Annals of the Academy of Romanian Scientists
Series on Mathematics and its Applications
ISSN 2066 - 6594 Volume 3, Number 1 / 2011

***In Memoriam Adelina Georgescu***

# $H_2$ OPTIMAL CONTROLLERS FOR A LARGE CLASS OF LINEAR STOCHASTIC SYSTEMS WITH PERIODIC COEFFICIENTS*

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**Abstract**

In this paper the $H_2$ type optimization problem for a class of time varying linear stochastic systems modeled by Ito differential equations and Markovian jumping with periodic coefficients is considered. The main goal of such an optimization problem is to minimize the effect of additive white noise perturbations on a suitable output of the controlled system. It is assumed that only an output is available for measurements. The solution of the considered optimization problem is constructed via the stabilizing solutions of some suitable systems of generalized Riccati differential equations with periodic coefficients.

**MSC**: 93E20, 93E15, 93E03.

**keywords:** $H_2$ norms; linear stochastic systems; periodic coefficients; output based controllers; Riccati differential equations.

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*Accepted for publication on December 15, 2010.
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# 1 Introduction

The $H_2$ and the linear quadratic control problems for linear stochastic systems have been widely studied in the current literature. A particular attention was paid to two classes of stochastic systems, namely Markov jump linear systems and systems subject to multiplicative white noise.When an important and unpredictible variation causes a discrete change in the plant characterization at isolated points in time, a Markov chain with a finite state space is a natural model for the plant parameter processes.

Some illustrative applications of these systems can be found for example in [2, 13, 16, 17] and their references, where stochastic stability properties and useful results concerning controllability, observability and optimal control are presented.

More recently, the $H_2$ control problem for Markov jump linear systems has been studied in [3] for the state feedback case and [11] for the output feedback case. The stochastic systems with multiplicative white noise naturally arise in control problems of linear uncertain systems with stochastic uncertainty (see [12, 15, 19] and the references therein). Results concerning the $H_2$ control problem for this type of systems are derived for instance in [4, 6]. In [8] the $H_2$ optimal state feedback control problem is addressed for time varying periodic linear stochastic systems subject to both Markov jumps and multiplicative white noise. The afore mentioned paper extends to the time varying periodic case a part of the results from [7].

In the present paper we extend the results of [8] to the case when only an output is available for measurements. Lately, there is an increasing interest in the consideration of control problems for systems modeled by differential equations with periodic coefficients. For the reader's convenience we refer to [1].

The outline of the paper is: Section 2 contains the description of the mathematical model of the considered controlled systems. Also the $H_2$ optimization problems are stated. Section 3 collects several auxiliary results which are required for the proof of the main result. Formulae for the computation of $H_2$-norms of a linear stochastic system with periodic coefficients are provided. The main result of the paper is given in Section 4.

## 2 The problem formulation

Consider the controlled system (**G**) modeled by a system of the Ito differential equations perturbed by a Markov process of the form:

$$
\begin{aligned}
dx(t) &= (A_0(t,\eta_t)x(t) + B_0(t,\eta_t)u(t))dt \\
&\quad + \sum_{k=1}^{r}(A_k(t,\eta_t)x(t) + B_k(t,\eta_t)u(t))dw_k(t) + B_v(t,\eta_t)dv(t) \\
dy(t) &= C_0(t,\eta_t)x(t)dt + \sum_{k=1}^{r} C_k(t,\eta_t)x(t)dw_k(t) + D_v(t,\eta_t)dv(t) \quad (2.1) \\
z(t) &= C_z(t,\eta_t)x(t) + D_z(t,\eta_t)u(t)
\end{aligned}
$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}^m$ are the control parameters, $y(t) \in \mathbf{R}^{n_y}$ are the measurements, while $z(t) \in \mathbf{R}^{n_z}$ is the controlled output. In (2.1) $\{\eta_t\}_{t\geq 0}$ is an homogeneous right continuous Markov process on a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with the set of the states $\mathfrak{S} = \{1, 2, ..., N\}$ and the transition probability matrix $P(t) = e^{Qt}, t \geq 0$, where $Q \in \mathbf{R}^{N\times N}$ is a matrix whose elements have the properties: $q_{ij} \geq 0$, if $i \neq j$ and $\sum\limits_{j=1}^{N} q_{ij} = 0$ for all $1 \leq i \leq N$. Also, the existence of $\lim_{t\to\infty} P(t)$ is valid. For details see for example [5]. Here, $(w^T(t), v^T(t))^T$ is an $(r + m_v)$-dimensional standard Wiener process. $w(t) = (w_1(t), ..., w_r(t))^T$, $v(t) = (v_1(t), ..., v_{m_v}(t))^T$ (see [14, 18]).

Throughout this paper, we make the following assumptions:

$\mathbf{H_1}$: $\{w(t)\}_{t\geq 0}$, $\{v(t)\}_{t\geq 0}$, $\{\eta_t\}_{t\geq 0}$ are independent stochastic processes and $\mathcal{P}\{\eta_0 = i\} > 0$, $1 \leq i \leq N$.

$\mathbf{H_2}$: $A_k(\cdot, i) : \mathbf{R} \to \mathbf{R}^{n\times n}$, $B_k(\cdot, i) : \mathbf{R} \to \mathbf{R}^{n\times m}$, $C_k(\cdot, i) : \mathbf{R} \to \mathbf{R}^{n_y\times n}$, $0 \leq k \leq r$, $B_v(\cdot, i) : \mathbf{R} \to \mathbf{R}^{n\times m_v}$, $D_v(\cdot, i) : \mathbf{R} \to \mathbf{R}^{n_y\times m_v}$, $C_z(\cdot, i) : \mathbf{R} \to \mathbf{R}^{n_z\times n}$, $D_z(\cdot, i) : \mathbf{R} \to \mathbf{R}^{n_z\times m}$, $1 \leq i \leq N$, are continuous matrix valued functions which are periodic with the period $\theta > 0$.

