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In Memoriam Adelina Georgescu

A FINITE VOLUME METHOD FOR SOLVING GENERALIZED NAVIER-STOKES EQUATIONS*

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Abstract

In this paper we set up a numerical algorithm for computing the flow of a class of pseudo-plastic fluids. The method uses the finite volume technique for space discretization and a semi-implicit two steps backward differentiation formula for time integration. As primitive variables the algorithm uses the velocity field and the pressure field. In this scheme quadrilateral structured primal-dual meshes are used. The velocity and the pressure fields are discretized on the primal mesh and the dual mesh respectively. A certain advantage of the method is that the velocity and pressure can be computed without any artificial boundary conditions and initial data for the pressure. Based on the numerical algorithm we have written a numerical code. We have also performed a series of numerical simulations.

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keywords: generalized Navier-Stokes equations, pseudo-plastic fluids, finite volume methods, admissible primal-dual mesh, discrete derivatives operators, discrete Hodge formula.

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1 Introduction

In this paper we are interested in the numerical approximation of a class of pseudo-plastic fluid flow. The motion of the fluid is described by the generalized incompressible Navier-Stokes equations

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} & = -\nabla p + \nabla\cdot\sigma(\mathbf{u}) + \mathbf{f}, \\ \nabla\cdot\mathbf{u} & = 0, \end{cases} \tag{1}$$

where $\mathbf{u}$ is the velocity vector field, p is the hydrodynamic pressure field, σ the extra stress tensor field and $\mathbf{f}$ is the body force. The extra stress tensor $\sigma(\mathbf{u})$ obeys a constitutive equation of the type

$$\sigma_{ab}(\mathbf{u}) = 2\nu(|\widetilde{\partial\mathbf{u}}|)\widetilde{\partial u}_{ab} \tag{2}$$

where $\widetilde{\partial u}$ is the strain rate tensor given by

$$\widetilde{\partial u}_{ab} = \frac{1}{2}\left(\partial_a u_b + \partial_b u_a\right),$$

∂_a standing for the partial derivative with respect to the space coordinate x_a, and for any square matrix $\mathbf{e}$, $|\mathbf{e}|$ being defined as

$$|\mathbf{e}| = \left(\sum_{i,j} e_{ij}^2\right)^{1/2}.$$

Concerning the viscosity function $\nu(s)$, we assume that it is a continuous differentiable, decreasing function, with bounded range

$$\begin{cases} 0 < \nu_\infty \le \nu(s) \le \nu_0 < \infty, \forall s > 0, \\ (\nu(s_1) - \nu(s_2))(s_1 - s_2) < 0, \forall s_1, s_2 > 0, \end{cases} \tag{3}$$

and it satisfies the constraint

$$\nu(s) + s\dot{\nu}(s) > c > 0. \tag{4}$$

The model of the Newtonian fluid corresponds to $\nu = constant$.

We consider the case when the flow takes place inside a fixed and bounded domain $\Omega \subset \mathbf{R}^2$ and we assume that the fluid adheres to its boundary $\partial\Omega$, hence we impose a Dirichlet type boundary condition for the velocity field,

$$\mathbf{u} = \mathbf{u}_D(x),\ x \in \partial\Omega, t > 0. \tag{5}$$

To the equations (1) we append the initial condition for the velocity

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x),\ x \in \Omega. \tag{6}$$

The initial boundary value problem (IBV), which we intend to solve numerically, consists in finding the velocity field $\mathbf{u}(x, t)$ and the pressure field $p(x, t)$ that satisfy the partial differential equations (1), boundary condition (5) and the initial condition (6).

A constitutive function as (2) is used , for example, to describe the behavior of polymeric fluids, [5], [8], [11], and the flow of the blood through the vessels, [19], [9], [6], [18].

In writing down a numerical algorithm for the non-stationary incompressible generalized Navier-Stokes equations three main difficulties occur, namely: (i) the velocity field and the pressure field are coupled by the incompressibility constraint [12], (ii) the presence of the nonlinear convection term and (iii) the nonlinear dependence of the viscosity on the share rate.

The first two problems are common to the Navier-Stokes equations and in the last fifty years several methods were developed to overcome them: the projection method, [12], [13], [7],[14], [3], and gauge method, [20]- to mention the most significant methods for our case.

When one deals with a non-Newtonian fluid, the nonlinearity of the viscosity rises a new problem in obtaining a discrete form for the generalized Navier-Stokes equations. The new issue is the development of an appropriate discrete form of the action of the stress tensor on the boundary of the volume-control. A similar dificulty is raised by the discretization of the p-laplacean, see [2] for that.

The outline of the paper is as follows. In Section 2 we define the weak solution of IBV (1), (5) and (6) and we present an existence theorem of the weak solution for a class of pseudo-plastic fluids that satisfy (4). In Section 3 we establish the semi-discrete, space discrete coordinates and continuum time variable form of the equation (1) and we present some general concepts concerning the space discretization and related notions like admissible mesh, primal and dual mesh, the discretization of the derivative operators etc. In Section 4 we present an algorithm for solving a 2D model. In the last section we present the results of some numerical simulations of the lid driven cavity flow.

2 The Existence of the Weak Solution

To define the weak solution we need the following functional frame, [16], [17].

By $L^p(\Omega)$ and $W^{m,p}(\Omega)$, $m = 0, 1, \cdots$, we denote the usual Lebesgue and Sobolev spaces, respectively. The scalar product in L^2 is indicated by $(\cdot, \cdot)$. For $\mathbf{u}$, $\mathbf{v}$ vector functions defined on Ω we put

$$(\mathbf{u}, \mathbf{v}) = \int_{\Omega} u^a v_a \mathrm{d}x,$$

$$(\nabla \mathbf{u}, \nabla \mathbf{v}) = \sum_{a,b=1} \int_{\Omega} \partial_a u^b \partial_a v^b \mathrm{d}x.$$

We denote by $||\cdot||$ the norm in L^2 associate to $(\cdot, \cdot)$. The norm in $W^{m,p}$ is denoted by $||\cdot||_{m,p}$. Consider the space

$$\mathcal{V} = \left\{\psi \in C_0^\infty(\Omega), \mathrm{div}\psi = 0\right\}.$$

We define $\mathbf{H}(\Omega)$ the completion of $\mathcal{V}$ in the space $\mathbf{L}^2(\Omega)$. We denote by $\mathbf{H}^1(\Omega)$ the completion of $\mathcal{V}$ in the space $\mathbf{W}^{1,2}$.

