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In Memoriam Adelina Georgescu

THE FLOW OF A PARTICULAR CLASS OF OLDROYD-B FLUIDS*

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Abstract

This paper deals with Taylor-Couette flow formation in a particular class of Oldroyd-B fluids filling the annular region between two infinitely long coaxial circular cylinders, due to a time-dependent axial shear applied on the outer surface of the inner cylinder. The obtained solution is presented as the sum of a related Newtonian solution and the specific non-Newtonian contribution. Afterwards, it was specialized to give the solution for second grade fluids and Maxwell fluids, as well. Some exact solutions for particular classes of Oldroyd-B fluids arise as limiting cases of our solution. These results were established as limiting cases of the solution of an initial-boundary problem in fractional derivatives which was obtained, in its turn, by using the Laplace and Hankel transformations.

MSC: 76A05

keywords: Taylor-Couette flow, Oldroyd-B fluid, Maxwell fluid, second grade fluid.

1 Introduction

Oldroyd-B model provides a simple linear viscoelastic model for dilute polymer solutions, based on the dumbbell model. A wide class of fluids, such as

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polymer solutions, petroleum products, oils, blood, etc., are non-Newtonian. Moreover, the non-Newtonian fluids arise in a large variety of industrial applications - such as chemical processes (e.g. the processing of synthetic fibres, foams), food industries, construction engineering and so on, what motivates the great interest in their study. Certainly, the analysis of the behavior of the fluid motion for non-Newtonian fluids is essentially more complex in comparison with that of Newtonian fluids. It is well known that for a wide class of flows of Newtonian fluids it is possible to give a closed form for their analytical solutions, while for non-Newtonian fluids such solutions are rarely found. On the other hand, some of the mathematical models do not fit well with experimental data. That is why some mathematical objects, obtained by placing some fractional derivatives instead of some time derivatives into the rheological constitutive equations that describe the rheological properties of some classes of materials, were tested. On this line we can quote the papers of Bagley [1], Friedrich [7], Makris and Constantinou [17], Glökle and Nonnenmacher [9], Mainardi [15], Mainardi and Gorenflo [16], Rossikhin and Y. A., Shitikova [19], [20] and so on; they had obtained results which are in a good agreement with experimental data. Unfortunately, as it was already remarked in [5], an initial-boundary problem for an equation with fractional derivatives (shortly, IBPEFD) is not necessarily the mathematical model for a real dynamical system, because the fractional derivatives have no always a tensorial character. Nevertheless, some of its limiting cases are the mathematical models for real phenomena. Therefore, it becomes important to solve such an IBPEFD because its solution gives the possibility to find the solutions for all its limiting cases, among them being the solutions of problems modelling real dynamical systems. For example, this is the case of limits for parameters which allow to avoid the presence of fractional derivative. In fact, the mathematical models for Newtonian fluids, ordinary Maxwell fluids, ordinary second grade fluids, ordinary Oldroyd-B fluids are limiting cases for the before mentioned IBPEFD.

Two important situations may arise in the limiting processes. A result of such a limit can be the disappearance of all fractional derivatives. As example, in the problem under consideration in the present paper (i.e., the IBPEFD [(7), (9), (10), (12)]), the Newtonian solution is obtained by making the relaxation time λ (and, necessarily, the retardation time λ_r) tends to zero. The second kind of results corresponds to the case when the orders of all fractional derivatives, here α or/and β, tend to 1; in this case the obtained equation contains ordinary or partial derivatives only. This time the limiting process is considered in the sense of Schwartz's distribution theory with respect to some appropriately classes of testing functions. For example, the solution for ordinary Maxwell fluids is obtained from the before mentioned IBPEFD when $\lambda_r \to 0$ and $\alpha \to 1$.

These remarks will be used in what follows in order to find the exact solution for Taylor-Couette flow of an incompressible Oldroyd-B fluid in a circular pipe. More exactly, the main purpose of this paper is to provide exact solutions for the velocity field and the shear stress corresponding to the large class of unsteady flows of incompressible Oldroyd-B fluids between two infinite coaxial circular cylinders, one of them being subject to a time-dependent rotational shear stress. More exactly, by the suggestion given in [13], we study the case when in the boundary condition (8) we put $a = 2$, so that this paper can be considered as a continuation of paper [5].

To this end, into the governing equations, corresponding to an Oldroyd-B fluid in the absence of body forces and a pressure gradient in the flow direction, some time derivatives are replaced by fractional derivatives. The obtained mathematical object was named by Tong and Liu [22] the governing equations of an incompressible "generalized" Oldroyd-B fluid. After making the similar replacement in the initial-boundary conditions, an IBPEFD is obtained. The governing equations for an incompressible "generalized" Maxwell fluid or for a "generalized" second grade fluid are similarly obtained. The attribute "generalized" will be used here for designing the hypothetical fluids that would be characterized by such IBPEFDs.

The solution of IBPEFD [(7), (9), (10), (12)] is presented as a sum of the Newtonian solution and the corresponding non-Newtonian contribution. It can be easily specialized to give the similar solutions for the second grade and Maxwell fluids. As it was already remarked, the Newtonian solutions can be also obtained as limiting cases of general solutions. Furthermore, the non-Newtonian contributions to the general solutions have been expressed in terms of the time derivatives of a Newtonian solution. The exact expressions for the fluid velocity and the shear stress are obtained by the successive use of the methods of Hankel and Laplace transforms.

In the particular cases $a = 0$ and $a = 1$, this problem was already solved in [5]. The present paper solve this problem in case $a = 2$. That is why this paper is really a continuation of [5]. We try to make its reading as selfcontained as possible.

2 Model and basic equations

Recall that the Oldroyd-B model is a classical model for dilute solutions of polymers suspended in a viscous incompressible solvent. The Oldroyd-B model can be derived from microscopic principles by assuming a linear Hook's Law for the restoring force under distention of immersed polymer coils. In the recent years the Oldroyd-B fluids has gained a special place among the fluids of rate types. They contain as special cases the classical Newtonian fluids and the Maxwell fluids as well as the second grade fluids.

Let us consider an incompressible Oldroyd-B fluid at rest filling the annular region between two infinitely long coaxial circular cylinders of radii $R_1,\ R_2$ $(0 < R_1 < R_2)$. The outer cylinder is always at rest, while at time $t = 0^+$ the inner cylinder is suddenly set in rotation around its axis by a time-dependent shear stress.

The equations governing the unsteady motion of an incompressible fluid are

$$div\ \mathbf{V} = 0, \qquad \rho\ \frac{d\mathbf{V}}{dt} = div\ \mathbf{T},$$

where $\mathbf{V}$ is the velocity field, ρ the density, $\mathbf{T}$ the Cauchy *stress tensor* and d/dt the material time derivative.

