = 2$).

From (23) we obtain on $\partial\Omega$

$$\frac{\partial^2 u}{\partial n^2} = 2K\frac{\partial u}{\partial n}.$$

Now it follows that

$$\frac{\partial u(P_0)}{\partial n} < K(P_0).$$

Since P_6 and $\frac{\partial u}{\partial n}$ take their maximum at the same point on the boundary it follows that either Ω is a sphere or

$$\max_{\partial\Omega} \frac{\partial u}{\partial n} < K(P_0) < \max_{\partial\Omega} K \equiv K_0. \tag{24}$$

A bound for the capacity C follows now from (22) and (24)

$$C \geq K_0^{-1}, \tag{25}$$

where the equality sign holds if Ω is a sphere.

An upper bound for the capacity is also given in [49]:

$$C \leq \frac{3|\Omega| K_0^2}{4\pi}, \tag{26}$$

where the equality sign holds if Ω is a sphere ($|\Omega| = \mathrm{vol}(\Omega)$).

Bounds for the derivatives of Green's function are also a consequence of Theorem 12. See for details [49].

4). Estimates for capillary free surfaces without gravity.

In the paper [36], Ma studied (using the P function method) the influence of boundary geometry and constant contact angle θ_0, $0 \leq \theta_0 < \pi/2$ (against the wall of the tube) on the size and shape for the capillary free surface without gravity.

Let Ω be a bounded, smooth and convex domain in $\mathbb{R}^2$ and let $K = \cos\theta_0 |\partial\Omega|/2|\Omega|$ be a given constant.

Consider the problem:

$$\begin{cases} \left(\frac{u_i}{\sqrt{1+|\nabla u|^2}}\right)_i = 2K & \text{in } \Omega \\ \partial u/\partial n = \cos\theta_0 \sqrt{1+|\nabla u|^2} & \text{on } \partial\Omega, \end{cases} \tag{27}$$

where $\partial u/\partial n$ denoted the directional derivative of u along the outer unit normal.

The graph of solution u of (27) describes a capillary free surface (having the nonparametric form $x_3 = u(x_1, x_2), (x_1, x_2) \in \Omega$) without gravity over the cross section Ω. We have the following result (Xi- Nan Ma)

Theorem 13. *If $u \in C^3(\Omega)$ is a solution of (27), then*

$$u(A) - u(C) \leq \frac{1-\sin\theta_0}{K}, \; k(A) \leq \frac{K}{\cos\theta_0} \tag{28}$$

$$u(B) - u(C) \geq \frac{1-\sin\theta_0}{K}, \; k(B) \geq \frac{K}{\cos\theta_0}. \tag{29}$$

If the equality sign holds in (28) and (29), then Ω is a disk of radius $\cos\theta_0/K$.

Here $A \in \partial\Omega$ is a point that corresponds to the minimum boundary value of u, $B \in \partial\Omega$ is a point that corresponds to the maximum boundary value of u and $C \in \Omega$ is the unique critical point of u.

If $S = \int_\Omega \sqrt{1 = |\nabla u|^2} dx$ is the area of the free capillary surface and $V = \int_\Omega u dx$ the volume of the liquid in the vertical tube, then we have the bound:

Theorem 14.

$$(\sin\theta_0 + 3Ku(A))|\Omega| - 3KV \leq S \leq (\sin\theta_0 + 3Ku(B))|\Omega| - 3KV.$$

Here and in the above mentioned result $A \in \partial\Omega$ is a point where u assumes its minimum on $\partial\Omega$, $B \in \partial\Omega$ is a point where u assumes its maximum on $\partial\Omega$, C is the unique critical point of u and K is the curvature of $\partial\Omega$.

The proofs follow from

Theorem 15. *If $u \in C^3(\Omega)$ is a solution of (27), then the function*

$$\mathrm{P}_7 = 2 - 2Ku - 2(1 + |\nabla u|^2)^{-\frac{1}{2}}$$

attains its minimum on the boundary of Ω.

Similar problems are treated in the paper of Payne and Philippin [46], e.g. equation of a surface of constant mean curvature, equation of the fluid in a capillary tube, equation of thin extensible film under the influence of gravity and surface tension. The authors obtain various bounds in terms of boundary data and geometry of Ω.

5). Equations of Monge - Ampère type.

We consider a class of Monge - Ampère equations

$$\det D^2 u = f(x, u, \nabla u) \tag{30}$$

with a prescribed contact angle boundary value on a bounded convex domain in two dimensions.

$$\partial u / \partial n = \cos\theta(x, u)\sqrt{1 + |\nabla u|^2} \quad \text{on } \partial\Omega,$$

where $D^2 u$ is the hessian matrix and $\theta(x, u) \in (0, \pi/2)$ is the wetting angle.

