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DEGENERATED HOPF BIFURCATIONS IN A MATHEMATICAL MODEL OF ECONOMICAL DYNAMICS *

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Abstract

It is assumed that the dynamics of the capital of a firm is governed by a Cauchy problem for a system of two nonlinear ordinary differential equations containing three real parameters. In this paper we determine a $k \geq 3$ order degenerated Hopf bifurcation point for this economical model. To this aim the normal form technique is used.

MSC: 37L10, 37G05, 91B55

keywords: nonlinear dynamics, Hopf bifurcation, normal form, Liapunov coefficients

1 Introduction

The nonlinear dynamics theory enables us to understand and develop more realistic processes and methods in economic models. The development of the theory of singularities and the theory of bifurcation has completed the multitude of ways at our disposal to analyze and represent more and more complex dynamics, giving us the possibility of analyzing some systems which were hard, if not impossible to approach by traditional methods. The study

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of nonlinear dynamics is of outmost interest because the economical systems are by excellence nonlinear systems. Many of these contain multiple discontinuities and incorporate inherent instability being permanently under shock actions, extern and intern perturbations.The classical methods based on continuity, linearity and stability have been proven unstable for representing economic phenomena and processes with a higher degree of complexity. The researchers are bound to follow these processes in a dynamic way, to study qualitatively the changes that interfere with the implicated economic variables as well as the results obtained with their help.There are several models describing microeconomical dynamics. One of them is shown by the subsequent model consisting in the Cauchy problem $x(0) = x_0$, $y(0) = y_0$ for the system o.d.e. in $\mathbf{R}^2$.

1.1 Mathematical model

Let K_t be the capital of a firm at the time t and let L_t be the number of workers. Then the production force reads $y_t = F(K_t, L_t)$. The dynamics of the capital depends on the politics of firm development involving the net profit π_t, the dividends covering by shareholders δ_t (where $\delta_t\pi_t$ represents the dividends and $(1-\delta_t)\pi_t$ are the remaining investments), the capital depreciation by a coefficient μ_t and the income obtained by liquidation of the depreciated assets at the revenue costs λ_t. Let γ_t be the rate of change of the capital, such that $\pi_t = \gamma_t y_t$. Then, according to Oprescu [6], Ungureanu [7]

$$\begin{cases} \dot{K}(t) = \gamma_t(1-\delta_t)F(K_t, L_t) - \mu_t(1-\lambda_t)K_t \\ \dot{L}(t) = \alpha_1 K_t + \alpha_2 L_t - \alpha_0 \end{cases}$$

where the dot over quantities represents the differentiation with respect to time. Within this system K and $L : R \to R$ are unknown functions depending on independent variable t (time), $K-$ the capital of a firm and L- the number of workers.

This study is made according to the simplifying assumption that the parameters are considered constant $\mu_t = \mu$, $\delta_t = \delta$, $\gamma_t = \gamma$, $\lambda_t = \lambda$. If $y_t = VK_t^\alpha L_t^\beta$ and the production has an increasing physical efficiency, i.e. $\alpha + \beta > 1$, the above equations become

$$\begin{cases} \dot{x} = cx^2y + bx \\ \dot{y} = x + \alpha_2 y - 1 \end{cases} \tag{1.1.1}$$

where we choose $\alpha = 2$, $\beta = 1$, $x = \beta_1 K_t$, $y = \beta_2 L_t$, $\beta_1 = \alpha_1/\alpha_0$, $\beta_2 = 1/\alpha_0$ for $\alpha_0 \neq 0$, $\alpha_1 \neq 0$, $a = V\gamma(1-\delta)$, $b = -\mu(1-\lambda)$, $c = a\alpha_0^2/\alpha_1$. In this way the new state functions x and y are proportional to the capital and working force respectively. In addition, the number of parameters was reduced from eight to three.

1.2 Equilibrium points

Here α_2,b,$c \in \mathbf{R}$ are constant economical parameters and x and y are two economical state functions which are proportional to the capital and working force respectively.

The dynamics generated by (1.1.1) strongly depends on the three parameters. However, it is qualitatively unchanged for parameters lying in some areas of the parameter space. Correspondingly, for various points in these areas, the phase portraits are topologically equivalent.

In phase portraits formation a particular influence is exercised by the equilibria. They are the starting points in the study of the dynamical bifurcation (understood as a negation of the structural stability).