To control the system (2.1) we consider the following admissible controllers $(\mathbf{G}_c)$ having the following state space representation:

$$
\begin{aligned}
dx_c(t) &= A_{c_0}(t,\eta_t)x_c(t)dt + \sum_{k=1}^{r} A_{ck}(t,\eta_t)x_c(t)dw_k(t) + B_c(t,\eta_t)du_c(t) \qquad (2.2)\\
y_c(t) &= C_c(t,\eta_t)x_c(t)
\end{aligned}
$$

where $x_c(t) \in \mathbf{R}^{n_c}$ is the vector of the state parameters of the controller, $u_c(t) \in \mathbf{R}^{n_y}$ is the vector of the inputs of the controller and $y_c \in \mathbf{R}^m$ is the output of the controller, $A_{ck}(\cdot,i)$, $0 \le k \le r$, $B_c(\cdot,i)$, $C_c(\cdot,i)$ are continuous matrix valued functions which are periodic with period $\theta$. As in the time invariant case, (see [7],[10]), the order $n_c$ of an admissible controller is not prefixed. It will be determined in the process of the construction of the solution of the optimization problems. The closed-loop system obtained when coupling an admissible controller (2.2) to the system (2.1) by taking $u_c(t) = y(t)$ and $u(t) = y_c(t)$ has the state space representation given by:

$$
(\mathbf{G}_{cl}) : \begin{cases} dx_{cl}(t) = A_{0cl}(t,\eta_t)x_{cl}(t)dt + \sum\limits_{k=1}^{r} A_{kcl}(t,\eta_t)x_{cl}(t)dw_k(t) + \\ \qquad\qquad\qquad + B_{vcl}(t,\eta_t)dv(t), \\ z_{cl}(t) = C_{cl}(t,\eta_t)x_{cl}(t). \end{cases} \tag{2.3}
$$

where,

$$
x_{cl}(t) = \begin{pmatrix} x^T(t) & x_c^T(t) \end{pmatrix}^T, A_{kcl}(t,i) = \begin{pmatrix} A_k(t,i) & B_k(t,i)C_c(t,i) \\ B_c(t,i)C_k(t,i) & A_{ck}(t,i) \end{pmatrix},
$$

$$
\begin{gathered}
0 \le k \le r, \quad B_{vcl}(t,i) = \begin{pmatrix} B_v(t,i) \\ B_c(t,i)D_v(t,i) \end{pmatrix}, \\
C_{cl}(t,i) = \begin{pmatrix} C_z(t,i) & D_z(t,i)C_c(t,i) \end{pmatrix}
\end{gathered} \tag{2.4}
$$

for all $t \in \mathbf{R}$, $1 \le i \le N$.

**Definition 2.1.** An admissible controller $(\mathbf{G}_c)$ of the form (2.2) is a stabilizing controller for the systems $(\mathbf{G})$ if the zero state equilibrium of the linear system

$$
dx_{cl}(t) = A_{0cl}(t,\eta_t)x_{cl}(t)dt + \sum_{k=1}^{r} A_{kcl}(t,\eta_t)x_{cl}dw_k(t) \tag{2.5}
$$

is exponentially stable in mean square (ESMS).

We denote $\mathcal{K}_s(\mathbf{G})$ the set of all stabilizing controllers of type (2.2).

Now, we construct the following two cost functionals associated to the system $(\mathbf{G})$:
$J_l : \mathcal{K}_s(\mathbf{G}) \to \mathbf{R}^+, \;\; l \in \{1, 2\}$ by

$$J_1(\mathbf{G}_c) = \lim_{\tau\to\infty} \frac{1}{\tau} \int_{t_0}^{t_0+\tau} E[|z_{cl}(t)|^2]dt \tag{2.6}$$

and

$$J_2(\mathbf{G}_c) = \lim_{\tau\to\infty} \frac{1}{\tau} \int_{t_0}^{t_0+\tau} \sum_{i=1}^{N} E[|z_{cl}(t)|^2/\eta_{t_0} = i]dt \tag{2.7}$$

In this paper we shall solve the following optimization problems, which will be called stochastic $H_2$ optimal control problems:

$\mathbf{OP_1}$ : Construct a stabilizing controller $\mathbf{G}_c^1 \in \mathcal{K}_s(\mathbf{G})$ with the property that

$$J_1(\mathbf{G}_c^1) = min\{J_1(\mathbf{G}_c)|\mathbf{G}_c \in \mathcal{K}_s(\mathbf{G})\} \tag{2.8}$$

$\mathbf{OP_2}$ : Construct an admissible controller $(\mathbf{G}_c^2) \in \mathcal{K}_s(\mathbf{G})$ with the property that

$$J_2(\mathbf{G}_c^2) = min\{J_2(\mathbf{G}_c)|\mathbf{G}_c \in \mathcal{K}_s(\mathbf{G})\}. \tag{2.9}$$

**Remark 2.1.** a) In the next section we shall see that both $J_1(\mathbf{G}_c)$ and $J_2(\mathbf{G}_c)$ do not depend upon the initial time $t_0$ and the initial state $x_{cl}(t_0)$. The values of these cost functionals are expressed in terms of bounded solutions of some suitable affine differential equations which extend to this framework the differential equations of the controllability Gramian and observability Gramian.

b) Also in Section 3 we shall see that the value of the cost functional $J_1(\mathbf{G}_c)$ depends upon the initial distribution $\pi_0 = (\pi_0(1), ..., \pi_0(N))$, $(\pi_0(i) = \mathcal{P}\{\eta_0 = i\})$ of the Markov process, while in the case of the second optimization problem, the value of the cost functional $J_2(\mathbf{G}_c)$ does not depend upon the initial distribution of the Markov process.