The existence of solutions for such boundary value problems is still open. Even the particular case is untreated in the literature.

$$\begin{cases} \det D^2u = c & \text{in } \Omega \subset \mathbb{R}^2 \\ \partial u/\partial n = \cos\theta_0\sqrt{1+|\nabla u|^2} & \text{on } \partial\Omega, \end{cases} \tag{31}$$

where Ω is convex, $c > 0$ is a constant and $\theta_0 \in (0, \pi/2)$.

Ma Xi-nan [35] gave a necessary condition of solvability for the problem (31).

Theorem 16. *Let $u \in C^2(\overline{\Omega}) \cap C^3(\Omega)$ be a strictly convex solution of problem (31). Under the above stated hypotheses on Ω, c, θ_0 we must have the relation*

$$K_0 \le \max\{\sqrt{c}\cdot\cos\theta_0, \sqrt{c}\cdot\tan\theta_0\},$$

where

$$K_0 = \min_{\partial\Omega} K > 0$$

and K is curvature of $\partial\Omega$.

The proof is achieved by using the P function $\mathrm{P}_8 = |\nabla u|^2 - 2\sqrt{c}u$ (which satisfies a maximum principle) and introducing a curvilinear coordinate system.

Bounds for solutions and gradient of general Monge - Ampère equations (30) are presented in the work of Philippin and Safoui [58].

II. The higher order case

Miranda [39] was the first that showed that for the biharmonic equation $\Delta^2 u = 0$, where $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$ is a function defined on a bounded plane domain the function $\mathrm{P}_9 = |\nabla u|^2 - u\Delta u$ takes its maximum value on the boundary of the domain, i.e.,

$$\max_{\overline{\Omega}} \mathrm{P}_9 = \max_{\partial\Omega} \mathrm{P}_9.$$

Since then many authors have extended the Miranda's result. For example, maximum principles for fourth order equations containing nonlinearities in u or Δu can be found in works of Payne [44], Schaefer [67], [70],[71]. Similar results are proved by H. Zhang and W. Zhang [84], Mareno [37], [38] (studied some equations from plate theory), Danet [8], Tseng and Lin [80], [10], [11] etc. (see the references). We will list only a few as an indication of the types of results that can be obtained.

1). Equations of fourth order arising in plate theory.

a). Von Kármán equations.

Assume that Ω is a bounded domain in the plane. We consider the von Kármán equations:

$$\begin{cases} \Delta^2\phi = -\frac{1}{2}[w, w] & \text{in } \Omega \\ \Delta^2 w = [w, \phi] + f(x, y) & \text{in } \Omega. \end{cases} \tag{32}$$

The equations (32) govern the equilibrium configuration of a thin elastic plate under stress. $f(x, y)$ represents nonconstant perpendicular loading terms. The function w denotes the deflection of the thin plate and ϕ represents the stress function. The operator $[\cdot, \cdot]$ is defined as follows:

$$[w, \phi] = w_{xx}\phi_{yy} - 2w_{xy}\phi_{xy} + w_{yy}\phi_{xx}.$$

Mareno [38] proved (the first that proved a maximum principle for such equations) that the P function

$$\mathrm{P}_{10} = |\nabla^2\phi|^2 + |\nabla^2 w|^2 - \phi_i\Delta\phi_i - w_i\Delta w_i + h(x, y)[|\nabla w|^2 + |\nabla\phi|^2] + f^2(x, y)$$

satisfies a maximum principle and as a consequence obtained the following bound:

$$\frac{2}{|\Omega|}\int_\Omega \Big(|\nabla^2\phi(x, y)|^2 + |\nabla^2 w(x, y)|^2\Big)dxdy \;\le\; |\nabla^2\phi(x_0, y_0)|^2 + |\nabla^2 w(x_0, y_0)|^2 + f^2(x_0, y_0),$$

for some point (x_0, y_0) on $\partial\Omega$, if $\phi = w = \partial\phi/\partial n = \partial w/\partial n = 0$ on $\partial\Omega$. Here $|\nabla^2 w| = w_{ij}w_{ij}$, and $h(x, y)$ is a smooth function.

b). An equation arising in plate theory.

We deal with the following equation

$$\Delta^2 u + k_1 u + k_2 u^3 = 0 \quad \text{in } \Omega \subset \mathbb{R}^{\mathrm{n}},\ \mathrm{n} \ge 2, \tag{33}$$

where $k_1, k_2 > 0$ are constants.