In the (x, y) phase plane they correspond to the equilibrium points denoted by $\overline{u}$.

The following cases hold:

a) $b = c = \alpha_2 = 0 \Rightarrow$ (1.1.1) has an infinity of equilibria $\overline{u} = (1, y_0)\,, \forall y_0 \in \mathbf{R}$ possessing the eigenvalues $s_1 = s_2 = 0$;

b) $b = c = 0$, $\alpha_2 \neq 0 \Rightarrow$ (1.1.1) has an infinity of equilibria $\overline{u} = (1 - \alpha_2 y_0, y_0)$, $\forall\, y_0 \in \mathbf{R}$ possessing the eigenvalues $s_1 = 0$, $s_2 = \alpha_2$;

c) $b = \alpha_2 = 0$, $c \neq 0 \Rightarrow$ (1.1.1) has a unique equilibrium $\overline{u} = (1, 0)$ possessing the eigenvalues $s_{1,2} = \pm\sqrt{c}$ for $c > 0$ and $s_{1,2} = \pm i\sqrt{-c}$ for $c < 0$;

d) $c = \alpha_2 = 0$, $b \neq 0 \Rightarrow$ (1.1.1) has no equilibrium;

e) $c = 0$, $b\alpha_2 \neq 0 \Rightarrow$ (1.1.1) has an equilibrium $\overline{u} = \left(0, \alpha_2^{-1}\right)$ possessing the eigenvalues $s_1 = b$, $s_2 = \alpha_2$;

f) $b = 0$, $c\alpha_2 \neq 0 \Rightarrow$ (1.1.1) has two equilibria $\overline{u}_1 = \overline{u}_2 = \left(0, \alpha_2^{-1}\right)$and $\overline{u}_3 = (1, 0)$ possessing the eigenvalues $s_1 = 0$,$s_2 = \alpha_2$ and $s_{1,2} = \left(\alpha_2 \pm \sqrt{\alpha_2^2 + 4c}\right)/2$, respectively;

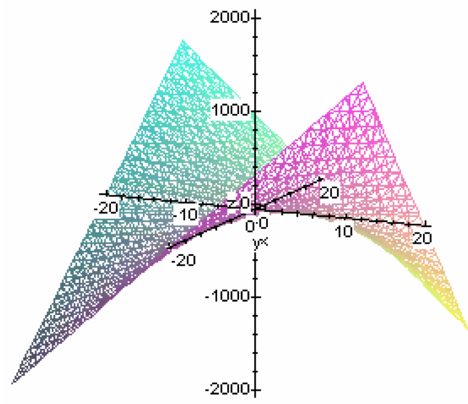

Figure 1: The surface S

g) $\alpha_2 = 0,\ bc \neq 0 \Rightarrow$ (1) has an equilibrium $\overline{u} = (1, -b/c)$ possessing the eigenvalues $s_{1,2} = \left(-b \pm \sqrt{b^2 + 4c}\right)/2$;

h) $\alpha_2 bc \neq 0 \Rightarrow$ (1.1.1) has three equilibria $\overline{u}_1 = \left(0, \alpha_2^{-1}\right)$,
$\overline{u}_2 = \left(\frac{c+\sqrt{c^2+4bc\alpha_2}}{2c}, \frac{c-\sqrt{c^2+4bc\alpha_2}}{2c\alpha_2}\right)$, $\overline{u}_3 = \left(\frac{c-\sqrt{c^2+4bc\alpha_2}}{2c}, \frac{c+\sqrt{c^2+4bc\alpha_2}}{2c\alpha_2}\right)$.

In the general case h), the three equilibria can never coincide, neither in the limit $b,\ c,\ \alpha_2 \to \pm\infty$. However, two of them can coincide at the points of the parameter space situated on a surface S (Figure 1). More exactly if $\overline{u}_1 = \overline{u}_2 = (1/2, 1/2\alpha_2)$. Therefore S is a hyperboloid with two sheets. It has the equation $c = -4b\alpha_2$, where $b\alpha_2 \neq 0$ and , and its sheets are situated in the octants characterized by $c > 0$, $b\alpha_2 < 0$, and $c < 0$, $b\alpha_2 > 0$, respectively.