The Cauchy *stress tensor* $\mathbf{T}$ for an incompressible Oldroyd-B fluid, is given by

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \qquad \mathbf{S} + \lambda\frac{D\mathbf{S}}{Dt} = \mu\left(\mathbf{A} + \lambda_r\frac{D\mathbf{A}}{Dt}\right), \tag{1}$$

where $-p\mathbf{I}$ is the *indeterminate spherical stress* (p is the *isotropic pressure*), $\mathbf{S}$ is the *extra-stress tensor*, $\mathbf{A}$ is the *first* Rivlin-Ericksen *tensor*, μ is the *dynamic viscosity* of the fluid, λ and $\lambda_r(< \lambda)$ are *material constants* (namely, the *relaxation time* and the *retardation time*, respectively), and the upper convected time derivatives are defined by

$$\frac{D\mathbf{S}}{Dt} = \frac{d\mathbf{S}}{dt} + (\mathbf{V}\cdot\nabla)\mathbf{S} - \mathbf{LS} - \mathbf{SL}^T, \qquad \frac{D\mathbf{A}}{Dt} = \frac{d\mathbf{A}}{dt} + (\mathbf{V}\cdot\nabla)\mathbf{A} - \mathbf{LA} - \mathbf{AL}^T. \tag{2}$$

Into above equation (2), ∇ is the gradient operator, $\mathbf{L}$ denotes the velocity gradient and the superscript T indicates the transpose operation. This time, the body forces have been neglected.

Since the motion is axial symmetric we shall use the cylindrical coordinates $(r,\ \theta,\ z)$. That is why, for the problem under consideration, we assume a velocity field $\mathbf{V}$ and an extra-stress tensor $\mathbf{S}$ of the form

$$\mathbf{V} = \mathbf{V}(r,t) = \omega(r,t)\mathbf{e}_\theta, \qquad \mathbf{S} = \mathbf{S}(r,t) \tag{3}$$

where $\mathbf{e}_\theta$ is the unit vector along the θ-direction of the cylindrical coordinate system. For such flows the constraint of incompressibility is automatically satisfied. Furthermore, if the fluid is at rest up to the moment $t = 0$, i.e.

$$\mathbf{V}(r,t) = \mathbf{0}, \quad \mathbf{S}(r,t) = \mathbf{0} \qquad for\ t \leq 0, \tag{4}$$

then the governing equations for an Oldroyd-B fluid, in the absence of body forces and a pressure gradient in the flow direction, are given for $r \in (R_1, R_2)$, $t > 0$ by

$$\lambda\frac{\partial^2\omega(r,t)}{\partial t^2} + \frac{\partial\omega(r,t)}{\partial t} = \nu\left(1 + \lambda_r\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\omega(r,t), \tag{5}$$

$$\left(1 + \lambda\frac{\partial}{\partial t}\right)\tau(r,t) = \mu\left(1 + \lambda_r\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\omega(r,t), \tag{6}$$

where $\tau(r,t) = S_{r\theta}(r,t)$ is the nonzero *shear stress*, $\nu = \mu/\rho$ is the *kinematic viscosity*, ρ is its constant *density*, while λ and λ_r are respectively the *relaxation* and *retardation times*. The system of equations (5)-(6) must be solved subject to the initial and boundary conditions

$$\omega(r,0) = \frac{\partial\omega(r,0)}{\partial t} = 0, \quad \tau(r,0) = 0, \tag{7}$$

respectively,

$$\left(1 + \lambda\frac{\partial}{\partial t}\right)\tau(R_1,t) = \mu\left(1 + \lambda_r\frac{\partial}{\partial t}\right)\left(\frac{\partial\omega(R_1,t)}{\partial r} - \frac{1}{R_1}\omega(R_1,t)\right) = ft^a, \tag{8}$$

for $r \in (R_1, R_2)$, $t > 0$ and $a \geq 0$.

By replacing some inner time derivatives by the fractional differential operators D_t^α and D_t^β $(0 < \beta \leq \alpha < 1)$, the governing equations (5) and (6) of an incompressible Oldroyd-B fluid become (see [22])

$$(1 + \lambda D_t^\alpha)\frac{\partial\omega(r,t)}{\partial t} = \nu(1 + \lambda_r D_t^\beta)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\omega(r,t), \tag{9}$$

$$(1 + \lambda D_t^\alpha)\tau(r,t) = \mu(1 + \lambda_r D_t^\beta)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\omega(r,t), \tag{10}$$

for $r \in (R_1, R_2),\ t > 0$; here the fractional derivatives are defined by [18]

$$D_t^p[f(t)] = \frac{1}{\Gamma(1-p)}\frac{d}{dt}\int_0^t \frac{f(\tau)}{(t-\tau)^p}d\tau, \qquad 0 < p < 1 \tag{11}$$

(where $\Gamma(\cdot)$ is EULER's Gamma function).

As before, by replacing in (8) the inner derivations with respect to t by the fractional differential operators D_t^α and D_t^β $(\beta \le \alpha)$, we get

$$(1 + \lambda D_t^\alpha)\tau(R_1, t) = \mu(1 + \lambda_r D_t^\beta)\left(\frac{\partial\omega(R_1, t)}{\partial r} - \frac{1}{R_1}\omega(R_1, t)\right) = ft^a \tag{12}$$

for $t > 0,\ a \ge 0$. Consequently, an IBPEFD, consisting of equations [(9), (7), (12)], is associated with the model of the Taylor-Couette flow of an Oldroyd-B fluid in an annulus due to a time depending couple and characterized by simultaneously equations [(5), (7), (8)]. It will be solved by using the integral transform techniques. More exactly, the Laplace and finite Hankel transforms are used to change the IBPEFD [(9), (7), (12)] into an algebraic system.

Moreover, the equations (9) and (10) contain as limiting cases the governing equations of the so called (see [22]) "generalized" second grade and Maxwell models (i.e. the models obtained by replacing some inner time derivatives by some fractional differential operators in the governing equations of a second grade or a Maxwell fluid), as well as the ordinary Oldroyd-B, Maxwell and second grade models.

In this paper, we are especially interested in the case when the boundary condition corresponds to $a = 2$; the cases $a = 0$ and $a = 1$ were already analyzed in [5].

COMMENT. In order to ensure the dimensional consistency of equations (7) and (8), the material constants λ and λ_r must have necessarily the dimensions of t^α and t^β, respectively. Into several papers (e.g. [13]) the authors (correctly) used λ^α and λ_r^β instead of λ and λ_r. However, for simplicity, we shall keep the notations λ and λ_r (like [11] or [22]) having in mind their correct significations.