For $T \in (0, \infty]$ we set $Q_T = \Omega \times [0, T)$ and define

$$\mathcal{V}_T = \left\{\phi \in C_0^\infty(Q_T);\ \mathrm{div}\phi(x, t) = 0 \text{ in } Q_T\right\}.$$

The weak solution of IBV is defined as follow.

Definition 1. *Let* $\mathbf{f} \in \mathbf{L}^2(\Omega)$. *Let* $\mathbf{u}_0(x) \in \mathbf{L}^2(\Omega)$ *and* u_D *be such that*

$$\begin{cases} \mathrm{div}\, \mathbf{u}_0 = 0, \\ \mathbf{u}_D \cdot \mathbf{n} = 0, x \in \partial\Omega, \\ \mathbf{u}_0 = \mathbf{u}_D,\ x \in \partial\Omega, \end{cases} \tag{7}$$

and there exists $\mathbf{v} \in \mathbf{W}^{1,2}(\Omega) \cap \mathbf{L}^4(\Omega)$ *a vector function that satisfies*

$$\begin{cases} \mathrm{div}\, \mathbf{v} = 0, \\ \mathbf{v} = \mathbf{u}_D,\ x \in \partial\Omega. \end{cases} \tag{8}$$

Then $\mathbf{u}$ *is a weak solution of* IBV (1,5,6) *if*

$$\mathbf{u} - \mathbf{v} \in L^2\left((0, T); \mathbf{H}^1(\Omega)\right) \cap L^\infty\left((0, T); \mathbf{H}(\Omega)\right) \tag{9}$$

and $\mathbf{u}$ *verifies*

$$-\int_0^\infty \left(\mathbf{u}, \frac{\partial \phi}{\partial t}\right) \mathrm{d}t - \int_0^\infty (\mathbf{u} \otimes \mathbf{u}, \nabla \phi)\, \mathrm{d}t + \int_0^\infty \left(\sigma(\mathbf{u}), \widetilde{\partial \phi}\right) \mathrm{d}t = \\ = \int_0^\infty (\mathbf{f}, \phi)\, \mathrm{d}t + (u_0, \phi) \tag{10}$$

for any test function $\phi \in \mathcal{V}_T$.

Concerning the existence of the weak solution of the IBV we proved the following result, [15]:

Theorem 1. *If the constitutive function* $\nu(\cdot)$ *satisfies the relations* (3) *and* (4) *then there exists a weak solution of the* IBV (1), (5) *and* (6).

3 Semi-discrete Finite Volume Method

The finite volume method (FVM) is a method for approximating the solution of a partial differential equation (PDE). It basically consists in partitioning the domain Ω, on which the PDE is formulated, into small polygonal domains ω_i (control volumes) on which the unknown is approximated by constant values, [10].

We consider a class of finite-volume schemes that includes two types of meshes: the *primal mesh*, $\mathcal{T} = \{\omega_{\mathcal{I}}, \mathbf{r}_{\mathcal{I}}\}$ and the *dual mesh*, $\widetilde{\mathcal{T}} = \{\widetilde{\omega}_{\mathcal{J}}, \widetilde{\mathbf{r}}_{\mathcal{J}}\}$. The space discrete form of the GNS equations are obtained from the integral form of the balance of momentum equation and mass balance equation on the primal mesh and the dual mesh respectively.

For any ω_i of the primal mesh $\mathcal{T}$ the integral form of the balance of momentum equation reads as,

$$\partial_t \int_{\omega_i} \mathbf{u}(\mathbf{x}, t) \mathrm{d}x + \int_{\partial \omega_i} \mathbf{u}\mathbf{u} \cdot \mathbf{n} \mathrm{d}s + \int_{\omega_i} \nabla p \mathrm{d}x = \int_{\partial \omega_i} \sigma \cdot \mathbf{n} \mathrm{d}s, \tag{11}$$

and for any $\widetilde{\omega}_\alpha$ of the dual mesh $\widetilde{\mathcal{T}}$ the integral form of mass balance equation is given by

$$\int_{\partial \widetilde{\omega}_\alpha} \mathbf{u} \cdot \mathbf{n} \mathrm{d}s = 0. \tag{12}$$

The velocity field $\mathbf{u}(\mathbf{x},t)$ and the pressure field $p(\mathbf{x},t)$ are approximated by the piecewise constant functions on the primal mesh and the dual mesh respectively,

$$\mathbf{u}(\mathbf{x},t) \approx \mathbf{u}_i(t),\ \forall \mathbf{x} \in \omega_i,\ \ p(\mathbf{x},t) \approx p_\alpha(t),\ \forall \mathbf{x} \in \widetilde{\omega}_\alpha.$$

By using certain approximation schemes of the integrals as functions of the discrete variables $\{u_i(t)\}_{i\in\mathcal{I}}$, $\{p_\alpha(t)\}_{\alpha\in\mathcal{J}}$ one can define:

$$\begin{aligned} &\mathcal{F}_i(\mathbf{u}) \approx \textstyle\int_{\partial\omega_i} \mathbf{u}\mathbf{u}\cdot\mathbf{n}\mathrm{d}s,\ \ \mathcal{S}_i(\mathbf{u}) \approx \int_{\partial\omega_i} \sigma\cdot\mathbf{n}\mathrm{d}s, \\ &\mathbf{Grad}_i(p) \approx \textstyle\int_{\omega_i} \nabla p \mathrm{d}x,\ \ ,\mathrm{Div}_\alpha(\mathbf{u}) \approx \int_{\partial\widetilde{\omega}_\alpha} \mathbf{u}\cdot\mathbf{n}\mathrm{d}s. \end{aligned} \tag{13}$$

The semi-discrete form of GNS equations, continuous with respect to time variable and discrete with respect to space variable, can be written as:

$$\begin{aligned} m_i\frac{\mathrm{d}\mathbf{u}_i}{\mathrm{d}t} + \mathcal{F}_i(\{\mathbf{u}\}) + \mathbf{Grad}_i(\{p\}) - \mathcal{S}_i(\{\mathbf{u}\}) = 0, \quad i\in\mathcal{I} \\ \mathrm{Div}_\alpha(\{\mathbf{u}\}) = 0, \quad \alpha\in\mathcal{J} \end{aligned} \tag{14}$$

where m_i stands for the volume of the ω_i.