In the domain determined by sheets of S and the plane on which it is supported,(b, α_2) , the system (1.1.1) has an equilibrium point. Outside this domain, at the points which do not belong to S or the three planes $b = 0$, $c = 0$, $\alpha_2 = 0$, the system (1.1.1) possesses three equilibria.

We recall that on the sheets of S (1.1.1) possesses two equilibria and S has no point in the plans of coordinates on the parameter space. We can have two equilibria only on S and in the $b = 0$ plane without axes, one equilibrium is double, namely $\overline{u}_2 = \overline{u}_3 = \left(0, \alpha_2^{-1}\right)$, and another one $\overline{u}_2 = (1, 0)$ simple. Let us notice that in this case $c \neq 0$.

Let us define the domains D_1 and D_2 determined by the sheets of S and the $c = 0$ plane ($b > 0$, $\alpha_2 < 0$ and $b < 0$, $\alpha_2 > 0$, respectively). The domains D_1 and D_2 do not contain Oc axis. There are three equilibria only for points of the parameter space situated outside the domains D_1 and D_2 .

System (1.1.1) can have one equilibrium only in the following three cases:

1) Points situated on the Oc axis without origin. In this case the equilibrium is $\overline{u}_2 = (1, 0)$;

2) The $c = 0$ plane without axis. In this case the equilibrium point is $\overline{u}_1 = \left(0, \alpha_2^{-1}\right)$;

3) The $\alpha_2 = 0$ plane. In this case the equilibrium is $\overline{u}_2 = (1, -b/c)$.

To points of the Ob axis without origin no equilibrium corresponds. For points of the $O\alpha_2$ axis including the origin there are an infinity of equilibria situated on the straight-line $x + \alpha_2 y - 1 = 0$. On the $O\alpha_2$ axis without origin the corresponding equilibria have the form $(x_0, (1 - x_0)/\alpha_2)$. Among them there is $\overline{u}_2 = (1, 0)$ (corresponding to $x_0 = 1$), $\overline{u}_1 = \overline{u}_3 = \left(0, \alpha_2^{-1}\right)$ (corresponding to $x_0 = 0$) and $\overline{u}_2 = \overline{u}_3 = (1/2, 1/2\alpha_2)$ (corresponding to $x_0 = 1/2$). It follows that to the points of the $O\alpha_2$ axis without origin the same equilibria $\overline{u}_1$and $\overline{u}_2 = \overline{u}_3$ correspond as for the points of S.

The half axes $\alpha_2 > 0$ and $\alpha_2 < 0$ consist of accumulation points for S. This is true both when S is considered as a topologic subspace of R^3 and when S possesses the above property concerning the equilibria (i.e. $\overline{u}_2 = \overline{u}_3$).

However, for points of the $O\alpha_2$ axis without origin, apart from the equilibria $\overline{u}_1$ and $\overline{u}_2 = \overline{u}_3$, there exists an infinity of other equilibria depending on the initial datum.

Finally, to the origin of the parameters space an infinity of equilibria of the form $(1, y_0)$corresponds. Among them, there is also the point $\overline{u}_2 = (1, 0)$ (corresponding to $y_0 = 0$).

2 Nonhyperbolic singularities of Hopf type

2.1 Normal forms

Using the eigenvalues and the eigenvectors of the nonhyperbolic point of equilibrium $\overline{\mathbf{u}}_3$ corresponding to the values of parameters $\alpha_2 = b$, $c < -4b^2$, we put the system (1.1.1) in the normal form and emphasises that it corresponds to a degenerated Hopf singularity.

Proposition 2.2.1. *Up to terms of degree greater than three system*

$$\begin{cases} \dot{x} = cx^2y + bx, \\ \dot{y} = x + by - 1 \end{cases} \tag{2.1.1}$$

has around the equilibrium $\overline{\mathbf{u}}_3 = (\frac{c-\sqrt{\Delta}}{2c}, \frac{c+\sqrt{\Delta}}{2bc}) \overset{not}{=} (u, v)$, *where* $\Delta = c^2 + 4b^2c$, *the normal form*

$$\begin{cases} \dot{x}_5 = irx_5 + Cx_5^2y_5, \\ \dot{y}_5 = -iry_5 + \overline{C}x_5y_5^2. \end{cases}$$