3 Exact solutions for the velocity field

In what follows, we shall use the modified Hankel transform, with respect to r, defined by means of the Bessel functions of index 1

$$B(r, r_n) = J_1(rr_n)Y_2(R_1r_n) - J_2(R_1r_n)Y_1(rr_n),$$

where $J_1(\cdot)$, $J_2(\cdot)$, $Y_1(\cdot)$ and $Y_2(\cdot)$ are Bessel functions (of index 1 and 2), $(r_n)_{n\in\mathbf{N}^*}$ is the increasing sequence of the positive roots of the transcedental equation $J_1(R_2x)Y_2(R_1x) - J_2(R_1x)Y_1(R_2x) = 0$ (i.e. $B(R_2, r_n) = 0$ for all $n \in \mathbf{N}^*$). We shall denote by $\omega_H(r_n, t)$ the image of $\omega(r, t)$ by the modified Hankel transform, defined by

$$\omega_H(r_n, t) = \int_{R_1}^{R_2} r\ \omega(r, t)\ B(r, r_n)\ dr\ . \tag{13}$$

Recall that the inverse of the modified Hankel transform (13) is defined by

$$f(r) = \frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{r_n^2 J_1^2(R_2r_n)B(r, r_n)}{J_2^2(R_1r_n) - J_1^2(R_2r_n)}f_H(r_n) = \frac{\pi^2}{2}\sum_{n=1}^{\infty} r_n^2 C_{Fn} f_H(r_n), \tag{14}$$

where $f_H(r_n)$ denotes the image of $f(r)$ by Hankel transform (13) and

$$C_{Fn} = \frac{J_1^2(R_2r_n)B(r, r_n)}{J_2^2(R_1r_n) - J_1^2(R_2r_n)} = D_{Fn}B(r, r_n). \tag{15}$$

By applying successively the Hankel transform (13) and the Laplace transform, the following expression for the velocity field $\omega(r, t)$ was obtained in [5]:

$$\begin{aligned}\omega(r, t) = {} & \omega_{N,a}(r, t) - \\ & -\frac{\pi f}{\rho}\Gamma(1 + a)\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\left(-\frac{\nu r_n^2}{\lambda}\right)^k r_n C_{Fn}\int_0^t F_k(s)e^{-\nu r_n^2(t-s)}ds,\end{aligned} \tag{16}$$

where

$$\begin{aligned}F_k(t) = \sum_{m=0}^{k}\frac{k!\lambda_r^m}{m!(k-m)!}\Bigg[& G_{\alpha,\alpha_m - a, k+1}\left(-\frac{1}{\lambda}, t\right) + \\ & + \nu r_n^2\frac{\lambda_r}{\lambda}G_{\alpha,\beta_m - a, k+1}\left(-\frac{1}{\lambda}, t\right)\Bigg]\end{aligned} \tag{17}$$

with (see [14])

$$G_{a,b,c}(d,t)=\sum_{j=0}^{\infty}\frac{\Gamma(j+c)t^{a(j+c)-b-1}}{\Gamma(j+1)\Gamma(c)\Gamma[a(j+c)-b]}d^j=\mathcal{L}^{-1}\left\{\frac{q^b}{(q^a-d)^c}\right\},\quad(18)$$

for $Re\ (ac-b)>0$, $\left|\dfrac{d}{q^a}\right|<1$, and

$$\omega_{N,a}(r,t)=\frac{\pi f}{\rho}\sum_{n=1}^{\infty}r_nC_{Fn}\int_0^t s^a e^{-\nu r_n^2(t-s)}ds,\quad(19)$$

represents the velocity field corresponding to a Newtonian fluid performing the same motion. More exactly, $\omega_{N,a}(r,t)$ for $R_1<r<R_2,\ t>0$ is the solution of the problem:

$$\begin{cases}\dfrac{\partial\omega_{N,a}(r,t)}{\partial t}=\nu\left[\dfrac{\partial^2\omega_{N,a}(r,t)}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial\omega_{N,a}(r,t)}{\partial r}-\dfrac{1}{r^2}\ \omega_{N,a}(r,t)\right],\\ \omega_{N,a}(r,0)=0,\quad\omega_{N,a}(R_2,t)=0,\\ \tau_{N,a}(R_1,t)=\mu\left[\dfrac{\partial\omega_{N,a}(R_1,t)}{\partial r}-\dfrac{1}{R_1}\omega_{N,a}(R_1,t)\right]=ft^a;\end{cases}$$

in last boundary condition, the presence of $\tau_{N,a}(R_1,t)$ can be ignored; it is just for helping us to motivate the form of the boundary condition. A special interest is for velocity field $\omega_N(r,t)=\omega_{N,0}(r,t)$. Recall that, it was proved in [5] that

$$\omega_N(r,t)=\omega_{N,0}(r,t)=\varphi_0(r)-\frac{\pi f}{\mu}\sum_{n=1}^{\infty}\frac{1}{r_n}C_{Fn}e^{-\nu r_n^2t},\quad(20)$$

where

$$\varphi_0(r)=-\frac{f}{2\mu}\left(\frac{R_1}{R_2}\right)^2\left(\frac{R_2^2}{r}-r\right).\quad(21)$$

Further, Eq. (20) gives

$$\frac{\partial\omega_N(r,t)}{\partial t}=\frac{\pi f}{\rho}\sum_{n=1}^{\infty}r_nC_{Fn}e^{-\nu r_n^2t}$$

and consequently (19) becomes

$$\omega_{N,a}(r,t)=t^a*\partial_t\omega_N(r,t).\quad(22)$$

Then, for $a = 1$ we get

$$\omega_{N,1}(r,t) = \varphi_0(r)t + \varphi_1(r) + \frac{\pi f}{\mu\nu}\sum_{n=1}^{\infty}\frac{1}{r_n^3}C_{Fn}e^{-\nu r_n^2 t}, \tag{23}$$

where

$$\varphi_1(r) = \frac{A}{r} + Br + Cr^3 + E\ r\ ln\ r, \tag{24}$$

with

$$\begin{aligned} A &= -\frac{fR_1^4}{8R_2^2\mu\nu}(2R_2^2 - R_1^2), & C &= \frac{f}{8\mu\nu}\left(\frac{R_1}{R_2}\right)^2, \\ B &= \frac{fR_1^2}{8R_2^2\mu\nu}\left[4R_2^4\ ln\ R_2 - (R_2^2 - R_1^2)^2\right], & E &= -\frac{fR_1^2}{2\mu\nu}. \end{aligned} \tag{25}$$

Similarly, for $a = 2$, we get

$$\omega_{N,2}(r,t) = \varphi_0(r)t^2 + \varphi_1(r)t + \varphi_2(r) - \frac{2\pi f}{\mu\nu^2}\sum_{n=1}^{\infty}\frac{1}{r_n^5}D_{Fn}B(r,r_n)e^{-\nu r_n^2 t}, \tag{26}$$

where $\varphi_2(r)$ is the solution of the boundary problem

$$\begin{cases} \varphi_2''(r) + \frac{1}{r}\varphi_2'(r) - \frac{1}{r^2}\varphi_2(r) = \frac{1}{\nu}\varphi_1(r), \quad R_1 < r < R_2, \\ \varphi_2(R_2) = 0, \\ \varphi_2'(R_1) - \frac{1}{R_1}\varphi_2(R_1) = 0. \end{cases}$$