The equation (33) arises in the plate theory and in the bending of cylindrical shells [67].

The next maximum principle ([10]) will be used to obtain solution and gradient bounds for the equation (33)

Theorem 17. *Let u be a classical solution of (33). Then the function*

$$\mathrm{P}_{11} = (\Delta u)^2 + \frac{k_2}{2} u^4 + k_1 u^2$$

attains its maximum value on $\partial\Omega$.

If u satisfies (33) then, we have the following bounds

a).

$$\max_{\overline{\Omega}} |u| \leq \sqrt{\frac{1}{k_1}} \left(\max_{\partial\Omega} |\Delta u| + \sqrt{\frac{k_2}{2}} \max_{\partial\Omega} u^2 + \sqrt{k_1} \max_{\partial\Omega} |u| \right), \tag{34}$$

where $n \geq 2$.

b).

$$\max_{\overline{\Omega}} |\nabla u|^2 \leq \max_{\partial\Omega} |\nabla u|^2 + \frac{3+k_1}{2} \max_{\partial\Omega} u^2 + \frac{k_2}{2k_1} \max_{\partial\Omega} u^4 + \frac{2k_1+1}{2k_1} \max_{\partial\Omega} (\Delta u)^2, \tag{35}$$

where $n = 2$.

The hypothesis that is assumed over and over again in plate theory is convexity. Under this assumption, Schaefer [67] proved the uniqueness for the solution of

$$\begin{cases} \Delta^2 u + k_1 u + k_2 u^3 = 0 & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{36}$$

where $\Omega \subset \mathbb{R}^2$ is a convex domain.

An application the maximum principle (Theorem 17) shows that the convexity assumption is redundant. Moreover, our result holds for $n \geq 2$.

The result reads as follows:

Theorem 18. *Let u be a classical solution of (36), where $\Omega \subset \mathbb{R}^n$ is an arbitrary domain. Then $u \equiv 0$ in Ω.*

Maximum principles for fourth and six order equations are presented in the author's paper [10] and [11].

2). The $m\,(>4)$ order case.

We conclude this paper with a result for the general case due to the author [11].

Theorem 19. *Let u be a classical solution of equation*

$$\Delta^m u + a_0 u = 0$$

in Ω, where $\Omega \subset \mathbb{R}^n, m$ even, $n \geq 2$.

Suppose that $a_0 > 0$, $\Delta a_0 \leq 0$ in Ω.

We define the function P_{10}

$$\mathrm{P}_{12} = \Big((\Delta^{m-1}u)^2 + (\Delta^{m-2}u)^2 + \cdots + u^2 \Big)/a_0.$$

a). If

$$\max\{1 + \sup_{\Omega} a_0, 2\} + \sup_{\Omega} \frac{\Delta a_0}{a_0} \leq 0, \tag{37}$$

then, the function P_{12} attains its maximum value on $\partial\Omega$.

b). If

$$\max\{1 + \sup_{\Omega} a_0, 2\} + \sup_{\Omega} \frac{\Delta a_0}{a_0} < \frac{4}{d^2 e^2} \tag{38}$$

and if there exists $i \in \{1, \dots, n\}$ such that $\frac{\partial}{\partial x_i}\big(\frac{1}{a_0}\big) \geq 0$ in Ω, then, the function P_{12}/w_3 attains its maximum value on $\partial\Omega$, where $w_3 = 1 - \beta e^{\alpha x_i}, \beta = \sup_{\Omega} c/\alpha^2$ and $\alpha > 0$ is a constant.

The proof follows from the generalized maximum principle, Theorem 7 and works also for the case m odd.

As an immediate consequence of the above mentioned maximum principle we obtain the uniqueness of the classical solution $(C^{2m}(\Omega) \cap C^{2m-2}(\overline{\Omega}), m \geq 3)$ of the boundary value problem

$$\begin{cases} \Delta^m u + (-1)^m a_0(x) u = f & \text{in } \Omega \\ u = g_1,\ \Delta u = g_2, \dots,\ \Delta^{m-1} u = g_m & \text{on } \partial\Omega. \end{cases} \tag{39}$$

Moreover the following classical maximum principle holds for solutions of (39), if $g_2 = \cdots = g_m = 0$ on $\partial\Omega$ and $f = 0$ in Ω.

$$\max_{\overline{\Omega}} |u| \leq C \max_{\partial\Omega} |u|, \tag{40}$$

where $C > 1$ is a constant.

Note that the problem was solved for a more general problem, but under the restriction Ω is of class C^2 (see [73]).

Final remark. Below we collected many papers concerning the P function method for the interested reader (not all are quoted in this paper).

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