Now the problem is to find the functions $\{\mathbf{u}_i(t)\}_{i\in\mathcal{I}}, \{p_\alpha(t)\}_{\alpha\in\mathcal{J}}$ that satisfy the differential algebraic system of equations (DAE) (14) and the initial condition

$$\mathbf{u}_i(t)|_{t=t_0} = \mathbf{u}_i^0,\ \forall i \in \mathcal{I}. \tag{15}$$

In solving the Cauchy problem (14) and (15), an essential step is to define a primal-dual mesh $\left(\mathcal{T},\widetilde{\mathcal{T}}\right)$ that allows one to calculate the velocity field independent of the pressure field.

In the next subsections we define a pair of quadrilateral admissible primal-dual (QAPD) meshes $\left(\mathcal{T},\widetilde{\mathcal{T}}\right)$, and we define the discrete gradient of the scalar functions and the discrete divergence of the vector functions such that the discrete space of the vector fields admits an orthogonal decomposition into two subspaces: one of discrete divergences free vectors fields, and other consisting of vectors that are the discrete gradient of some scalar fields.

3.1 Quadrilateral primal-dual meshes

Let Ω be a polygonal domain in $\mathbf{R}^2$. Let $\mathcal{T} = \{\omega_\mathcal{I}, \mathbf{r}_\mathcal{I}\}$ be a quadrilateral mesh defined as follows:

$$\left\|\begin{array}{l}(1)\ \omega_i \text{ is a quadrilateral, } \overline{\cup_{i\in I}\omega_i} = \overline{\Omega},\\ (2)\ \forall i \neq j \in I \text{ and } \overline{\omega}_i \cap \overline{\omega}_j \neq \Phi, \text{ either } \mathcal{H}_1(\overline{\omega}_i \cap \overline{\omega}_j) = 0, \text{ or}\\ \qquad \sigma_{ij} := \overline{\omega}_i \cap \overline{\omega}_j \text{ is a common } (n-1) - \text{face of } \omega_i \text{ and } \omega_j,\\ (3)\ \mathbf{r}_i \in \omega_i, \text{if } \omega_i = [ABCD], \text{ then } \mathbf{r}_i = [M_{AB}M_{DC}] \cap [M_{AD}M_{BC}],\\ (4) \text{ for any vertex P} \in \Omega \text{ there exists only four quadrilateral } \omega\\ \qquad \text{with the common vertex P,}\end{array}\right.$$

where $\mathcal{H}_1$ is the one-dimensional Hausdorff measure, and M_{AB} denotes the midpoint of the line segment $[AB]$.

Let $\widetilde{\mathcal{T}} = \{\widetilde{\omega}_{\mathcal{J}}, \widetilde{\mathbf{r}}_{\mathcal{J}}\}$ be another mesh defined as follows:

$$\left\|\begin{array}{l}(1)\ \forall\alpha \in \mathcal{J},\ \widetilde{\mathbf{r}}_\alpha \text{ is a vertex of } \mathcal{T},\\ (2)\ \widetilde{\mathbf{r}}_\alpha \in \widetilde{\omega}_\alpha, \forall\alpha \in \mathcal{J},\\ (3)\ \forall\widetilde{\mathbf{r}}_\alpha \in \overline{\Omega}, \text{the poligon } \widetilde{\omega}_\alpha \text{ has the vertexes :}\\ \qquad \text{the centers of the quadrilaterals with the common vertex } \widetilde{\mathbf{r}}_\alpha\\ \qquad \text{and the midpoints of the sides emerging from } \widetilde{\mathbf{r}}_\alpha,\end{array}\right.$$

where by "center" of the quadrilateral we understand the intersection of the two segments determined by the midpoints of two opposed sides.

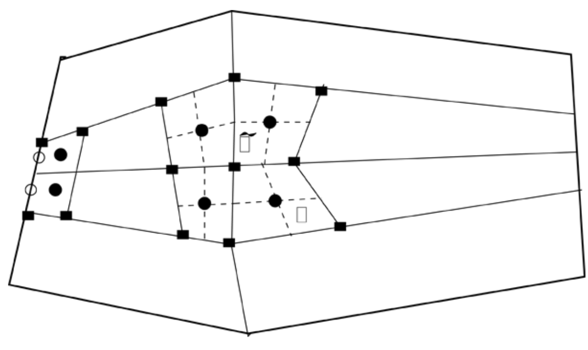


Figure 1: Quadrilateral mesh.

We call $(\mathcal{T}, \widetilde{\mathcal{T}})$ - a pair of QAPD meshes.

We denote by $H_{\widetilde{\mathcal{T}}}(\Omega)$ the space of piecewise constant scalar functions that are constant on each volume $\widetilde{\omega}_\alpha \in \widetilde{\omega}_{\mathcal{J}}$, by $\mathbf{H}_{\mathcal{T}}(\Omega)$ the space of piecewise constant vectorial functions that are constant on each volume $\omega_i \in \omega_{\mathcal{I}}$ and by $\mathbf{H} \otimes \mathbf{H}_{\widetilde{\mathcal{T}}}(\Omega)$ the space of piecewise constant tensorial functions of order two that are constant on each volume $\widetilde{\omega}_a \in \widetilde{\omega}_{\mathcal{J}}$.

For any quantity ψ that is piecewise constant on $\widetilde{\omega}_{\mathcal{J}}$ we denote by ψ_α the constant value of ψ on $\widetilde{\omega}_a$, analogously ψ_i stands for the constant value of a piecewise constant quantity ψ on $\omega_{\mathcal{I}}$.