The solution of this last problem is:

$$\begin{aligned} \varphi_2(r) &= \frac{C_1}{r} + C_2 r + \\ &+ \frac{1}{96}\left(48A\ ln\ r + 12B\ r^2 + 4C\ r^4 + 12E\ r^2 ln\ r - 9E\ r^2\right)r, \end{aligned} \tag{27}$$

where

$$\begin{aligned} C_1 &= -\frac{fR_1^6}{192\mu\nu^2R_2^4}\left[12\ R_2^4 - \right. \\ &\left. -14R_1^2R_2^2 + 12R_2^4\ ln\ R_1 - 12R_2^4\ lnR_2 + 3R_1^4\right], \\ C_2 &= \frac{fR_1^2}{192\mu\nu^2R_2^6}\left[15R_1^4R_2^4 - 14R_1^6R_2^2 + 12R_1^4R_2^4\ ln\ R_1 - \right. \\ &\left. -24R_1^4R_2^4\ ln\ R_2 + 3R_1^8 - 7R_2^8 + 24R_1^2R_2^6\ ln\ R_2 - 6R_1^2R_2^6\right]. \end{aligned} \tag{28}$$

Finally, having in mind the expression (20) of the Newtonian solution $\omega_N(r,t)$, it is easy to show that the general solution $\omega(r,t)$ can be written in a suitable form in terms of its time derivatives, namely

$$\begin{aligned}\omega(r,t) &= t^a * \partial_t \omega_N(r,t) - \\ &-\Gamma(1+a)\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!\lambda_r^m}{m!(k-m)!\lambda^k}\partial_t^{k+1}\omega_N(r,t) * G_{\alpha,\alpha_m-a,k+1}\left(-\frac{1}{\lambda},t\right) + \\ &+\frac{\lambda_r}{\lambda}\Gamma(1+a)\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!\lambda_r^m}{m!(k-m)!\lambda^k}\partial_t^{k+2}\omega_N(r,t) * G_{\alpha,\beta_m-a,k+1}\left(-\frac{1}{\lambda},t\right),\end{aligned} \tag{29}$$

where $\alpha_m = m\beta + \alpha - k - 1,\ \beta_m = m\beta + \beta - k - 2$. In case $a = 2$, it seem natural to express the general solution $\omega(r,t)$ in terms of $\omega_{N,2}(r,t)$ and its time derivatives. As

$$\begin{aligned}\frac{\partial^3 \omega_{N,2}(r,t)}{\partial t^3} &= \frac{2\pi f}{\rho}\sum_{n=1}^{\infty} r_n D_{Fn} B(r,r_n) e^{-\nu r_n^2 t} = \\ &= \frac{2\pi f}{\rho}\sum_{n=1}^{\infty} r_n C_{Fn} e^{-\nu r_n^2 t} = 2\frac{\partial \omega_N(r,t)}{\partial t}\end{aligned}$$

it results

$$\begin{aligned}\omega(r,t) &= \omega_{N,2}(r,t) - \\ &-\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!\lambda_r^m}{m!(k-m)!\lambda^k}\partial_t^{k+3}\omega_{N,2}(r,t) * G_{\alpha,\alpha_m-2,k+1}\left(-\frac{1}{\lambda},t\right) + \\ &+\frac{\lambda_r}{\lambda}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!\lambda_r^m}{m!(k-m)!\lambda^k}\partial_t^{k+4}\omega_{N,2}(r,t) * G_{\alpha,\beta_m-2,k+1}\left(-\frac{1}{\lambda},t\right).\end{aligned} \tag{30}$$

4 Calculations of the shear stress

By using Eq. (29), the following form of the shear stress was already obtained in [5]:

$$\tau(r,t) = \tau_{N,a}(r,t) + \mu\Gamma(1+a)A\left(-\frac{1}{\lambda},t\right) * \partial_t \Omega_N(r,t) -$$

$$-\mu\Gamma(1+a)\sum_{k=0}^{\infty}\left[B_k\left(-\frac{1}{\lambda},t\right)*\partial_t^{k+1}\Omega_N(r,t)-\right.$$
$$\left.-\frac{\lambda_r}{\lambda}C_k\left(-\frac{1}{\lambda},t\right)*\partial_t^{k+2}\Omega_N(r,t)\right] \tag{31}$$

where

$$\Omega_N(r,t)=\frac{\partial\omega_N(r,t)}{\partial r}-\frac{1}{r}\omega_N(r,t), \tag{32}$$

$$\tau_{N,a}(r,t)=\mu t^a*\partial_t\Omega_N(r,t)=$$
$$=-\pi f\nu\sum_{k=0}^{\infty}\frac{r_n^2[J_2(rr_n)Y_2(R_1r_n)-J_2(R_1r_n)Y_2(rr_n)]}{J_2^2(R_1r_n)-J_1^2(R_2r_n)}\int_0^t s^a e^{-\nu r_n^2(t-s)}\,ds \tag{33}$$

represents the shear stress corresponding to a Newtonian fluid and

$$A\left(-\frac{1}{\lambda},t\right)=\frac{\lambda_r}{\lambda}R_{\alpha,\beta-a-1}\left(-\frac{1}{\lambda},t\right)-R_{\alpha,\alpha-a-1}\left(-\frac{1}{\lambda},t\right),$$

$$R_{\alpha,\beta}(a,t)=\sum_{k=0}^{\infty}\frac{a^k t^{(k+1)\alpha-\beta-1}}{\Gamma((k+1)\alpha-\beta)},\quad Re\ (\alpha-\beta)>0,\quad |at^\alpha|<1,$$

$$B_k\left(-\frac{1}{\lambda},t\right)=\frac{1}{\lambda^k}\sum_{m=0}^{k}\frac{k!\lambda_r^m}{m!(k-m)!}\left[G_{\alpha,\alpha_m-a,k+1}\left(-\frac{1}{\lambda},t\right)+\right.$$
$$\left.+\frac{\lambda_r}{\lambda}G_{\alpha,\alpha_m+\beta-a,k+2}\left(-\frac{1}{\lambda},t\right)-G_{\alpha,\alpha_m+\alpha-a,k+2}\left(-\frac{1}{\lambda},t\right)\right],$$

$$C_k\left(-\frac{1}{\lambda},t\right)=\frac{1}{\lambda^k}\sum_{m=0}^{k}\frac{k!\lambda_r^m}{m!(k-m)!}\left[G_{\alpha,\beta_m-a,k+1}\left(-\frac{1}{\lambda},t\right)+\right.$$
$$\left.+\frac{\lambda_r}{\lambda}G_{\alpha,\beta_m+\beta-a,k+2}\left(-\frac{1}{\lambda},t\right)-G_{\alpha,\beta_m+\alpha-a,k+2}\left(-\frac{1}{\lambda},t\right)\right].$$

Starting from Eq. (16), the following equivalent form of the shear stress is obtained

$$\tau(r,t)=\tau_{N,a}(r,t)+\mu\Gamma(1+a)A\left(-\frac{1}{\lambda},t\right)*\partial_t\Omega_N(r,t)+$$

$$+\pi f\nu\ \Gamma(1+a)\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}r_n\widetilde{C}_{Fn}(-\nu r_n^2)^k\left[B_k\left(-\frac{1}{\lambda},t\right)+\right. \tag{34}$$

$$\left.+\nu r_n^2\frac{\lambda_r}{\lambda}C\left(-\frac{1}{\lambda},t\right)\right]*e^{-\nu r_n^2 t},$$

where

$$\widetilde{C}_{Fn}=\frac{J_2(rr_n)Y_2(R_1r_n)-J_2(R_1r_n)Y_2(rr_n)}{J_2^2(R_1r_n)-J_1^2(R_2r_n)}J_1^2(R_2r_n).$$