## 3 Several preliminary results

Let $\mathcal{S}_n \subset \mathbf{R}^{n\times n}$ be the linear subspace of the real symmetric matrices. Define $\mathcal{S}_n^N$ by $\mathcal{S}_n^N = \mathcal{S}_n \oplus \mathcal{S}_n \oplus ... \oplus \mathcal{S}_n$. We recall that $\mathcal{S}_n^N$ is a real Hilbert space with respect to the inner product

$$< \mathbf{X}, \mathbf{Y} >= \sum_{i=1}^{N} Tr[X(i)Y(i)] \tag{3.1}$$

for all $\mathbf{X} = (X(1), ..., X(N))$, $\mathbf{Y} = (Y(1), ..., Y(N)) \in \mathcal{S}_n^N$.

Additionally, $\mathcal{S}_n^N$ is the ordered linear space, via the ordering induced by the cone

$$\mathcal{S}_n^{N+} = \{\mathbf{X} \in \mathcal{S}_n^N | \mathbf{X} = (X(1), X(2), ..., X(N)),\ X(i) \geq 0, 1 \leq i \leq N\}. \tag{3.2}$$

Here $X(i) \geq 0$ means that $X(i)$ is positive semidefinite. For more details concerning the properties of the cone $\mathcal{S}_n^{N+}$ we refer to [9].

Based on the coefficients of the linear system (2.5) we construct the following operator valued function $t \to \mathcal{L}_{cl}(t)$ by $\mathcal{L}_{cl}(t)\mathbf{X} = ((\mathcal{L}_{cl}(t)X)(1), (\mathcal{L}_{cl}(t)X)(2),$ $..., (\mathcal{L}_{cl}(t)X)(N))$ where

$$\begin{aligned}(\mathcal{L}_{cl}(t)\mathbf{X})(i) &= A_{0cl}(t,i)X(i) + X(i)A_{0cl}^T(t,i) + \\ &+ \sum_{k=1}^{r} A_{kcl}(t,i)X(i)A_{kcl}^T(t,i) + \sum_{j=1}^{N} q_{ji}X(j)\end{aligned} \tag{3.3}$$

for all $\mathbf{X} = (X(1), ..., X(N)) \in \mathcal{S}_{n+n_c}^N$. By direct calculation one obtains that the adjoint operator of $\mathcal{L}_{cl}(t)$ with respect to the inner product (3.1) is given by
$\mathcal{L}_{cl}^*(t)\mathbf{X} = ((\mathcal{L}_{cl}^*(t)\mathbf{X})(1), ..., (\mathcal{L}_{cl}^*(t)\mathbf{X})(N))$,

$$\begin{aligned}(\mathcal{L}_{cl}^*\mathbf{X})(i) &= A_{0cl}^T(t,i)X(i) + X(i)A_{0cl}(t,i) + \\ &+ \sum_{k=1}^{r} A_{kcl}^T(t,i)X(i)A_{kcl}(t,i) + \sum_{j=1}^{N} q_{ij}X(j)\end{aligned} \tag{3.4}$$

for all $\mathbf{X} \in \mathcal{S}_{n+n_c}^N$.

In our developments an important role is played by the following affine differential equations on $\mathcal{S}_{n+n_c}^N$:

$$\frac{d}{dt}Y(t) = \mathcal{L}_{cl}(t)Y(t) + \mathcal{B}^\epsilon(t) \tag{3.5}$$

$$\frac{d}{dt}X(t) + \mathcal{L}_{cl}^*(t)X(t) + \mathcal{C}(t) = 0 \tag{3.6}$$

where $\mathcal{B}^\epsilon(t) = (\mathcal{B}^\epsilon(t,1), \mathcal{B}^\epsilon(t,2), ..., \mathcal{B}^\epsilon(t,N))$,

$$\mathcal{B}^\epsilon(t,i) = \varepsilon(i)B_{vcl}(t,i)B_{vcl}^T(t,i) \tag{3.7}$$

and $\mathcal{C}(t) = (\mathcal{C}(t,1), \mathcal{C}(t,2), ..., \mathcal{C}(t,N))$,

$$\mathcal{C}(t,i) = C_{cl}^T(t,i)C_{cl}(t,i) \tag{3.8}$$

In (3.7) $\varepsilon(i)$ are given nonnegative scalars. Applying Theorem 4.9 and Theorem 4.7 in [9] in the case of the equations (3.5) and (3.6), respectively, we obtain:

**Corollary 3.1.** *Under the considered assumptions, if the zero state equilibrium of the linear system (2.5) is (ESMS), each of the affine differential equations (3.5) and (3.6), has a unique bounded on* $\mathbf{R}$ *solution* $\mathcal{Y}_{cl}^\epsilon(t)$ *and* $\mathcal{X}_{cl}(t)$*, respectively. Additionally, these solutions have the properties:*

(i) $\mathcal{Y}_{cl}^\epsilon(t) \in \mathcal{S}_{n+n_c}^{N+}$, $\mathcal{X}_{cl}(t) \in \mathcal{S}_{n+n_c}^{N+}$ for all $t \in \mathbf{R}$.

(ii) $\mathcal{Y}_{cl}^\epsilon(t+\theta) = \mathcal{Y}_{cl}^\epsilon(t)$, $\mathcal{X}_{cl}(t+\theta) = \mathcal{X}_{cl}(t), \forall t \in \mathbf{R}$.

**Remark 3.1.** In the special case of $N = 1$, $A_k(t,1) = 0$, $1 \le k \le r$ the differential equations (3.5) and (3.6) reduce to the well known differential equations of the controllability Gramian and observability Gramian, respectively from the deterministic case.