We define the discrete derivative operators:
$\mathrm{Div}_{(\mathcal{T},\widetilde{\mathcal{T}})} : \mathbf{H}_{\mathcal{T}}(\Omega) \to H_{\widetilde{\mathcal{T}}}(\Omega)$, by

$$\mathrm{Div}_\alpha(\mathbf{u}) := \int\limits_{\partial\widetilde{\omega}_\alpha} \mathbf{u}\cdot\mathbf{n}\mathrm{d}s = \sum_i u_i^a \int\limits_{\partial\widetilde{\omega}_\alpha\cap\omega_i} n_a \mathrm{d}s, \tag{16}$$

$\partial_{(\mathcal{T},\widetilde{\mathcal{T}})} : \mathbf{H}_{\mathcal{T}}(\Omega) \to \mathbf{H}\otimes\mathbf{H}_{\widetilde{\mathcal{T}}}(\Omega)$ by

$$\partial_b\, u^a\big|_\alpha =: \frac{1}{\mathrm{m}(\widetilde{\omega}_\alpha)} \int\limits_{\partial\widetilde{\omega}_\alpha} u^b n_a \mathrm{d}s = \frac{1}{\mathrm{m}(\widetilde{\omega}_a)} \sum_i u_i^b \int\limits_{\partial\widetilde{\omega}_\alpha\cap\omega_i} n_a \mathrm{d}s, \tag{17}$$

$\mathbf{Grad}_{(\mathcal{T},\widetilde{\mathcal{T}})} : H_{\widetilde{\mathcal{T}}}(\Omega) \to \mathbf{H}_{\mathcal{T}}(\Omega)$ by

$$\mathbf{Grad}_i(\phi) \int\limits_{\partial\omega_i} \phi\,\mathbf{n}\mathrm{d}s = \sum_\alpha \phi_\alpha \int\limits_{\partial\omega_i\cap\widetilde{\omega}_\alpha} \mathbf{n}\mathrm{d}s, \tag{18}$$

$\mathbf{rot}_{(\mathcal{T},\widetilde{\mathcal{T}})} : H_{\widetilde{\mathcal{T}}}(\Omega) \longrightarrow \mathbf{H}_{\mathcal{T}}(\Omega)$ by

$$\mathbf{rot}_i(\phi) := \frac{1}{\mathrm{m}(\omega_i)} \int\limits_{\partial\omega_i} \phi \mathrm{d}\mathbf{r} = \frac{1}{\mathrm{m}(\omega_i)} \sum_\alpha \phi_\alpha \int\limits_{\partial\omega_i\cap\widetilde{\omega}_\alpha} \mathrm{d}\mathbf{r}. \tag{19}$$

On the space $\mathbf{H}_{\mathcal{T}}(\Omega)$ we define the scalar product $\langle\langle\cdot,\cdot\rangle\rangle$ by

$$\langle\langle\mathbf{u},\mathbf{v}\rangle\rangle = \sum_{i\in\mathcal{I}} \mathbf{u}_i\cdot\mathbf{v}_i, \tag{20}$$

and on the space $H_{\widetilde{\mathcal{T}}}(\Omega)$ we define the scalar product $\langle\cdot,\cdot\rangle$ by

$$\langle\phi,\psi\rangle = \sum_{\alpha\in\mathcal{J}} \phi_\alpha\psi_\alpha. \tag{21}$$

In the next lemma we prove certain properties of the discrete derivative operators.

Lemma 1. *Let* $\left(\mathcal{T},\widetilde{\mathcal{T}}\right)$ *be a pair of QAPD meshes and the discrete divergence, the discrete gradient and the discrete rotation be defined, respectively, by* (16), (18), *and* (19). *Then:*

(a1) *Discrete Stokes formula. For any* $\mathbf{u} \in \mathbf{H}_{\mathcal{T}}(\Omega)$ *and any* $\phi \in H_{\widetilde{\mathcal{T}}}(\Omega)$, *a discrete integration by parts formula holds, that is*

$$\left\langle \mathrm{Div}_{(\mathcal{T},\widetilde{\mathcal{T}})}(\mathbf{u}), \phi \right\rangle + \left\langle\left\langle \mathbf{u}, \mathbf{Grad}_{(\mathcal{T},\widetilde{\mathcal{T}})}(\phi) \right\rangle\right\rangle = 0. \tag{22}$$

(a2) *For any* $\psi \in H_{\widetilde{\mathcal{T}}}(\Omega)$, $\psi|_{\partial\Omega} = 0$ *one has*

$$\mathrm{Div}_{(\mathcal{T},\widetilde{\mathcal{T}})}\mathbf{rot}_{(\mathcal{T},\widetilde{\mathcal{T}})}\,\psi = 0. \tag{23}$$

Proof. To prove (a1) we use the fact that for any domain ω

$$\int_{\partial\omega} \mathbf{n} \mathrm{d}s = 0$$

and the definitions of the two operators.

To prove (a2), we note firstly that

$$\mathrm{Div}_\alpha(\mathbf{rot}_{(\mathcal{T},\widetilde{\mathcal{T}})}\psi) = \sum_i \mathbf{rot}_i(\psi) \cdot \int_{\partial\widetilde{\omega}_\alpha \cap \omega_i} \mathbf{n} \mathrm{d}s =$$

$$= \sum_i \frac{1}{\mathrm{m}(\omega_i)} \sum_\beta \psi_\beta \int_{\widetilde{\omega}_\beta \cap \partial\omega_i} \mathrm{d}\mathbf{r} \cdot \int_{\partial\widetilde{\omega}_\alpha \cap \omega_i} \mathbf{n} \mathrm{d}s.$$

Then, let $\omega_{i_a^\alpha}$, $a = \overline{1,4}$ be the primal volumes with the common vertex P_α and numbered such that $\omega_{i_a^\alpha}$ and $\omega_{i_{a+1}^\alpha}$ have a common side. For each i_a^α let $P_{\alpha_b^{i_a^\alpha}}$, $b = \overline{1,4}$ be the vertexes of the quadrilateral $\omega_{i_a^\alpha}$ anticlockwise numbered and $P_{\alpha_1^{i_a^\alpha}} = P_\alpha$. We have

$$\frac{1}{\mathrm{m}(\omega_{i_a})} \sum_b \psi_{\alpha_b^{i_a^\alpha}} \int_{\widetilde{\omega}_{\alpha_b^{i_a^\alpha}} \cap \partial\omega_{i_a^\alpha}} \mathrm{d}\mathbf{r} \cdot \int_{\partial\widetilde{\omega}_\alpha \cap \omega_{i_a}} \mathbf{n} \mathrm{d}s = \psi_{\alpha_2^{i_a^\alpha}} - \psi_{\alpha_4^{i_a^\alpha}}.$$

Finally, by summing up for $a = \overline{1,4}$, we have

$$\mathrm{Div}_\alpha(\mathbf{rot}_{(\mathcal{T},\widetilde{\mathcal{T}})}\psi) = \sum_a (\psi_{\alpha_2^{i_a^\alpha}} - \psi_{\alpha_4^{i_a^\alpha}}) = 0,$$

for any α such that $P_\alpha \in \Omega$. If for some α, $P_\alpha \in \partial\Omega$, we use the fact that $\psi_\beta = 0$ on any boundary dual-volumes $\widetilde{\omega}_\beta$.