Proof. We carry the point $\overline{\mathbf{u}}_3$ at the origin by means of the change of coordinates $x_1 = x - u$, $y_1 = y - v$. Then (2.1.1) becomes

$$\begin{cases} \dot{x}_1 = -bx_1 + cu^2y_1 + cvx_1^2 + 2cux_1y_1 + cx_1^2y_1, \\ \dot{y}_1 = x_1 + by_1. \end{cases} \tag*{$(2.1.1)'$}$$

The matrix associated to the system linearized around the point $(x_1, y_1) = (0,0)$ is $Q = \begin{pmatrix} -b & cu^2 \\ 1 & b \end{pmatrix}$ and it admits the purely imaginary eigenvalues $s_{1,2} = \pm i\sqrt{\frac{-c-4b^2+\sqrt{c^2+4bc}}{2}} \overset{not}{=} \pm ir$. Hence $\overline{\mathbf{u}}_3$ is a Hopf singularity. Let $\overline{\mathbf{p}} = (s_1 - b, 1)$ be an eigenvector of Q corresponding to the eigenvalue $s_1 = ir$. Then, $\overline{\mathbf{p}}$ may be written in the form $\overline{\mathbf{p}} = \overline{\mathbf{q}} + i\overline{\mathbf{t}}$ where $\overline{\mathbf{q}} = (-b, 1)$,and $\overline{\mathbf{t}} = (r, 0)$. The matrix $P = \begin{pmatrix} r & -b \\ 0 & 1 \end{pmatrix}$ is nonsingular and so, we may perform the transformation

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = P^{-1} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \frac{1}{r} \begin{pmatrix} 1 & b \\ 0 & r \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

and obtain a system in (x_2, y_2). As the linearized system corresponding at (x_2, y_2) has not a matrix in a diagonal form we perform the change $\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = M_c \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ where $M_c = \begin{pmatrix} 0,5 & 0,5 \\ -0,5i & 0,5i \end{pmatrix}$. Therefore, $\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = M_cP^{-1} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$. As M_c is a matrix in the complex field it follows that $x_3, y_3 \in \mathbf{C}$ namely $x_3 = \overline{y}_3$, ($\overline{y}_3$ is the complex conjugate of y_3). We have $x_3 = \frac{1}{r}[x_1 + (b + ir)y_1]$, $y_3 = \frac{1}{r}[x_1 + (b - ir)y_1]$ or, $x_1 = \frac{r+ib}{2}x_3 + \frac{r-ib}{2}y_3$, $y_1 = \frac{i}{2}(y_3 - x_3)$. Thus, (2.1.1) becomes

$$\begin{cases} \dot{x}_3 = irx_3 + \frac{T}{2}, \\ \dot{y}_3 = -iry_3 + \frac{T}{2}, \end{cases} \tag{2.1.2}$$

where $T = a_{20}x_3^2 + \overline{a}_{20}y_3^2 + a_{11}x_3y_3 + a_{30}x_3^3 + \overline{a}_{30}y_3^3 + a_{21}x_3^2y_3 + \overline{a}_{21}x_3y_3^2$, $a_{20} = \frac{bc-5b\sqrt{\Delta}}{4r} + i\sqrt{\Delta}$, $a_{11} = \frac{b(\sqrt{\Delta}-c)}{2r}$, $a_{21} = \frac{ci(c-2b^2-\sqrt{\Delta}+4irb)}{8r}$, $a_{30} = \frac{-ci(r+ib)^2}{4r}$.

In order to eliminate the nonresonant terms of second degree it is necessary to complete the following table, where $\Lambda_{m,i} = (\mathbf{m}\cdot s) - s_i$ and $h_{m,1} = \frac{X_{m,i}}{(\mathbf{m}\cdot s)-s_i}$, $s = (s_1, s_2)$ [1].

Table 2.1

m_1	m_2	$X_{m,1}$	$X_{m,2}$	$\Lambda_{m,1}$	$\Lambda_{m,2}$	$h_{m,1}$	$h_{m,2}$
2	0	$\frac{a_{20}}{2}$	$\frac{a_{20}}{2}$	ir	$3ir$	$\frac{a_{20}}{2ir}$	$\frac{a_{20}}{6ir}$
1	1	$\frac{a_{11}}{2}$	$\frac{a_{11}}{2}$	$-ir$	ir	$-\frac{a_{11}}{2ir}$	$\frac{a_{11}}{2ir}$
0	2	$\frac{\overline{a}_{20}}{2}$	$\frac{\overline{a}_{20}}{2}$	$-3ir$	$-ir$	$-\frac{\overline{a}_{20}}{6ir}$	$-\frac{\overline{a}_{20}}{2ir}$