The following result provides values of the cost functionals (2.6), (2.7) respectively, in terms of the bounded solutions of the affine differential equations (3.5) and (3.6).

**Theorem 3.2.** *Under the assumptions* $\mathbf{H_1}$ *and* $\mathbf{H_2}$ *for each stabilizing controller* $\mathbf{G}_c$) *the following equalities hold:*

*(i)*

$$J_1(\mathbf{G}_c) = \frac{1}{\theta}\int_0^\theta \sum_{j=1}^N \pi_{j\infty} Tr[B_{vcl}^T(s,j)\mathcal{X}_{cl}(s,j)B_{vcl}(s,j)]ds \tag{3.9}$$

$$= \frac{1}{\theta}\int_0^\theta \sum_{j=1}^N Tr[C_{cl}(s,j)\mathcal{Y}_{cl}^{\pi\infty}(s,j)C_{cl}^T(s,j)]ds$$

*(ii)*

$$J_2(\mathbf{G}_c) = \frac{1}{\theta}\int_0^\theta \sum_{j=1}^N \delta(j) Tr[B_{vcl}^T(s,j)\mathcal{X}_{cl}(s,j)B_{vcl}(s,j)]ds \tag{3.10}$$

$$= \frac{1}{\theta}\int_0^\theta \sum_{j=1}^N Tr[C_{cl}(s,j)\mathcal{Y}_{cl}^{\delta}(s,j)C_{cl}^T(s,j)]ds$$

*where $\mathcal{X}_{cl}(t) = (\mathcal{X}_{cl}(t,1),...,\mathcal{X}_{cl}(t,N))$ is the unique bounded solution of the affine differential equation (3.6), (3.8), while $\mathcal{Y}_{cl}^{\pi\infty}(t) = (\mathcal{Y}_{cl}^{\pi\infty}(t,1),...,\mathcal{Y}_{cl}^{\pi\infty}(t,N))$ is the unique bounded on $\mathbf{R}$ solution of affine differential equation (3.5), (3.7) with $\varepsilon(j) = \pi_{j\infty}$ and $\mathcal{Y}_{cl}^{\delta}(t) = (\mathcal{Y}_{cl}^{\delta}(t,1),...,\mathcal{Y}_{cl}^{\delta}(t,N))$ is the unique bounded on $\mathbf{R}$ solution of the affine differential equation (3.5), (3.7) for $\varepsilon(j) = \delta(j)$. The scalars $\pi_{j\infty}$ and $\delta(j)$ are defined by*

$$\pi_{j\infty} = \sum_{i=1}^N \tilde{p}_{ij}\mathcal{P}\{\eta_0 = i\}, \quad \delta(j) = \sum_{i=1}^N \tilde{p}_{ij} \tag{3.11}$$

*where $\tilde{p}_{ij}$ are the elements of the matrix $\tilde{P} = \lim_{t\to\infty} P(t)$.*

**Proof.** The first equalities of (3.9) and (3.10), respectively, are obtained directly applying Theorem 4.2 and Theorem 4.3, respectively in [8].

We rewrite the second part of (3.9) and (3.10) in an unified manner, as follows:

$$\frac{1}{\theta}\int_0^\theta \sum_{j=1}^N \varepsilon(j) Tr[B_{vcl}^T(s,j)\mathcal{X}_{cl}(s,j)B_{vcl}(s,j)]ds \tag{3.12}$$

$$= \frac{1}{\theta}\int_0^\theta \sum_{j=1}^N Tr[C_{cl}(s,j)\mathcal{Y}_{cl}^{\epsilon}(s,j)C_{cl}^T(s,j)]ds$$

So, to complete the proof of the theorem it is sufficient to show that (3.12) is true for some given nonnegative scalars $\varepsilon(j)$.

Using (3.7) together with (3.1) we obtain

$$\sum_{j=1}^{N} \varepsilon(j) Tr[B_{vcl}^T(s,j)\mathcal{X}_{cl}(s,j)B_{vcl}(s,j)] = < \mathcal{X}_{cl}(s), \mathcal{B}^{\epsilon}(s) > .$$

Using successively equations (3.5) and (3.6) we deduce

$$\sum_{j=1}^{N} \varepsilon(j) Tr[B_{vcl}^T(s,j)\mathcal{X}_{cl}(s,j)B_{vcl}(s,j)] = \frac{d}{ds} < \mathcal{X}_{cl}(s), \mathcal{Y}_{cl}^{\epsilon}(s) > + < \mathcal{C}(s), \mathcal{Y}_{cl}^{\epsilon}(s) > .$$

Integrating the last equality from $s = 0$ to $s = \theta$ we obtain via Corollary 3.1. (ii) that

$$\int_0^{\theta} \sum_{j=1}^{N} \varepsilon(j) Tr[B_{vcl}^T(s,j)\mathcal{X}_{cl}(s,j)B_{vcl}(s,j)]ds = \int_0^{\theta} < \mathcal{C}(s), \mathcal{Y}_{cl}^{\epsilon}(s) > ds =$$

$$= \int_0^{\theta} \sum_{j=1}^{N} Tr[C_{cl}(s,j)\mathcal{Y}_{cl}^{\epsilon}(s,j)C_{cl}^T(s,j)].$$

For the last equality we used (3.1) together with (3.8). Thus the proof is complete.

**Remark 3.2.** From (3.9) and (3.10) one sees that the values of the cost functionals (2.6) and (2.7), respectively, do not depend upon the initial conditions $(t_0, x_{cl}(t_0))$ of the trajectories of the closed loop system $(\mathbf{G}_{cl})$. These values may be seen as measures of the effect of the additional white noise on an output of the closed-loop system. So, the optimization problems we want to solve in this work minimize the effect of the additive white noise perturbations on a suitable output of the closed-loop system.