Making $a=0,\ 1$ and 2 into (31) and (34), the shear stresses corresponding to $f,\ ft$ and ft^2 into (7) are obtained. For instance, the shear stresses for the corresponding Newtonian solutions are

$$\tau_N(r,t)=\tau_{N,0}(r,t)=\tau_0(r)+\pi f\sum_{n=1}^{\infty}\widetilde{C}_{Fn}\ e^{-\nu r_n^2 t}, \tag{35}$$

$$\tau_{N,1}(r,t)=t\tau_0(r)+\tau_1(r)-\frac{\pi f}{\nu}\sum_{n=1}^{\infty}\widetilde{C}_{Fn}e^{-\nu r_n^2 t}, \tag{36}$$

$$\tau_{N,2}(r,t)=t^2\tau_0(r)+t\tau_1(r)+\tau_2(r)-\frac{\pi f}{\nu}\sum_{n=1}^{\infty}\widetilde{C}_{Fn}e^{-\nu r_n^2 t}, \tag{37}$$

where

$$\begin{aligned}
\tau_0(r)&=\frac{fR_1^2}{r^2},\\
\tau_1(r)&=\frac{fR_1^2\left[(R_2^2-r^2)^2-(R_2^2-R_1^2)^2\right]}{8\nu R_2^2r^2},\\
\tau_2(r)&=\mu[\varphi_2'(r)-\frac{1}{r}\varphi_2'(r)]=\\
&=\frac{\mu}{48r^2}(-96C_1+24Ar^2+12Br^4+8Cr^6+12Er^4\ ln\ r-3Er^4).
\end{aligned}$$

5 Limiting cases

1. Making the limit of Eqs. (16), (29), (31) and (34) as $\lambda_r \to 0$, we get the similar solutions corresponding to the so-called "generalized" Maxwell fluids, namely

$$\begin{aligned}\omega(r,t) &= \omega_{N,2}(r,t) - \\ -2\frac{\pi f}{\rho}\sum_{n=1}^{\infty} r_n C_{Fn} \sum_{k=0}^{\infty}\left(-\frac{\nu r_n^2}{\lambda}\right)^k &\int_0^t G_{\alpha,\gamma_k-2,k+1}\left(-\frac{1}{\lambda},t\right) e^{-\nu r_n^2 (t-s)}\, ds = \\ &= \omega_{N,2}(r,t) - 2\sum_{k=0}^{\infty}\frac{1}{\lambda^k} G_{\alpha,\gamma_k-2,k+1}\left(-\frac{1}{\lambda},t\right) * \partial_t^{k+1}\omega_{N,2}(r,t),\end{aligned} \tag{38}$$

$$\begin{aligned}\tau(r,t) = \tau_{N,2}(r,t) - 2\mu\int_0^t \partial_s \Omega_N(r,s) R_{\alpha,\alpha-31}\left(-\frac{1}{\lambda}, t-s\right)\, ds + \\ +2\pi f\nu \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} r_n \widetilde{C}_{Fn}\left(\frac{-\nu r_n^2}{\lambda}\right)^k \left[G_{\alpha,\gamma_k-2,k+1}\left(-\frac{1}{\lambda},t\right) - \right. \\ \left. -G_{\alpha,\gamma_k+\alpha-2,k+2}\left(-\frac{1}{\lambda},t\right)\right] * e^{-\nu r_n^2 t} = \\ = \tau_{N,2}(r,t) - 2\mu\partial_t \Omega_N(r,t) * R_{\alpha,\alpha-3}\left(-\frac{1}{\lambda},t\right) - \\ -2\mu\sum_{k=0}^{\infty}\frac{1}{\lambda^k}\partial_t^{k+1}\Omega_N(r,t) * \left[G_{\alpha,\gamma_k-2,k+1}\left(-\frac{1}{\lambda},t\right) - \right. \\ \left. -G_{\alpha,\gamma_k+\alpha-2,k+2}\left(-\frac{1}{\lambda},t\right)\right],\end{aligned} \tag{39}$$

where $\gamma_k = \alpha - k - 1$. Furthermore, making $\lambda \to 0$ in (38) and (39) and taking into account of

$$\lim_{\lambda\to 0}\frac{1}{\lambda^k} G_{a,b,k}(-1/\lambda,t) = \frac{t^{-b-1}}{\Gamma(-b)}, \quad \lim_{\lambda\to 0}\frac{1}{\lambda} R_{a,b}(-1/\lambda,t) = \frac{t^{-b-1}}{\Gamma(-b)}, \tag{40}$$

the Newtonian solutions

$$\omega_{N,2}(r,t) = t^2 * \partial_t \omega_N(r,t), \qquad \tau_{N,2}(r,t) = \mu t^2 * \partial_t \Omega_N(r,t) \tag{41}$$

are recovered.

2. By making now $\alpha \to 1$ into (38) and (39), the solutions for ordinary Maxwell fluids are obtained, namely

$$\begin{aligned}
&\omega(r,t) = \omega_{N,2}(r,t) - \\
&-2\frac{\pi f}{\rho}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\left(-\frac{\nu r_n^2}{\lambda}\right)^k r_n C_{Fn}\int_0^t G_{1,-k-2,k+1}\left(-\frac{1}{\lambda},t\right) e^{-\nu r_n^2 (t-s)}\, ds = \\
&= \omega_{N,2}(r,t) - 2\sum_{k=0}^{\infty}\frac{1}{\lambda^k}\partial_t^{k+1}\omega_N(r,t) * G_{1,-k-2,k+1}\left(-\frac{1}{\lambda},t\right),
\end{aligned} \tag{42}$$

$$\begin{aligned}
&\tau(r,t) = \tau_{N,2}(r,t) - 2\mu\int_0^t \partial_s \Omega_N(r,s) R_{1,-2}\left(-\frac{1}{\lambda},t-s\right)\, ds + \\
&\qquad +2\pi f \nu \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} r_n \widetilde{C}_{Fn}\left(\frac{-\nu r_n^2}{\lambda}\right)^k \times \\
&\times \int_0^t e^{-\nu r_n^2 t}\left[G_{1,-k-2,k+1}\left(-\frac{1}{\lambda},t-s\right) - G_{1,-k-1,k+2}\left(-\frac{1}{\lambda},t-s\right)\right]\, ds = \\
&\qquad = \tau_{N,2}(r,t) - 2\mu\partial_t\Omega_N(r,t) * R_{1,-2}\left(-\frac{1}{\lambda},t\right) - \\
&-2\mu\sum_{k=0}^{\infty}\frac{1}{\lambda^k}\partial_t^{k+1}\Omega_N(r,t) * \left[G_{1,-k-2,k+1}\left(-\frac{1}{\lambda},t\right) - G_{1,-k-1,k+2}\left(-\frac{1}{\lambda},t\right)\right].
\end{aligned} \tag{43}$$