It follows the transformation

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_4 \\ y_4 \end{pmatrix} + \begin{pmatrix} \frac{a_{20}}{2ir}x_4^2 - \frac{a_{11}}{2ir}x_4y_4 - \frac{\overline{a}_{20}}{6ir}y_4^2 \\ \frac{a_{20}}{6ir}x_4^2 + \frac{a_{11}}{2ir}x_4y_4 - \frac{\overline{a}_{20}}{2ir}y_4^2 \end{pmatrix},$$

which introduced in (2.1.2), leads to the system

$$\begin{cases} \dot{x}_4 = irx_4 + Ax_4^3 + \overline{A}y_4^3 + Cx_4^2y_4 + \overline{C}x_4y_4^2, \\ \dot{y}_4 = -iry_4 + Ax_4^3 + \overline{A}y_4^3 + Cx_4^2y_4 + \overline{C}x_4y_4^2, \end{cases} \tag{2.1.3}$$

where $A = \frac{6a_{20}^2 + a_{11}a_{20} + 6ira_{30}}{12ir}$ and $C = \frac{2a_{20}\overline{a}_{20} - 3a_{11}a_{20} + 3a_{11}^2 + 6ira_{21}}{12ir}$.

In order to reduce the nonresonant terms of order three (2.1.3) we use the table.

Table 2.2

m_1	m_2	$X_{m,1}$	$X_{m,2}$	$\Lambda_{m,1}$	$\Lambda_{m,2}$	$h_{m,1}$	$h_{m,2}$
3	0	A	A	$2ir$	$4ir$	$\frac{A}{2ir}$	$\frac{A}{4ir}$
2	1	C	C	0	$2ir$	–	$\frac{C}{2ir}$
1	2	$\overline{C}$	$\overline{C}$	$-2ir$	0	$-\frac{\overline{C}}{2ir}$	–
0	3	$\overline{A}$	$\overline{A}$	$-4ir$	$-2ir$	$-\frac{\overline{A}}{4ir}$	$-\frac{\overline{A}}{2ir}$

Thus we obtain the transformation

$$\begin{pmatrix} x_4 \\ y_4 \end{pmatrix} = \begin{pmatrix} x_5 \\ y_5 \end{pmatrix} + \begin{pmatrix} \frac{A}{2ir}x_5^3 - \frac{\overline{C}}{2ir}x_5y_5^2 - \frac{\overline{A}}{4ir}y_5^3 \\ \frac{A}{4ir}x_5^3 + \frac{C}{2ir}x_5^2y_5 - \frac{\overline{A}}{2ir}y_5^3 \end{pmatrix}$$

leading to the system

$$\begin{cases} \dot{x}_5 = irx_5 + Cx_5^2y_5, \\ \dot{y}_5 = -iry_5 + \overline{C}x_5y_5^2. \end{cases} \tag{2.1.4}$$

In this system we retained terms up to the third degree. Thus we obtained the normal form in **C**. Obviously the second equation is the conjugate of the first, therefore, up to terms of the third degree the normal form is $(2.1.4)_1$.

Theorem 2.1.1 *The Hopf singularity $\overline{\mathbf{u}}_3$ is degenerated of order $k \geq 2$.*

Proof. Taking into account the expressions of a_{20}, a_{11}, a_{21}, r, Δ a direct computation leads us to the expression of C :

$C = -\frac{ic}{48r^3}\left(16b^4 + 5b^2c - c^2 + c\sqrt{\Delta} - 7b^2\sqrt{\Delta}\right)$. Since $c < -4b^2$ it follows that $16b^4 + 5b^2c - c^2 + c\sqrt{\Delta} - 7b^2\sqrt{\Delta} < -4b^2 - c^2 - 3b^2\sqrt{\Delta} < 0$, hence $C \neq 0$ and C is purely imaginary. Then (2.1.4) has the follow normal form, according to Arrowsmith [1]:

$$\begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \sum_{k=1}^{[(N-1)/2]} (y_1^2 + y_2^2)^k \left\{ a_k \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + b_k \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix} \right\} +$$

$O\left(|y|^{N+1}\right)$, $\beta = \sqrt{\det A}, N \geq 3$, $[.]$ represents integer part and $a_k,\ b_k \in \mathbf{R}$ where $a_1 = 0$ and $b_1 = \operatorname{Im} C \neq 0$, whence the conclusion of the theorem.(Figure 1)

Corollary 2.1.1 *The first Liapunov coefficient associated to system $(2.1.1)'$ is null* ($\operatorname{Re} C = 0$).