To construct the optimal controllers of the two optimization problems stated before, we need the stabilizing solution of the following systems of Riccati equations:

a) System of generalized Riccati differential equations of control SGRDE-C

$$\frac{d}{dt}X(t,i)+A_0^T(t,i)X(t,i)+X(t,i)A_0(t,i)+\sum_{k=1}^{r}A_k^T(t,i)X(t,i)A_k(t,i)+$$
$$+\sum_{j=1}^{N}q_{ij}X(t,j)-(X(t,i)B_0(t,i)+\sum_{k=1}^{r}A_k^T(t,i)X(t,i)B_k(t,i)+$$
$$+C_z^T(t,i)D_z(t,i))(D_z^T(t,i)D_z(t,i)+$$
$$+\sum_{k=1}^{r}B_k^T(t,i)X(t,i)B_k(t,i))^{-1}(B_0^T(t,i)X(t,i)+ \tag{3.13}$$
$$+\sum_{k=1}^{r}B_k^T(t,i)X(t,i)A_k(t,i)+D_z^T(t,i)C_z(t,i))+$$
$$+C_z^T(t,i)C_z(t,i)=0, \quad 1\le i\le N.$$

b) System of generalized Riccati differential equations of filtering SGRDE-F

$$\frac{d}{dt}Y(t,i)=A_0(t,i)Y(t,i)+Y(t,i)A_0^T(t,i)+\sum_{k=1}^{r}A_k(t,i)Y(t,i)A_k^T(t,i)+$$
$$+\sum_{j=1}^{N}q_{ji}Y(t,j)-(Y(t,i)C_0^T(t,i)+\sum_{k=1}^{r}A_k(t,i)Y(t,i)C_k^T(t,i)+$$
$$\varepsilon(i)B_v(t,i)D_v^T(t,i))(\varepsilon(i)D_v(t,i)D_v^T(t,i) \tag{3.14}$$
$$+\sum_{k=1}^{r}C_k(t,i)Y(t,i)C_k^T(t,i))^{-1}(C_0(t,i)Y(t,i)$$
$$+\sum_{k=1}^{r}C_k(t,i)Y(t,i)A_k^T(t,i)+\varepsilon(i)D_v(t,i)B_v^T(t,i))+$$
$$+\varepsilon(i)B_v(t,i)B_v^T(t,i), \quad 1\le i\le N.$$

We recall that a global solution $\mathbf{X}_s:\mathbf{R}\to\mathcal{S}_n^N$ of SGRDE-C (3.13) is called stabilizing solution if the zero state equilibrium of the following closed-loop system

$$dx(t)=(A_0(t,\eta_t)+B_0(t,\eta_t)F_s(t,\eta_t))x(t)dt \tag{3.15}$$
$$+\sum_{k=1}^{r}(A_k(t,\eta_t)+B_k(t,\eta_t)F_s(t,\eta_t))x(t)dw_k(t)$$

is ESMS, where

$$F_s(t,i) = -(D_z^T(t,i)D_z(t,i)+\sum_{k=1}^r B_k^T(t,i)X_s(t,i)B_k(t,i))^{-1}(B_0^T(t,i)X_s(t,i)$$
$$+\sum_{k=1}^r B_k^T(t,i)X_s(t,i)A_k(t,i)+D_z^T(t,i)C_z(t,i)), 1\le i\le N. \quad (3.16)$$

Also, a global solution $\mathbf{Y}_s : \mathbf{R} \to \mathcal{S}_n^N$ of SGRDE-F (3.14) is called stabilizing solution if the zero state equilibrium of the closed-loop system

$$dx(t) = (A_0(t,\eta_t) + K_s(t,\eta_t)C_0(t,\eta_t))x(t)dt \quad (3.17)$$
$$+\sum_{k=1}^r (A_k(t,\eta_t) + K_s(t,\eta_t)C_k(t,\eta_t))x(t)dw_k(t)$$

is ESMS, where

$$K_s(t,i) = -(Y_s(t,i)C_0^T(t,i)+\sum_{k=1}^r A_k(t,i)Y_s(t,i)C_k^T(t,i) + \quad (3.18)$$
$$+\varepsilon(i)B_v(t,i)D_v^T(t,i))(\sum_{k=1}^r C_k(t,i)Y_s(t,i)C_k^T(t,i)+\varepsilon(i)D_v(t,i)D_v^T(t,i))^{-1},$$
$$1 \le i \le N.$$

It must be remarked that in (3.14) and (3.18), $\varepsilon(i)$ is replaced by $\pi_{i\infty}$ in the case of $\mathbf{OP_1}$ and by $\delta(i)$, in the case of $\mathbf{OP_2}$, respectively.

Applying Theorem 7 from Chapter 4 in [10] one obtains a set of necessary and sufficient conditions for the existence of the bounded on $\mathbf{R}$ and stabilizing solution
$\mathbf{X}_s(t) = (X_s(t,1), ..., X_s(t,N))$ of SGRDE-C (3.13) which satisfies the condition

$$D_z^T(t,i)D_z(t,i) +\sum_{k=1}^r B_k^T(t,i)X_s(t,i)B_k(t,i)>0,\ 0 \le t \le \theta,\ 1 \le i \le N. \quad (3.19)$$

Moreover, $X_s(t+\theta, i) = X_s(t,i) \ \ \forall t \in \mathbf{R}, \ \ 1 \le i \le N.$

Also, applying Theorem 7 from Chapter 4 in [10] to a suitable dual equation one obtains a set of necessary and sufficient conditions for the existence of a bounded on $\mathbf{R}$ and stabilizing solution $\mathbf{Y}_s(t) = (Y_s(t,1),...,Y_s(t,N))$ of the SGRDE-F (3.14) which satisfies the following sign condition:

$$\varepsilon(i)D_v(t,i)D_v^T(t,i)+\sum_{k=1}^{r}C_k(t,i)Y_s(t,i)C_k^T(t,i)>0,\ 0\leq t\leq\theta,\ 1\leq i\leq N. \quad (3.20)$$

Additionally, we have $Y_s(t+\theta,i) = Y_s(t,i) \quad (\forall)t \in \mathbf{R},\ \ i \in \mathfrak{S}$.