Now we prove an orthogonal decomposition of the space $\mathbf{H}_{\mathcal{T}}(\Omega)$ that resembles the one for the non-discrete case. Let $\{\Psi^\alpha\}_{\alpha\in\mathcal{J}}, \Psi^\alpha \in H_{\widetilde{\mathcal{T}}}(\Omega)$ be a basis of the space $H_{\widetilde{\mathcal{T}}}(\Omega)$ given by

$$\Psi^\alpha(x) = \begin{cases} 1, & \text{if } x \in \widetilde{w}^\alpha, \\ 0, & \text{if } , x \notin \widetilde{w}^\alpha. \end{cases} \tag{24}$$

Define the discrete vector field $\mathcal{U}^\alpha \in \mathbf{H}_{\mathcal{T}}(\Omega)$ by

$$\mathcal{U}^\alpha = \mathbf{rot}(\Psi^\alpha). \tag{25}$$

Let $\mathbf{W}_{\mathcal{T}}(\Omega)$ be the linear closure of the set $\{\mathcal{U}^\alpha; \alpha \in \mathrm{Int}(\mathcal{J})\}$ in the space $\mathbf{H}_{\mathcal{T}}(\Omega)$ and let $\mathbf{G}_{\mathcal{T}}(\Omega)$ be the subspace orthogonal to it, so that

$$\mathbf{H}_{\mathcal{T}}(\Omega) = \mathbf{W}_{\mathcal{T}}(\Omega) \oplus \mathbf{G}_{\mathcal{T}}(\Omega). \tag{26}$$

We state and prove the following proposition.

Proposition 1. $\mathbf{G}_{\mathcal{T}}(\Omega)$ *consists of elements* $\mathbf{Grad}_{(\mathcal{T},\widetilde{\mathcal{T}})}$ *with* $\phi \in H_{\widetilde{\mathcal{T}}}(\Omega)$.

Proof. Let $\mathbf{u} \in \mathbf{G}_{\mathcal{T}}(\Omega)$, i.e.

$$\langle\langle \mathbf{u}, \mathcal{U}^\alpha \rangle\rangle = 0, \forall \alpha \in \mathrm{Int}(\mathcal{J}). \tag{27}$$

We construct a function $\phi \in H_{\widetilde{\mathcal{T}}}(\Omega)$ such that

$$\mathbf{Grad}_i(\phi) = \mathbf{u}_i,\ \forall i \in \mathcal{I}.$$

For a given ω_i we denote by $P_{\alpha_b^i}$, $b = \overline{1,4}$ its vertexes counterclockwise numbered. The gradient of a scalar field ϕ can be written as

$$\mathbf{Grad}_i(\phi) = \overrightarrow{\tau}_{1,3}(\phi_{\alpha_3^i} - \phi_{\alpha_1^i}) + \overrightarrow{\tau}_{2,4}(\phi_{\alpha_4^i} - \phi_{\alpha_2^i}),$$

where $\overrightarrow{\tau}_{1,3}$ is a vector orthogonal to $\overrightarrow{P_{\alpha_2^i}P_{\alpha_4^i}}$ oriented from $P_{\alpha_1^i}$ to $P_{\alpha_3^i}$ and $|\overrightarrow{\tau}_{1,3}| = \left|\overrightarrow{P_{\alpha_2^i}P_{\alpha_4^i}}\right|/2$ and $\overrightarrow{\tau}_{2,4}$ is a vector orthogonal to $\overrightarrow{P_{\alpha_1^i}P_{\alpha_3^i}}$ oriented from $P_{\alpha_2^i}$ to $P_{\alpha_4^i}$ and $|\overrightarrow{\tau}_{2,4}| = \left|\overrightarrow{P_{\alpha_1^i}P_{\alpha_3^i}}\right|/2$. Hence, we have

$$\begin{aligned} \frac{\mathbf{u}_i \cdot \overrightarrow{P_{\alpha_2^i}P_{\alpha_4^i}}}{\mathrm{m}(\omega_i)} &= \phi_{\alpha_4^i} - \phi_{\alpha_2^i}, \\ \frac{\mathbf{u}_i \cdot \overrightarrow{P_{\alpha_1^i}P_{\alpha_3^i}}}{\mathrm{m}(\omega_i)} &= \phi_{\alpha_3^i} - \phi_{\alpha_1^i}. \end{aligned} \tag{28}$$

The point is that if $\mathbf{u}$ satisfies the ortogonalty conditions (27) then one can solve the equations (28) inductively, i.e. starting from two adjacent values and following some path of continuation. For a general discrete field $\mathbf{u}$ the different paths lead to different values!

Corollary 1 (Discrete Hodge formula)**.** *Let* $(\mathcal{T}, \widetilde{\mathcal{T}})$ *be a pair of QAPD meshes. Then for any* $\mathbf{w} \in \mathbf{H}_{\mathcal{T}}(\Omega)$ *there exists an element* $\mathbf{u} \in \mathbf{H}_{\mathcal{T}}(\Omega)$ *and a scalar function* $\phi \in H_{\widetilde{\mathcal{T}}}(\Omega)$ *such that*

$$\mathbf{w} = \mathbf{u} + \mathbf{Grad}(\phi) \quad \textit{with } \mathrm{Div}_{(\mathcal{T},\widetilde{\mathcal{T}})}(\mathbf{u}) = 0. \tag{29}$$

Proof. We search for a divergence free vector $\mathbf{u}$ of the form

$$\mathbf{u} = \sum_{a \in \mathcal{J}} \alpha_a \mathcal{U}^a.$$

By inserting this form into (29), one obtains a linear algebraic system of equation for the determination of the unknowns $\{\alpha_a\}_{a \in \mathcal{J}}$,

$$\left\langle\left\langle \mathbf{w}, \mathcal{U}^b \right\rangle\right\rangle = \sum_{a \in \mathcal{J}} \alpha_a \left\langle\left\langle \mathcal{U}^a, \mathcal{U}^b \right\rangle\right\rangle.$$

The matrix of the system is the Gram matrix of a linear independent family, hence there exists an unique solution $\mathbf{u}$.