Indeed, direct computations implying suitable grouping of terms and the use of equation

$$\sum_{k=0}^{\infty}\left(-\frac{\nu r_n^2}{\lambda}\right)^k G_{1,-k-a,k+1}\left(-\frac{1}{\lambda},t\right) = \lambda \mathcal{L}^{-1}\left(\frac{1}{q^{a-1}}\frac{1}{\lambda q^2 + q + \nu r_n^2}\right) \tag{44}$$

shows that Eq. (42) can be respectively written in the form

$$\omega(r,t) = \omega_{N,2}(r,t) - \\ - \frac{4f\lambda}{R\rho}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{J_1(Rr_n)}e^{-\nu r_n^2 t} * \mathcal{L}^{-1}\left(\frac{1}{q}\cdot\frac{1}{\lambda q^2+q+\nu r_n^2}\right), \tag{45}$$

Then, by using the formula

$$e^{-\nu r_n^2 t} * \mathcal{L}^{-1}\left(\frac{1}{q}\frac{1}{\lambda q^2+q+\nu r_n^2}\right) = \\ = \frac{1}{(\nu r_n^2)^3}\left[e^{-\nu r_n^2 t} + \lambda^2\frac{q_{n2}^3 e^{q_{n1}t} - q_{n1}^3 e^{q_{n2}t}}{q_{n1}-q_{n2}} - \lambda\nu r_n^2\right], \tag{46}$$

one obtains

$$\omega(r,t) = \omega_{N,2}(r,t) - \frac{4f\lambda}{R\rho}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{J_1(Rr_n)}\left\{\frac{1}{(\nu r_n^2)^2} - \frac{1}{\nu r_n^2(\lambda-\lambda_r)}\left[e^{-\nu r_n^2 t} + \right.\right. \\ \left.\left. + \frac{1}{q_{n1}-q_{n2}}\left(\frac{e^{q_{n1}t}}{q_{n1}(1-\nu r_n^2\lambda)+\nu r_n^2} - \frac{e^{q_{n1}t}}{q_{n2}(1-\nu r_n^2\lambda)+\nu r_n^2}\right)\right]\right\}, \tag{47}$$

where q_{n1}, q_{n2} are the roots of equation $\lambda q^2 + q + \nu r_n^2 = 0$. Taking into account Eq. (20), the solution (47) can be written in the following simpler form:

$$\omega(r,t) = \omega_{N,2}(r,t) + \frac{f\lambda}{\mu\nu}\frac{r^3(2R^2-r^2)}{24R^2} - \frac{4f}{R\mu\nu^2}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n^6 J_1(Rr_n)}\left[e^{-\nu r_n^2 t} + \right. \\ \left. + \frac{[\nu r_n^2 + (1-\nu r_n^2\lambda)q_{n2}]e^{q_{n1}t} - [\nu r_n^2 + (1-\nu r_n^2\lambda)q_{n1}]e^{q_{n2}t}}{q_{n1}-q_{n2}}\right]. \tag{48}$$

A similar procedure, applied to Eq. (43), yields

$$\tau(r,t) = \tau_{N,2}(r,t) - 2\mu\left(\lambda^2 e^{-\frac{t}{\lambda}} - \lambda^2 + \lambda t\right) * \partial\Omega_N(r,t) + \\ + \mu\frac{4f}{R\rho}\sum_{n=1}^{\infty}\frac{r_n J_2(rr_n)}{J_1(Rr_n)}\, e^{-\frac{t}{\lambda}} * e^{-\nu r_n^2 t} * \mathcal{L}^{-1}\left(\frac{1}{q}\frac{1}{\lambda q^2+q+\nu r_n^2}\right). \tag{49}$$

After a straightforward computation we get

$$\tau(r,t)=\tau_{N,2}(r,t)-2\mu\lambda\left(\lambda^2 e^{-\frac{t}{\lambda}}-\lambda^2+\lambda t\right)*\partial\Omega_N(r,t)+ \\ +\frac{4f}{R}\sum_{n=1}^{\infty}\frac{J_2(rr_n)}{r_nJ_1(Rr_n)}\left(\frac{\lambda}{(\nu r_n^2)^2}\frac{q_{n2}^2e^{q_{n1}t}-q_{n1}^2e^{q_{n2}t}}{q_{n1}-q_{n2}}+\right. \\ \left.+\frac{e^{-\nu r_n^2 t}}{(\nu r_n^2)^2(\lambda\nu r_n^2-1)}-\frac{\lambda^2 e^{-\frac{t}{\lambda}}}{\lambda\nu r_n^1-1}+\frac{\lambda^2}{\nu r_n^2}\right). \tag{50}$$

3. In the special case when $\lambda\to 0$ into (16) and (29), the solutions

$$\omega(r,t)=\omega_{N,2}(r,t)+ \\ +2\frac{\pi f}{\rho}\lambda_r\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{k}C_{Fn}\frac{k!\lambda_r^m(-\nu r_n^2)^{k+1}}{m!(k-m)!\Gamma(2-\beta_m)}\int_0^t s^{1-\beta_m}e^{-\nu r_n^2(t-s)}\,ds= \tag{51} \\ =\omega_{N,2}(r,t)+2\lambda_r\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!\lambda_r^m}{m!(k-m)!}\int_0^t\partial_s^{k+2}\omega_N(r,s)\;\frac{(t-s)^{1-\beta_m}}{\Gamma(2-\beta_m)}\;ds$$

and

$$\tau(r,t)=\tau_{N,2}(r,t)+2\mu\lambda_r\int_0^t\frac{(t-s)^{2-\beta}}{\Gamma(3-\beta)}\partial_s\Omega_N(r,t)\;ds- \\ -2\pi f\nu\lambda_r\sum_{n=1}^{\infty}r_n\widetilde{C}_{Fn}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!\lambda_r^m(-\nu r_n^2)^{k+1}}{m!(k-m)!}\times \\ \times\int_0^t\left[\frac{s^{1-\beta_m}}{\Gamma(2-\beta_m)}+\lambda_r\frac{s^{2-\beta_m-\beta-1}}{\Gamma(2-\beta_m-\beta)}\right]e^{-\nu r_2^2(t-s)}\;ds= \\ =\tau_{N,2}(r,t)+2\mu\lambda_r\int_0^t\frac{(t-s)^{2-\beta}}{\Gamma(3-\beta)}\;\partial_s\Omega_N(r,s)\;ds+2\mu\lambda_r\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!\lambda_r^k}{m!(k-m)!}\times$$

$$\times\int\limits_0^t \left[\frac{(t-s)^{1-\beta_m}}{\Gamma(2-\beta_m)} + \lambda_r \frac{(t-s)^{1-\beta_m-\beta}}{\Gamma(2-\beta_m-\beta)}\right]\ \partial_s^{k+2}\Omega_N(r,s)\ ds \tag{52}$$

corresponding to a "generalized" second grade fluid are obtained.