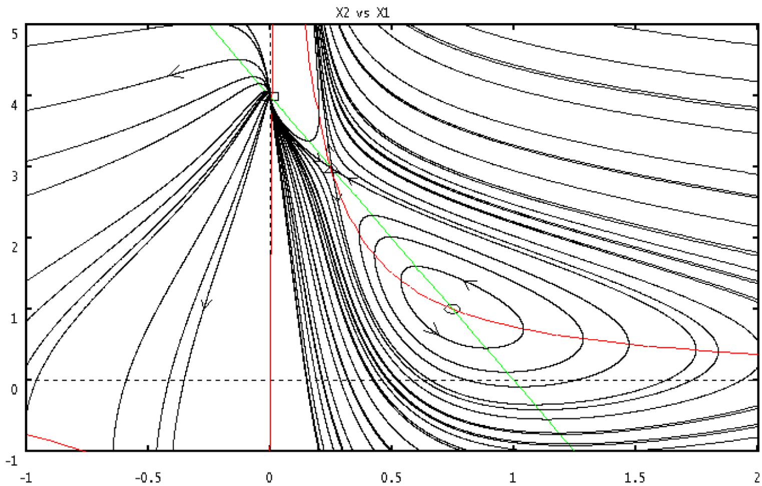

Figure 2: Local phase portrait in the degenerated Hopf bifurcation, for $\alpha_2 = b = -0.1$, $c = -0.04$.

2.2 Computation of the Liapunov coefficients

Proposition 2.2.1. *The system (2.1.1)′ is topologically equivalent to system*

$$\begin{cases} \dot{x}_2 = -ry_2 + cvrx_2^2 + (2cu - 2vb)x_2y_2 + \frac{cvb^2 - 2cub}{r}y_2^2 + crx_2^2y_2 \\ -2cbx_2y_2^2 + \frac{cb^2}{r}y_2^3, \\ \dot{y}_2 = rx_2. \end{cases} \tag{2.2.1}$$

Proof. The transformation of coordinates

$$\begin{cases} x_1 = rx_2 - by_2, \\ y_1 = y_2 \end{cases}$$

carries system (2.1.1)′ in (2.2.1). In this conditions, according to Chow and Wang [2] there exists a smooth function $F(x) = \frac{r}{2}(x_2^2 + y_2^2) + O(|x, y|^3)$ such that

$$\langle \mathrm{gradF}, \mathrm{X}_0 \rangle = \sum_{i=1}^{m} V_i (x_2^2 + y_2^2)^{i+1} + O(|x, y|^{m+1}) \tag{2.2.2}$$

where X_0 is the vector field corresponding to (2.2.1), and V_i are the Liapunov coefficients.

Proposition 2.2.2. *For the system (2.2.1) we have* $V_1 = 0$ *and*
$V_2 = -\frac{b^2c^2}{24r^2}\left[13c^2 + 78b^2c + 5bc + 104b^4 - 90b^3 - (3b + 52b^2 + 13c)\sqrt{c^2 + 4b^2c}\right].$

Proof. We look for F to the form $F(x) = \frac{r}{2}(x_2^2 + y_2^2) + \sum\limits_{i+j=k}\sum\limits_{k>3} c_{ij}x_2^i y_2^j$.