Since $\langle\langle \mathbf{w} - \mathbf{u}, \mathcal{U}^a \rangle\rangle = 0$ for any function in the basis, it follows that $\mathbf{w} - \mathbf{u}$ is orthogonal to $\mathbf{G}^{\perp}$, thus $\mathbf{w} - \mathbf{u} \in \mathbf{G}$. Hence there exists $\phi \in H_{\widetilde{\mathcal{T}}}(\Omega)$ such that

$$\mathbf{w} - \mathbf{u} = \mathbf{Grad}(\phi).$$

3.2 Discrete convective flux and discrete stress flux

To cope with the boundary value problems one defines a partition $\{\partial_k \omega\}_{k \in \mathcal{K}}$ of the boundary $\partial\Omega$ mesh induced by the primal mesh i.e

$$\partial_k \omega = \partial\Omega \cup \partial\omega_{i_k}, \partial\Omega = \cup_{k \in \mathcal{K}} \partial_k \omega.$$

On each $\partial_k \omega$ the boundary data $\mathbf{u}_D$ are approximated by constant values $\mathbf{u}_{D\,k}$.

Several formulas to calculate the numerical convective flux (NCF) are available, most of them derived from the theory of hyperbolic equations.

In the case of a hyperbolic equation, the numerical convective flux must satisfy, besides the accuracy of the approximation requirements, a number of conditions in order that the implied solution be physically relevant. In the case of Navier-Stokes equation at high Reynolds number, the way in which NCF is evaluated is also very important. We propose to define the NCF as follow. Consider the tensorial product $\mathbf{u}\oplus\mathbf{u}$ constant on the dual mesh and, for any control volume ω_i that does not lie on the boundary $\mathcal{F}$, set for the NCF:

$$\mathcal{F}_i^a = \sum_\alpha (u^a u^b)_\alpha \int_{\widetilde{\omega}_\alpha \cap \partial\omega_i} n_b \mathrm{d}s. \tag{30}$$

The tensorial product $\mathbf{u}\oplus\mathbf{u}$ is approximated by

$$(u^a u^b)_\alpha = \frac{1}{\mathrm{m}(\widetilde{\omega}_\alpha)} \int_{\widetilde{\omega}_\alpha} u^a \mathrm{d}x \, \frac{1}{\mathrm{m}(\widetilde{\omega}_\alpha)} \int_{\widetilde{\omega}_\alpha} u^b \mathrm{d}x. \tag{31}$$

The numerical stress flux is set up by considering that the gradient of the velocity is piecewise constant on the dual mesh. This fact implies that the stress tensor is also piecewise constant on the dual mesh. So we can write for the numerical stress flux

$$\mathcal{S}_i(\mathbf{u}) = \sum_\alpha \sigma_\alpha(\mathbf{u}) \cdot \int\limits_{\partial\omega_i \cap \widetilde{\omega}_\alpha} \mathbf{n} \mathrm{d}s. \tag{32}$$

The values of $\sigma_\alpha(\mathbf{u})$ are evaluated as

$$\sigma_\alpha(\mathbf{u}) = 2\nu(|\mathbf{D}_\alpha(\mathbf{u})|)\mathbf{D}_\alpha(\mathbf{u}), \tag{33}$$

where the discrete strain rate tensor $\mathbf{D}_\alpha$ is given by

$$D_{ab}(\mathbf{u})|_\alpha = \frac{1}{2} \left(\partial_a u^b + \partial_b u^a\right)\Big|_\alpha . \tag{34}$$

Dirichlet Boundary conditions. The boundary conditions for the velocity are taken into account through the numerical convective flux and numerical stress flux. If for some α the dual volume $\widetilde{\omega}_\alpha$ intersects the boundary $\partial\Omega$ the gradient of the velocity is given by:

$$\partial_a u^b\Big|_\alpha = \frac{1}{\mathrm{m}(\widetilde{\omega}_\alpha)} \int\limits_{\partial\widetilde{\omega}_\alpha} u^b n_a \mathrm{d}s =$$

$$= \frac{1}{\mathrm{m}(\widetilde{\omega}_\alpha)} \left(\int\limits_{\partial_{\mathrm{ext}}\widetilde{\omega}_\alpha} u_D^b n_a \mathrm{d}s + \sum_i u_i^b \int\limits_{\partial_{\mathrm{int}}\widetilde{\omega}_\alpha \cap \omega_i} n_a \mathrm{d}s \right). \tag{35}$$

For a primal volume ω_i adjacent to the boundary $\partial\Omega$ the NCF (30) is given by

$$\mathcal{F}_i^a = u_D^a u_D^b \int\limits_{\partial\Omega \cap \partial\omega_i} n_b \mathrm{d}s + \sum_\alpha (u^a u^b)_\alpha \int\limits_{\Omega \cap \partial\omega_i} u n_b \mathrm{d}s. \tag{36}$$

4 Fully-Discrete Finite Volume Method

We set up a time integration scheme of the Cauchy problem (14) and (15) that determines the velocity field independently on the pressure field. The pressure field results from the discrete balance momentum equation (14-1). The scheme resembles the Galerkin method and it makes use of the orthogonal decomposition (26) of the space $\mathbf{H}_{\mathcal{T}}(\Omega)$ and of the set of divergence free vectorial fields $\{\mathcal{U}^\alpha\}_{\alpha\in\mathcal{J}^0}$.