Several aspects concerning the numerical computation of the stabilizing solutions of (3.13) and (3.14), respectively, via some Lyapunov iterations can be found in [8] or [10] Chapter 4.

## 4 The main result

Let us introduce the following performance index $W_\epsilon : \mathcal{K}_s(\mathbf{G}) \to \mathbf{R}^+$ defined by:

$$W_\epsilon(\mathbf{G}_c) = \frac{1}{\theta}\int_0^\theta \sum_{j=1}^{N} \varepsilon(j)Tr[B_{vcl}^T(s,j)\mathcal{X}_{cl}(s,j)B_{vcl}(s,j)]ds \quad (4.1)$$

where $\varepsilon(i) \geq 0$ are given and $\mathcal{X}_{cl}(t) = (\mathcal{X}_{cl}(t,1),...,\mathcal{X}_{cl}(t,N))$ is the unique bounded on $\mathbf{R}$ solution of the affine differential equation on $\mathcal{S}_{n+n_c}^N$ (3.6), (3.8).

From Theorem 3.2 we deduce that if $\varepsilon(i) = \pi_{i\infty}$ then $W_\epsilon(\mathbf{G}_c)$ coincides with $J_1(\mathbf{G}_c)$ while if $\varepsilon(i) = \delta(i)$ then $W_\epsilon(\mathbf{G}_c)$ recovers $J_2(\mathbf{G}_c)$. Therefore, the finding of a controller which minimizes (4.1) allows us to obtain in an unified manner the solutions of the two optimization problems stated in Section 2.

**Theorem 4.1.** *Assume: a) the assumptions* $\mathbf{H_1}$) *and* $\mathbf{H_2}$) *are fulfilled.*

*b) The SGRDE-C (3.13) has a* $\theta$ *periodic and stabilizing solution* $\mathbf{X}_s(\cdot)$ *which verifies condition (3.19).*

*c) The SGRDE-F (3.14) has a* $\theta$*-periodic and stabilizing solution* $\mathbf{Y}_s(\cdot)$ *which verifies condition (3.20).*

*Consider the controller* $\tilde{\mathbf{G}}_c^\epsilon$ *having the state space representation*

$$\begin{aligned} d\tilde{x}_c(t) &= \tilde{A}_{c0}(t,\eta_t)\tilde{x}_c(t)dt + \sum_{k=1}^{r}\tilde{A}_{ck}(t,\eta_t)\tilde{x}_c(t)dw_k(t) + \tilde{B}_c(t,\eta_t)du_c(t) \\ dy_c(t) &= \tilde{C}_c(t,\eta_t)\tilde{x}_c(t) \end{aligned} \tag{4.2}$$

*where*

$$\begin{aligned} \tilde{A}_{ck}(t,i) &= A_k(t,i) + B_k(t,i)F_s(t,i) + K_s(t,i)C_k(t,i), \;\; 0 \le k \le r \\ \tilde{B}_c(t,i) &= -K_s(t,i), \qquad \tilde{C}_c(t,i) = F_s(t,i). \end{aligned} \tag{4.3}$$

$F_s(t,i)$ *and* $K_s(t,i)$ *being constructed as in (3.16), (3.18) respectively. Under the considered assumptions* $\tilde{\mathbf{G}}_c^\epsilon \in \mathcal{K}_s(\mathbf{G})$ *and* $W_\epsilon(\tilde{\mathbf{G}}_c^\epsilon) \le W_\epsilon(\mathbf{G}_c)$*, for all* $\mathbf{G}_c \in \mathcal{K}_s(\mathbf{G})$.

*The minimal value achieved by the cost (4.1) is*

$$\begin{aligned} W_\epsilon(\tilde{\mathbf{G}}_c^\epsilon) = \frac{1}{\theta}\int_0^\theta \sum_{j=1}^{N}\{\varepsilon(j)Tr[B_v^T(s,j)X_s(s,j)B_v(s,j)] \\ +Tr[V(s,j)F_s(s,j)Y_s(s,j)F_s^T(s,j)V(s,j)]\}ds \end{aligned} \tag{4.4}$$

*where*

$$V(s,j) = (D_z^T(s,j)D_z(s,j) + \sum_{k=1}^{r} B_k^T(s,j)X_s(s,j)B_k(s,j))^{\frac{1}{2}}. \tag{4.5}$$

**Proof.** From (4.3) one sees that $\tilde{\mathbf{G}}_c^\epsilon$ depends upon $\epsilon$, via $K_s(t,i)$. In the sequel we do not write explicitly the dependence of $\tilde{\mathbf{G}}_c$ upon the parameter $\epsilon$.

To show that $\tilde{\mathbf{G}}_c \in \mathcal{K}_s(\mathbf{G})$ we consider the linear system of type (2.5) obtained when coupling (4.2), (4.3) to (2.1), taking $u_c(t) = y(t)$ and $u(t) = y_c(t)$.