Therefore, we have

$$\begin{aligned}\langle \mathrm{gradF}, \mathrm{X_0}\rangle \;=\; & r^2x_2y_2 + r\sum_{i+j=3} jc_{ij}x_2^{i+1}y_2^{j-1} - r^2x_2y_2 - r\sum_{i+j=4} ic_{ij}x_2^{i-1}y_2^{j+1} + \\ & cvr^2x_2^3 + cvr\sum_{i+j=3} ic_{ij}x_2^{i+1}y_2^j + 2c(u-vb)rx_2^2y_2 + \\ & 2c(u-vb)\sum_{i+j=4} ic_{ij}x_2^iy_2^{j+1} + \frac{cvb^2-2cub}{r}rx_2y_2^2 + \\ & \frac{cvb^2-2cub}{r}y_2^2\sum_{i+j=3} ic_{ij}x_2^{i-1}y_2^{j+2} + cr^2x_2^3y_2 + \\ & cr\sum_{i+j=4} ic_{ij}x_2^{i+1}y_2^{j+1} - 2cbrx_2^2y_2^2 - 2cb\sum_{i+j=3} ic_{ij}x_2^iy_2^{j+2} + \\ & cb^2x_2y_2^3 + \frac{cb^2}{r}\sum_{i+j=4} ic_{ij}x_2^{i-1}y_2^{j+3}\end{aligned}$$

Identifying the monomials of degree three coefficients in (2.2.2) we obtain the following system in unknowns c_{ij}, $i + j = 3$:

$$\begin{cases} rc_{21} + cvr^2 = 0, \\ -rc_{12} = 0, \\ 3rc_{03} - 2rc_{21} + cvb^2 - 2cub = 0, \\ 2rc_{12} - 3rc_{30} + 2cru - 2cvbr = 0. \end{cases}$$

The solution of this system reads $c_{12} = 0$, $c_{21} = -cvr$, $c_{03} = \frac{2cub-cvb^2-2r^2cv}{3r}$, $c_{30} = \frac{2cu-2cvb}{3}$.

Identifying the monomials of degree four coefficients in (2.2.2) we obtain the following system in unknowns c_{ij}, $i + j = 4$ and V_1

$$\begin{cases} rc_{31} + 3cvrc_{30} = V_1, \\ -rc_{13} + \frac{cvb^2-2cub}{r}c_{12} = V_1, \\ 4rc_{04} - 2rc_{22} + 2c(u-vb)c_{12} + 2\frac{cvb^2-2cub}{r}c_{21} + cb^2 = 0, \\ 3rc_{13} - 3rc_{31} + cvrc_{12} + 4c(u-vb)c_{21} + 3\frac{cvb^2-2cub}{r}c_{30} - 2cbr = 2V_1, \\ 2rc_{22} - 4rc_{40} + 2cvrc_{21} + 6c(u-vb)c_{30} + cr^2 = 0, \end{cases}$$

the solution of which is $V_1 = 0$, $c_{13} = 0$, $c_{31} = -2c^2uv + 2c^2v^2b$, $c_{04} = \frac{c_{22}}{2} + \frac{2c^2v^2b^2-4c^2uvb-cb^2}{4r}$, $c_{40} = \frac{c_{22}}{2} + \frac{4c^2u^2-8c^2uvb+4c^2v^2b^2-2c^2v^2r^2+cr^2}{4r}$.

Remark 2.2.1. The result $V_1 = 0$ represents a new proof for Theorem 2.1.1.

Identifying the monomials of degree five coefficients in (2.2.2) we obtain the following system in unknowns c_{ij}, $i+j=5$

$$\begin{cases} rc_{41} + 4cvrc_{40} = 0, \\ rc_{14} + \frac{cvb^2-2cub}{r}c_{13} + \frac{cb^2}{r}c_{12} = 0, \\ 5rc_{05} - 2rc_{23} + 2c(u-vb)c_{13} + 2\frac{cvb^2-2cub}{r}c_{22} - 2bcc_{12} + \frac{2cb^2}{r}c_{21} = 0 \\ 4rc_{14} - 3rc_{32} + cvrc_{13} + 4c(u-vb)c_{22} + 3\frac{cvb^2-2cub}{r}c_{31} + crc_{12} \\ -4cbc_{21} + \frac{3cb^2}{r}c_{30} = 0, \\ 3rc_{23} - 4rc_{41} + 2cvrc_{22} + 6c(u-vb)c_{31} + 4\frac{cvb^2-2cub}{r}c_{40} \\ +2crc_{21} - 6bcc_{30} = 0, \\ 2rc_{32} - 5rc_{50} + 3cvrc_{31} + 8c(u-vb)c_{40} + 3crc_{30} = 0 \end{cases}$$