Of course, making $\lambda_r \to 0$ into Eqs. (51) and (52), we again attain to the Newtonian solutions given by Eq. (41). Moreover, in the special case when $\beta \to 1$, Eqs. (51) and (52) reduce to the solutions for an ordinary second grade fluid, namely

$$\omega(r,t) = \omega_{N,2}(r,t) +$$

$$+2\frac{\pi f \lambda_r}{\rho}\lambda_r \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{k} C_{Fn}\frac{k!\lambda_r^m(-\nu r_n^2)^{k+1}}{m!(k-m)!\Gamma(k-m+3)}\int\limits_0^t s^{k-m+2}e^{-\nu r_n^2(t-s)}\ ds =$$

$$= \omega_{N,2}(r,t) + 2\lambda_r \sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!\lambda_r^m}{m!(k-m)!}\int\limits_0^t \partial_s^{k+2}\omega_N(r,s)\ \frac{(t-s)^{k-m+2}}{\Gamma(k-m+3)}\ ds \tag{53}$$

and

$$\tau(r,t) = \tau_{N,2}(r,t) + 2\mu\lambda_r \int\limits_0^t (t-s)\partial_s\Omega_N(r,t)\ ds -$$

$$-2\pi f \nu\lambda_r \sum_{n=1}^{\infty} r_n\ \widetilde{C}_{Fn}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!\lambda_r^m}{m!(k-m)!}(-\nu r_2^2)^{k+1}\times$$

$$\times \int_0^t \left[\frac{s^{k-m+2}}{\Gamma(ka-m+3)} + \lambda_r \frac{s^{k-m+1}}{\Gamma(k-m+2)}\right] e^{-\nu r_n^2(t-s)}\ ds = \tag{54}$$

$$= \tau_{N,2}(r,t) + 2\mu\lambda_r t * \partial_t\Omega_N(r,t) +$$

$$+2\mu\lambda_r \sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!\lambda_r^k}{m!(k-m)!}\left[\frac{t^{k-m+2}}{\Gamma(k-m+3)} + \right.$$

$$\left. +\lambda_r \frac{t^{k-m+1}}{\Gamma(k-m+2)}\right] * \partial_t^{k+2}\Omega_N(r,t)$$

i.e.

$$\omega(r,t)=\omega_{N,2}(r,t)-\frac{f\lambda_r r^3}{\mu R^2}t-\frac{f\lambda_r}{\mu\nu}\frac{r^3(r^2-2R^2)}{12R^2}+\frac{f\lambda_r^2 r^3}{\mu R^2}-$$
$$-\frac{4f}{\mu\nu^2 R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n^6 J_1(Rr_n)}\left(e^{-\frac{\nu r_n^2 t}{1+\nu r_n^2\lambda_r}}-e^{-\nu r_n^2 t}\right)-$$
$$-\frac{8f\lambda_r}{\mu\nu R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n^2 J_1(Rr_n)}e^{-\frac{\nu r_n^2 t}{1+\nu r_n^2\lambda_r}}-\frac{4f\lambda_r^2}{\mu R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n^2 J_1(Rr_n)}e^{-\frac{\nu r_n^2 t}{1+\nu r_n^2\lambda_r}},$$

$$\tau(r,t)=\tau_{N,2}(r,t)+2\mu\lambda_r t*\partial_t\Omega(r,t)+$$
$$+\frac{2f\lambda_r}{R}\sum_{n=1}^{\infty}\frac{r_n^3 J_2(rr_n)}{r_n J_1(Rr_n)}\left(1-\frac{2}{\nu r_n^2}-\right.$$
$$\left.-\frac{1+\lambda_r\nu r_n^2}{\lambda_r\nu^2 r_n^4}e^{-\frac{\nu r_n^2 t}{1+\lambda_r\nu r_n^2}}+\frac{\lambda_r\nu r_n^2-1}{\lambda_r\nu^2 r_n^4}e^{-\nu r_n^2 t}\right). \tag{55}$$

4. In the special case when $\alpha\to 1$ and $\beta\to 1$ into (16), (29), (32) and (34), the solutions for an Oldroyd-B fluid are obtained, namely:

$$\omega(r,t)=\omega_{N,2}(r,t)-$$
$$-\Gamma(1+a)\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!\lambda_r^m}{m!(k-m)!\lambda^k}\partial_t^{k+1}\omega_N(r,t)*G_{1,m-k-2,k+1}\left(-\frac{1}{\lambda},t\right)+$$
$$+2\frac{\lambda_r}{\lambda}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!\lambda_r^m}{m!(k-m)!\lambda^k}\partial_t^{k+2}\omega_N(r,t)*G_{1,m-k-3,k+1}=$$
$$=\omega_{N,2}(r,t)-2\frac{\pi f}{\rho}\sum_{n=1}^{\infty}r_n\;C_{Fn}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!\lambda_r^m}{m!(k-m)!}\left(-\frac{\nu r_n^2}{\lambda}\right)^k\times$$
$$\times\left[G_{1,m-k-2,k+1}\left(-\frac{1}{\lambda},t\right)+\nu r_n^2\frac{\lambda_r}{\lambda}G_{1,m-k-3,k+1}\left(-\frac{1}{\lambda},t\right)\right], \tag{56}$$

$$\tau(r,t) = \tau_{N,2}(r,t) + 2\mu\frac{\lambda_r-\lambda}{\lambda}R_{1,-2}\left(-\frac{1}{\lambda},t\right) * \partial_t\Omega_N(r,t) -$$
$$-2\mu\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!\lambda_r^m}{m!(k-m)!\lambda^k}\left[G_{1,m-k-2,k+1}(-1/\lambda,t) + \right.$$
$$\left. +\frac{\lambda_r}{\lambda}G_{1,m-k-1,k+2}(-1/\lambda,t) - G_{1,m-k-1,k+2}\left(-\frac{1}{\lambda},t\right)\right] * \partial_t^{k+1}\Omega_N(r,t) +$$
$$+2\mu\frac{\lambda_r}{\lambda}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!\lambda_r^m}{m!(k-m)!\lambda^k}\left[G_{1,m-k-3,k+1}(-1/\lambda,t) + \right.$$
$$\left. +\frac{\lambda_r}{\lambda}G_{1,m-k-2,k+2}(-1/\lambda,t) - G_{1,m-k-2,k+2}(-1/\lambda,t)\right] * \partial_t^{k+2}\Omega_N(r,t) =$$
$$= \tau_{N,2}(r,t) + 2\mu\frac{\lambda_r-\lambda}{\lambda}R_{1,-2}\left(-\frac{1}{\lambda},t\right) * \partial_t\Omega_N(r,t) +$$
$$+2\pi f\nu\sum_{n=1}^{\infty}r_n\widetilde{C}_{Fn}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!\lambda_r^m}{m!(k-m)!}\left(-\frac{\nu r_n^2}{\lambda}\right)^k \times$$
$$\times\left\{\left[G_{1,m-k-2,k+1}\left(-\frac{1}{\lambda},t\right) + \frac{\lambda_r-\lambda}{\lambda}G_{1,m-k-1,k+2}\left(-\frac{1}{\lambda},t\right)\right] + \right.$$
$$\left. \nu r_n^2\frac{\lambda_r}{\lambda}\left[G_{1,m-k-3,k+1}\left(-\frac{1}{\lambda},t\right) + \frac{\lambda_r-\lambda}{\lambda}G_{1,m-k-1,k+2}\left(-\frac{1}{\lambda},t\right)\right]\right\} * e^{-\nu r_n^2 t} \tag{57}$$