We write the unknown velocity field $\mathbf{u}(t)$ as linear combination of $\{\mathcal{U}^\alpha\}_{\alpha\in\mathcal{J}^0}$

$$\mathbf{u} = \sum_\alpha \xi_\alpha(t) \mathcal{U}^\alpha \tag{37}$$

where the coefficients $\xi_\alpha(t)$ are required to satisfy the ordinary differential equations

$$\sum_\alpha \frac{\mathrm{d}\xi_\alpha}{\mathrm{d}t} \left\langle\!\left\langle m\mathcal{U}^\alpha, \mathcal{U}^\beta \right\rangle\!\right\rangle + \left\langle\!\left\langle \mathcal{F}(\xi), \mathcal{U}^\beta \right\rangle\!\right\rangle - \left\langle\!\left\langle \mathcal{S}(\xi), \mathcal{U}^\beta \right\rangle\!\right\rangle = 0,\ \forall\beta \in \mathcal{J}^0, \tag{38}$$

with the initial conditions

$$\sum_\alpha \xi_\alpha(0) \left\langle\!\left\langle \mathcal{U}^\alpha, \mathcal{U}^\beta \right\rangle\!\right\rangle = \left\langle\!\left\langle \mathbf{u}^0, \mathcal{U}^\beta \right\rangle\!\right\rangle,\ \forall\beta \in \mathcal{J}^0. \tag{39}$$

If the functions ξ_α satisfy (38) and (39) then $m\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \mathcal{F}(\{\mathbf{u}\}) - \mathcal{S}(\{\mathbf{u}\})$ belongs to the space $\mathbf{G}_{\mathcal{T}}(\Omega)$ which implies that there exists a scalar field $p(t)$ such that

$$-\mathbf{Grad}_{(\mathcal{T},\widetilde{\mathcal{T}})} p = m\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \mathcal{F}(\{\mathbf{u}\}) - \mathcal{S}(\{\mathbf{u}\}). \tag{40}$$

Concerning the initial conditions (15), we note that for $t = 0$ the solution (37) equals not u^0 but its projection on the space $\mathbf{W}_{\mathcal{T}}(\Omega)$.

Now we develop a time integration scheme for the equation (38) derived from two steps implicit backward differentiation formulae (BDF).

Let $\{t^n\}$ be an increasing sequence of moments of time. We make the notations: $\xi_\alpha^n = \xi_\alpha(t^n)$, $\mathbf{u}^n = \sum\limits_\alpha \xi_\alpha^n \mathcal{U}^\alpha$. Supposing that one knows the values $\{\xi^{n-1}, \xi^n\}$ one calculates the values ξ^{n+1} at the next moment of time t_{n+1} as follows. Define the second degree polynomial $P(t)$ which interpolates the unknown ξ^{n+1} and known quantities $\{\xi^{n-1}, \xi^n\}$ at the moments of time t^{n+1}, t^n, t^{n-1}, respectively. The unknowns ξ^{n+1} are determined by imposing to the polynomial $P(t)$ to satisfy the equations (38).

For a constant time step $\triangle t$ one has

$$\frac{\mathrm{d}P_\alpha(t^{n+1})}{\mathrm{d}t} = \left(\frac{3}{2}\xi_\alpha^{n+1} - 2\xi_\alpha^n + \frac{1}{2}\xi_\alpha^{n-1}\right)/\triangle t$$

that leads to the following nonlinear equations for ξ^{n+1}

$$\begin{aligned}\sum_I \frac{3}{2}\xi_\alpha^{n+1}\left\langle\left\langle m\mathcal{U}^\alpha, \mathcal{U}^\beta\right\rangle\right\rangle + \quad & \triangle t\left\langle\left\langle \mathcal{F}(\xi^{n+1}), \mathcal{U}^\beta\right\rangle\right\rangle - \triangle t\left\langle\left\langle \mathcal{S}(\xi^{n+1}), \mathcal{U}^\beta\right\rangle\right\rangle = \\ & = \left\langle\left\langle 2\mathbf{u}^n - 0.5\mathbf{u}^{n-1}, m\mathcal{U}^\beta\right\rangle\right\rangle.\end{aligned} \tag{41}$$

To overcome the difficulties implied by the nonlinearity, we consider a linear algorithm:

$$\begin{gathered}\sum_\alpha \frac{3}{2}\lambda_\alpha^{n+1}\left\langle\left\langle m\mathcal{U}^\alpha, \mathcal{U}^\beta\right\rangle\right\rangle - \triangle t\left\langle\left\langle \mathcal{S}(\mathbf{u}; \lambda^{n+1}), \mathcal{U}^\beta\right\rangle\right\rangle = \\ 0.5\left\langle\left\langle \mathbf{u}^n - \mathbf{u}^{n-1}, m\mathcal{U}^\beta\right\rangle\right\rangle - \\ -\triangle t\left\langle\left\langle \frac{3}{2}\mathcal{F}(\mathbf{u}^n) - \frac{1}{2}\mathcal{F}(\mathbf{u}^{n-1}), \mathcal{U}^\beta\right\rangle\right\rangle + \triangle t\left\langle\left\langle \mathcal{S}(\mathbf{u}^n), \mathcal{U}^\beta\right\rangle\right\rangle,\end{gathered} \tag{42}$$

where

$$\lambda^{n+1} := \xi^{n+1} - \xi^n.$$

For the first step one can use a Euler step

$$\begin{gathered}\sum_\alpha \lambda_\alpha^{n+1}\left\langle\left\langle m\mathcal{U}^\alpha, \mathcal{U}^\beta\right\rangle\right\rangle - \triangle t\left\langle\left\langle \mathcal{S}(\mathbf{u}^n; \lambda^{n+1}), \mathcal{U}^\beta\right\rangle\right\rangle = \\ -\triangle t\left\langle\left\langle \mathcal{F}(\mathbf{u}^n), \mathcal{U}^\beta\right\rangle\right\rangle + \triangle t\left\langle\left\langle \mathcal{S}(\mathbf{u}^n), \mathcal{U}^\beta\right\rangle\right\rangle.\end{gathered} \tag{43}$$

In both (42) and (43) schemes we use the notation

$$\mathcal{S}(\mathbf{u}^n; \lambda^{n+1}) = 2\nu(|\mathbf{D}(\mathbf{u}^n)|) \sum_{\alpha} \lambda_{\alpha}^{n+1} \mathbf{D}(\mathcal{U}^{\alpha}).$$

5 Numerical Simulations

We present the results of some numerical experiments designed to test the numerical method presented in the previous sections.