If $\tilde{x}_{cl}(t) = (\tilde{x}(t), \tilde{x}_c^T(t))^T$ is the state vector of this system, we perform the change of the state variables as:

$$\tilde{e}(t) = \tilde{x}(t) - \tilde{x}_c(t).$$

Thus, we obtain the system of stochastic differential equations:

$$\begin{aligned} d\tilde{x}(t) &= [(A_0(t,\eta_t)+B_0(t,\eta_t)F_s(t,\eta_t))\tilde{x}(t)-B_0(t,\eta_t)F_s(t,\eta_t)\tilde{e}(t)]dt \\ &+\sum_{k=1}^{r}[(A_k(t,\eta_t)+B_k(t,\eta_t)F_s(t,\eta_t))\tilde{x}(t) \\ &-B_k(t,\eta_t)F_s(t,\eta_t)\tilde{e}(t)]dw_k(t) \qquad (4.6)\\ d\tilde{e}(t) &= [A_0(t,\eta_t)+K_s(t,\eta_t)C_0(t,\eta_t)]\tilde{e}(t)dt \\ &+\sum_{k=1}^{r}[A_k(t,\eta_t)+K_s(t,\eta_t)C_k(t,\eta_t)]\tilde{e}(t)dw_k(t) \end{aligned}$$

The exponential stability in mean square of the closed loop systems (3.15) and (3.17), respectively, together with Theorem 32, (iii) Chapter 2 in [10] allows us to deduce that the trajectories of (4.6) satisfy:

$$\lim_{t\to\infty} E[|\tilde{x}(t)|^2+|\tilde{e}(t)|^2|\eta_0=i]=0$$

or equivalently,

$$\lim_{t\to\infty} E[|\tilde{x}_{cl}(t)|^2|\eta_0=i]=0 \qquad (4.7)$$

for all $1\le i\le N$ and all $\tilde{x}_{cl}(0)\in\mathbf{R}^{2n}$.

Further (4.7) together with Theorem 23 in Chapter 2 in [10] lead to

$$E[|\tilde{x}_{cl}(t)|^2|\eta_{t_0}=i]\le\beta e^{-\alpha(t-t_0)}|\tilde{x}_{cl}(t_0)|^2$$

for all $t\ge t_0\ge 0$, $\tilde{x}_{cl}(t_0)\in\mathbf{R}^{2n}$, for some $\beta>0$ and $\alpha>0$.

This shows that the controller (4.2), (4.3) is stabilizing. It remains to prove that the controller $\tilde{\mathbf{G}}_c$ minimizes the cost (4.1). First we rewrite (4.1) in the form

$$W_\epsilon(\mathbf{G}_c)=\frac{1}{\theta}\int_0^\theta\sum_{j=1}^{N}\varepsilon(j)Tr[B_v^T(s,j)X_s(s,j)B_v(s,j)]ds \qquad (4.8)$$
$$+\frac{1}{\theta}\int_0^\theta\sum_{j=1}^{N}\varepsilon(j)Tr[B_{vcl}^T(s,j)\hat{X}(s,j)B_{vcl}(s,j)]ds$$

for all $\mathbf{G}_c \in \mathcal{K}_s(\mathbf{G})$, where $\hat{X}(t,j) = \mathcal{X}_{cl}(t,j) - diag(X_s(t,j),0)$. By direct calculations one obtains that $t \to \hat{X}(t) = (\hat{X}(t,\cdot),...,\hat{X}(t,N))$ verifies the affine differential equation

$$\frac{d}{dt}\hat{\mathbf{X}}(t) + \mathcal{L}_{cl}^*(t)\hat{\mathbf{X}}(t) + \mathbf{\Theta}(t) = 0 \tag{4.9}$$

where $\mathbf{\Theta}(t) = (\mathbf{\Theta}(t,1),...,\mathbf{\Theta}(t,N))$ with

$$\begin{aligned}\mathbf{\Theta}(t,i) &= (\Theta^T(t,i)\Theta(t,i))\\ \Theta(t,i) &= V(t,i)\begin{pmatrix} F_s(t,i) & -C_c(t,i)\end{pmatrix}.\end{aligned} \tag{4.10}$$

So $\hat{X}(t,i) \geq 0, \quad (\forall)\ t \in \mathbf{R}, 1 \leq i \leq N$. Reasoning as in the proof of the equality (3.12), one gets

$$\sum_{j=1}^{N}\varepsilon(j)Tr[B_{vcl}^T(s,j)\hat{X}(s,j)B_{vcl}(s,j)] = \sum_{j=1}^{N}Tr[\Theta(s,j)\mathcal{Y}_{cl}^{\varepsilon}(s,j)\Theta^T(s,j)].$$

This allows us to transform (4.8) as follows:

$$\begin{aligned}W_\epsilon(\mathbf{G}_c) = \frac{1}{\theta}\int_0^\theta \sum_{j=1}^{N}\varepsilon(j)Tr[B_v^T(s,j)X_s(s,j)B_v(s,j)]ds\\ +\frac{1}{\theta}\int_0^\theta \sum_{j=1}^{N}Tr[\Theta(s,j)\mathcal{Y}_{cl}^{\epsilon}(s,j)\Theta^T(s,j)]ds.\end{aligned}$$

Further, we write

$$W_\epsilon(\mathbf{G}_c) = \tilde{\mu} + \frac{1}{\theta}\int_0^\theta \sum_{j=1}^{N}Tr[\Theta(s,j)\hat{\mathcal{Y}}(s,j)\Theta^T(s,j)]ds \tag{4.11}$$

where we denote by

$$\begin{aligned}\tilde{\mu} = \frac{1}{\theta}\int_0^\theta \sum_{j=1}^{N}\{\varepsilon(j)Tr[B_v^T(s,j)X_s(s,j)B_v(s,j)]\\ +Tr[V(s,j)F_s(s,j)Y_s(s,j)F_s^T(s,j)V(s,j)]\}ds\end{aligned} \tag{4.12}$$

and $\hat{Y}(s,j) = \mathcal{Y}_{cl}^{\epsilon}(s,j) - diag(Y_s(s,j),0)$.