As before we get

$$\omega(r,t) = \omega_{N,2}(r,t) - \frac{4fR}{\mu}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n^2 J_1(Rr_n)}\left[\lambda_r\frac{t^2}{2} + \frac{\lambda - 2\lambda_r - \lambda_r^2\nu r_n^2}{\nu r_n^2}t + \right.$$
$$+\frac{\lambda_r^2(\nu r_n^2)^2 + (1+\lambda_r\nu r_n^2)(3\lambda_r - 2\lambda)}{(\nu r_n^2)^2} + \frac{1}{(\nu r_n^2)^3}e^{-\nu r_n^2 t} +$$
$$\left. \lambda^3\frac{q_{n2}^3(1+\lambda q_{n1}+\lambda_r q_{n2}+\lambda_r\nu r_n^2)e^{q_{n1}t} - q_{n1}^3(1+\lambda q_{n2}+\lambda_r q_{n1}+\lambda_r\nu r_n^2)e^{q_{n2}t}}{(\lambda-\lambda_r)(\nu r_n^2)^4(q_{n1}-q_{n2})}\right]$$

$$= \frac{4fR}{\mu}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n^2 J_1(Rr_n)}\left[\frac{t^2}{2} - \frac{1-\lambda_r\nu r_n^2}{\nu r_n^2}t + \frac{1+(2\lambda_r-\lambda)\nu r_n^2 + \lambda_r^2(\nu r_n^2)^2}{(\nu r_n^2)^2} + \right.$$
$$\left. +\frac{1+(2\lambda_r-\lambda)\nu r_n^2}{(\nu r_n^2)^2}\frac{q_{n2}e^{q_{n1}t} - q_{n1}e^{q_{n2}t}}{q_{n1}-q_{n2}} - (1+\lambda_r\nu r_n^2)\frac{e^{q_{n1}t}-e^{q_{n2}t}}{q_{n1}-q_{n2}}\right], \tag{58}$$

$$\tau(r,t) = \tau_{N,2}(r,t) + 2\mu(\lambda\lambda_r - \lambda)\left(t - \lambda + \lambda e^{-\frac{t}{\lambda}}\right) * \partial_t \Omega_N(r,t) +$$
$$+\frac{4f\nu}{R}\sum_{n=1}^{\infty}\frac{r_n J_2(rr_n)}{J_1(Rr_n)}\left[\frac{\lambda_r}{\nu r_n^2}t + \frac{\lambda - 2\lambda_r - \nu r_n^2\lambda\lambda_r}{(\nu r_n^2)^2} +\right.$$
$$+\frac{\lambda^2}{\nu r_n^2}\frac{1-\nu r_n^2\lambda_r}{1-\nu r_n^2\lambda}e^{-\frac{t}{\lambda}} + \frac{\lambda^2}{(\nu r_n^2)^3}\frac{1-\nu r_n^2\lambda_r}{1-\nu r_n^2\lambda}e^{-\nu r_n^2 t} +$$
$$\left.+\frac{\lambda(\lambda-\lambda_r)}{q_{n1}-q_{n2}}\left(\frac{q_{n1}e^{q_{n1}t}}{A_n q_{n1}+B_n} - \frac{q_{n2}e^{q_{n2}t}}{A_n q_{n2}+B_n}\right)\right] =$$
$$= 4f\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n^2 J_1(Rr_n)}\left[-\frac{t^2}{2} + \frac{1+\lambda_r\nu r_n^2+\lambda-\lambda_r}{\nu r_n^2} +\right.$$
$$+\frac{(\lambda+\lambda_r)(1+\lambda\nu r_n^2)}{\nu r_n^2} - \frac{\lambda-\lambda_r}{\lambda}\nu r_n^2 e^{-\frac{t}{\lambda}} +$$
$$\left.+\frac{\lambda^2}{(\nu r_n^2)^3}\frac{(1+\lambda_r q_{n1})q_{n2}^3(1+\lambda q_{n2})e^{q_{n1}t} - (1+\lambda_r q_{n2})q_{n1}^3(1+\lambda q_{n1})e^{q_{n2}t}}{q_{n1}-q_{n2}}\right], \tag{59}$$

where q_{n1} and q_{n12} are the real roots of $\lambda q^2 + (1+\nu r_n^2\lambda_r)q + \nu r_n^2 = 0$ (they are real negative numbers because $(1+\lambda_r\nu r_n^2)^2 - 4\lambda\nu r_n^2 > 0$, $q_{n1}\cdot q_{n2}\lambda\nu r_n^2 > 0$ and $q_{n1}+q_{n2} = -\dfrac{1+\nu r_n^2\lambda_r}{\lambda} < 0$).

6 Conclusions

The main purpose of this paper is to provide exact solution for the unsteady flow of an incompressible Oldroyd-B fluid filling the annular region between two infinitely long co-axial cylinders subject to a particular time-dependent shear stress. Such solutions, obtained by using the Hankel and Laplace transforms, are presented as sums between the Newtonian solutions and the corresponding non-Newtonian contributions. Furthermore, the non-Newtonian contributions of the general solutions are also presented in equivalent forms, under series form in terms of the time derivative of the (simplest) Newtonian solution ω_N and $\omega_{N,2}$ as well. For $\lambda \to 0$ (and, consequently, $\lambda_r \to 0$) these contributions tend to zero, such that the general solutions become Newtonian solutions corresponding to the given initial-boundary conditions.

It is remarkable that the general solutions can be easily specialized to give both the similar solutions for "generalized" second grade and Maxwell fluids and the solutions for all ordinary fluids (Oldroyd-B, Maxwell and second grade) performing the same motions. Direct computations shows that the solutions which have been obtained certainly satisfy both the governing equations and all imposed initial and boundary conditions. Furthermore, the solutions corresponding to ordinary Maxwell and second grade fluids can be also obtained as limiting cases of those for ordinary Oldroyd-B fluids. As regard the Newtonian solutions, given under simple forms (20), (22), and (37), they can be obtained as limiting cases of the previous solutions.

From our general solutions, corresponding to non-Newtonian fluids, it clearly results that the non-Newtonian contributions of these solutions exponentially decrease in time, the motion of the non-Newtonian fluids being well approximated, for large values of t, by the motion of the corresponding Newtonian fluid.

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