We consider the pseudo-plastic fluid modeled by the Carreau-Yasuda law,

$$\nu(\dot{\gamma}) = \nu_{\infty} + (\nu_0 - \nu_{\infty}) \left(1 + (\Lambda\dot{\gamma})^a\right)^{(n-1)/a}.$$

In the current study the problem was solved for a series of rectangular regular or non-regular meshes. The code incorporates the time integration scheme (42) and (43); the numerical convective flux $\mathcal{F}$ defined by the formulae (30), (31), (36) and the numerical stress flux $\mathcal{S}$ defined by formulae (32), (33), (34), (17), (35). In all the numerical simulations we consider that at the initial time the fluid is at rest.

Lid Driven Cavity Flow

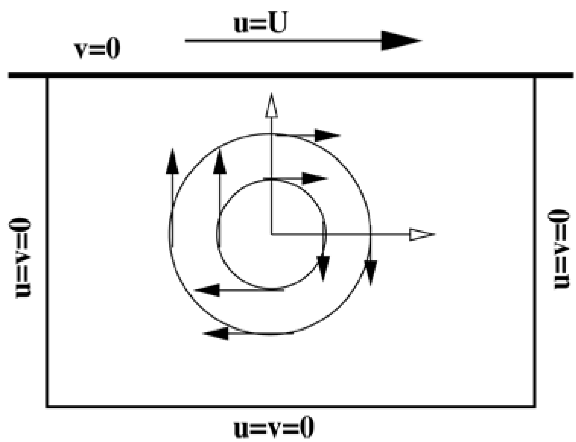


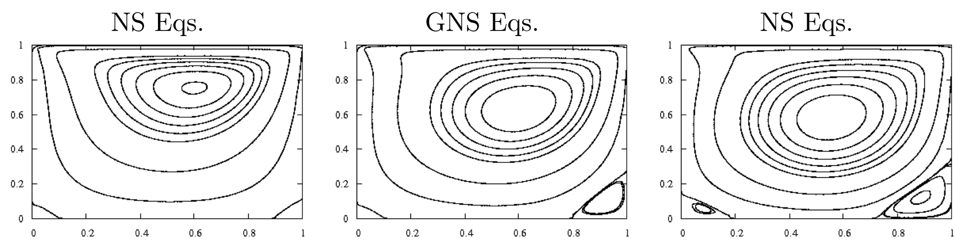
Figure 2: Lid Driven 2D Cavity Flow.

The fluid is moving in a rectangular box, the side and bottom walls are static while the top wall is moving across the cavity with a constant velocity

$u = U, v = 0$ as in Fig. 2. We assume the non-slip boundary conditions on the walls.

In the first set of computations we test the capabilty of the method to catch the behavior of the pseudo-plastic fluid. To be more precise, we chose a pseudo-plastic fluid and two Navier Stokes fluids having the viscosities equal to ν_0, and ν_∞, respectively.

Figure 3 shows the contours plots of the steady solutions for the three type of fluids. Each flow consists of a core of fluid undergoing solid body rotation and small regions in the bottom corners of counter-rotating vortex. The intensity of the counter-rotating vortex is decreasing with respect to viscosity. The velocity profile along the vertical centerline is shown in Figure 4. We observe that, in the lower part of the cavity, the fluid is moving in contrary sense to the sense of the motion of the top wall. The maximum of the negative velocity depends decreasingly on the viscosity of the fluid.



$\mathrm{Re}_0 = 10^2/1.57$ $\mathrm{Re}_0 \leq \mathrm{Re} \leq \mathrm{Re}_\infty$ $\mathrm{Re}_\infty = 10^3/1.57$

Figure 3: $U = 0.01\mathrm{ms}^{-1}$, $a = 0.144$. Contour plot of stream functions, steady solutions. Regular grid, 51×51 grid points.

The second set of computations analyzes the response of the numerical method to the variation of the parameters of the fluid. The results are shown in Figure 5.

Final Remarks

A certain advantage of our method is that there is no need to introduce artificial boundary conditions for the pressure field or supplementary boundary conditions for additional velocity field as in the projection methods or gauge methods. The preliminary numerical results prove a good agreement with

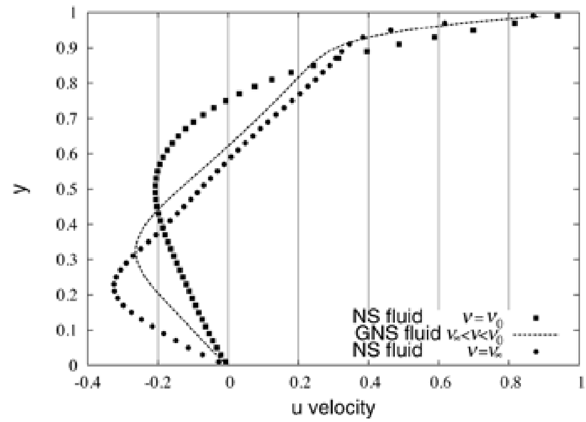
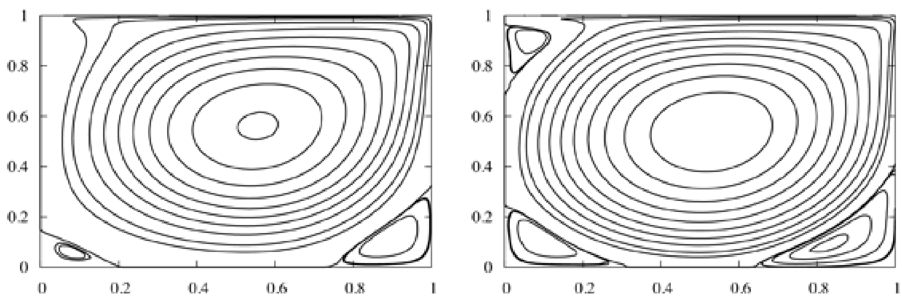


Figure 4: $U = 0.01\mathrm{ms}^{-1}$, $a = 0.144$. Distribution of u-velocity along at vertical centre line of the cavity. Regular grid, 51×51 grid points.



$a = 0.644$
steady solution

$a = 0.144$
$t = 200$

Figure 5: GNS Eqs. $U = 0.1\mathrm{ms}^{-1}$ $\mathrm{Re}_\infty = 10^4/1.57, \mathrm{Re}_0 = 10^3/1.57$. Stretched grid, 51×51 grid points.

the results obtained by other methods. At the present moment we do not know if it is possible to extended the method to the 3D case and this is a drawback of the method. The study of this extension might be a task for our future work.

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