By direct calculations one obtains that $t \to \hat{\mathbf{Y}}(t) = (\hat{Y}(t,1),...,\hat{Y}(t,N))$ is a bounded solution of the affine differential equation on $\mathcal{S}_{n+n_c}^N$:

$$\frac{d}{dt}\hat{\mathbf{Y}}(t) = \mathcal{L}_{cl}(t)\hat{\mathbf{Y}}(t) + \mathbf{\Psi}(t) \tag{4.13}$$

where $\mathbf{\Psi}(t) = (\mathbf{\Psi}(t,1),...,\mathbf{\Psi}(t,N))$ with

$$\begin{aligned} \mathbf{\Psi}(t,i) &= \Psi(t,i)\Psi^T(t,i) \\ \Psi(t,i) &= \begin{pmatrix} K_s(t,i) \\ -B_c(t,i) \end{pmatrix} \hat{V}(t,i) \\ \hat{V}(t,i) &= (\varepsilon(i)D_v(t,i)D_v^T(t,i) + \sum_{k=1}^{r} C_k(t,i)Y_s(t,i)C_k^T(t,i))^{\frac{1}{2}}. \end{aligned} \tag{4.14}$$

Applying Theorem 4.9 in [9] to the equation (4.13), (4.14) we deduce that

$$\hat{Y}(t,i) \geq 0 \tag{4.15}$$

for all $t \in \mathbf{R}$, $1 \leq i \leq N$ and for all $\mathbf{G}_c \in \mathcal{K}_s(\mathbf{G})$.

From (4.12) one sees that $\tilde{\mu}$ does not depend upon the controller $\mathbf{G}_c$. Moreover, from (4.11) and (4.15) we deduce that

$$W_\epsilon(\mathbf{G}_c) \geq \tilde{\mu} \tag{4.16}$$

for all $\mathbf{G}_c \in \mathcal{K}_s(\mathbf{G})$. To complete the proof we have to show that in (4.16) the equality takes place if $\mathbf{G}_c = \tilde{\mathbf{G}}_c$. To this end let us remark that in the case of the controller $\tilde{\mathbf{G}}_c$ we have

$$\begin{aligned} \Theta(s,j)\hat{Y}(s,j)\Theta^T(s,j) &= V(s,j)F_s(s,j)\mathcal{J}\hat{Y}(s,j)\mathcal{J}^T F_s^T(s,j)V(s,j) = \\ &= V(s,j)F_s(s,j)Z_{11}(s,j)F_s^T(s,j)V(s,j) \end{aligned} \tag{4.17}$$

where $\mathcal{J} = \begin{pmatrix} I_n & -I_n \end{pmatrix}$ and $Z_{11}(s,j)$ is the 11-block of the matrix $Z(t,j) = \mathcal{T}\hat{\mathcal{Y}}(s,j)\mathcal{T}^T$, with $\mathcal{T} = \begin{pmatrix} I_n & -I_n \\ 0 & I_n \end{pmatrix}$. One obtains the equation

$$\begin{aligned} \frac{d}{dt}Z(t,i) = \hat{A}_0(t,i)Z(t,i) + Z(t,i)\hat{A}_0^T(t,i) + \sum_{k=1}^{r} \hat{A}_k(t,i)Z(t,i)\hat{A}_k^T(t,i) \\ + \sum_{j=1}^{N} q_{ji}Z(t,j) + \mathcal{T}\Psi(t,i)\Psi^T(t,i)\mathcal{T}^T \end{aligned} \tag{4.18}$$

where $\hat{A}_k(t,i) \in \mathbf{R}^{2n\times 2n}$, $\hat{A}_k(t,i) = \mathcal{T}\tilde{A}_{kcl}(t,i)\mathcal{T}^{-1}$, $\tilde{A}_{kcl}(t,i)$ being constructed via (2.4) using (4.3). Taking the (1,1)-block of (4.18) one obtains that $t \to (Z_{11}(t,1), ..., Z_{11}(t,N))$ is a bounded solution of the differential equation on $\mathcal{S}_n^N$:

$$\begin{aligned} \frac{d}{dt}Z_{11}(t,i) = (A_0(t,i)+&K_s(t,i)C_0(t,i))Z_{11}(t,i) \\ &+Z_{11}(t,i)(A_0(t,i)+K_s(t,i)C_0(t,i))^T \\ +\sum_{k=1}^{r}(A_k(t,i) + K_s(t,i)&C_k(t,i))Z_{11}(t,i)(A_k(t,i) \\ +K_s(t,i)C_k(t,i))^T + \sum_{j=1}^{N} &q_{ji}Z_{11}(t,j), \;\; 1 \le i \le N \end{aligned} \tag{4.19}$$

Having in mind the fact that $K_s(t,i)$ is the stabilizing injection associated to the stabilizing solution $\mathbf{Y}_s(\cdot)$ of SGRDE-F (3.14) we conclude that (4.19) admits a unique bounded on $\mathbf{R}$ solution. Hence $Z_{11}(t,i) = 0$ for all $t \in \mathbf{R}$, $1 \le i \le N$. So, we deduce that in (4.16) we have equality if $\mathbf{G}_c = \tilde{\mathbf{G}}_c$. This completes the proof.

**Remark 4.1.** In the special case $N = 1$, $A_k(t,1) = 0$, $B_k(t,1) = 0$, $C_k(t,1) = 0$, $1 \le k \le r$, $t \in \mathbf{R}$ the optimal controller (4.2), (4.3) reduces to the well known Kalman filter (see e.g. [20]).

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