whence

$$\begin{aligned} &c_{14} = 0, \\ &c_{41} = -4cvc_{40}, \\ &c_{23} = \Big[\tfrac{1}{3r}(-16cvr - 4\tfrac{cvb^2-2cub}{r})c_{40} - 2cvrc_{22} - 6c(u-vb)c_{31} - 2crc_{21} \\ &+6cbc_{30}\big], \\ &c_{32} = \big[\tfrac{1}{3r^2}4cr(u-vb)c_{22} + 6c^3v^3b^3 - 18c^3v^2ub^2 + 12c^3vu^2b + 4c^2vbr^2 \\ &+2c^2ub^2 - 2c^2vb^3\big], \\ &c_{50} = \tfrac{1}{5r}\left[2rc_{32} + 3cvrc_{31} + 8c(u-vb)c_{40} + 3crc_{30}\right], \\ &c_{05} = \tfrac{2}{5}c_{23} - \tfrac{2cb^2}{5r^2}c_{21} - \tfrac{2}{5r^2}(cvb^2 - 2cub)c_{22}. \end{aligned}$$

By identifying the coefficients of the monomials of degree six in (2.2.2) we obtain the following system in unknowns V_2 and c_{ij}, $i+j=6$

$$\begin{cases} rc_{51} + 5cvrc_{50} = V_2, \\ -rc_{15} + \frac{cvb^2-2cub}{r}c_{14} + \frac{cb^2}{r}c_{13} = V_2 \\ 2rc_{42} - 6rc_{60} + 4cvrc_{41} + 10c(u-vb)c_{50} + 4crc_{40} = 0 \\ 3rc_{33} - 5rc_{51} + 3cvrc_{32} + 8c(u-vb)c_{41} + 5\frac{cvb^2-2cub}{r}c_{50} \\ +3crc_{31} - 8cbc_{40} = 3V_2, \\ 4rc_{24} + 2cvrc_{23} + 6c(u-vb)c_{32} - 4rc_{42} + 4\frac{cvb^2-2cub}{r}c_{41}+ \\ 2crc_{22} - 6bcc_{31} + \frac{4cb^2}{r}c_{40} = 0, \\ 5rc_{15} - 3rc_{33} + cvrc_{14} + 4c(u-vb)c_{23} + 3\frac{cvb^2-2cub}{r}c_{32} + crc_{13} \\ -4cbc_{22} + \frac{3cb^2}{r}c_{31} = 3V_2, \\ 6rc_{06} - 2rc_{24} + 2c(u-vb)c_{14} + 2\frac{cvb^2-2cub}{r}c_{23} - 2bcc_{13} + \frac{2cb^2}{r}c_{22} = 0. \end{cases}$$

Since $c_{15} = -\frac{V_2}{r}$, $c_{51} = \frac{V_2}{r} - 5cvrc_{50}$, by replacing the found value for $3rc_{33}$ from the sixth equation in the fourth equation and taking into account the found values for c_{ij}, $i+j=5$, we have

$$\begin{aligned} V_2 \;&=\; -\frac{b^2c^2}{24r^2}13c^2 + 78b^2c + 5bc + 104b^4 - 90b^3 \\ &\quad\; -(3b + 52b^2 + 13c)\sqrt{c^2 + 4b^2c}. \end{aligned}$$

The set $V_2 = 0$ intersects the domain $\left\{\alpha_2 = b, c < -4b^2\right\}$ along a curve γ_3 the existence of which is proved by studying the sign of V_2 in the domain considered, and also by numerical methods (figure 3).

It is important to remark that the equilibrium $\bar{\mathbf{u}}_3$ exists also on the half-axis $c < 0$ and in this case $b = 0$ implies $V_2 = 0$. The curve γ_3 and the negative half-axis $c < 0$ divide the interior of the parabola $\alpha_2 = b$, $c = -4b^2$ in three regions:

$U_1 = \left\{(b, b, c) |\; c < 0, -\frac{\sqrt{-c}}{2} < b < 0\right\}$,

$U_2 = \left\{(b, b, c) |\; c < 0, 0 < b < b(c)\right\}$,

$U_3 = \left\{(b, b, c) |\; c < 0, b(c) < b < \frac{\sqrt{-c}}{2}\right\}$,

where $\alpha_2 = b$, $b = b(c)$ are the equations of the curve γ_3. It is easy to see that $V_2 < 0$ on $U_1 \cup U_2$, $V_2 > 0$ on U_3. As $V_2 = 0$ on γ_3 and on the half-axis $c < 0$ it follows

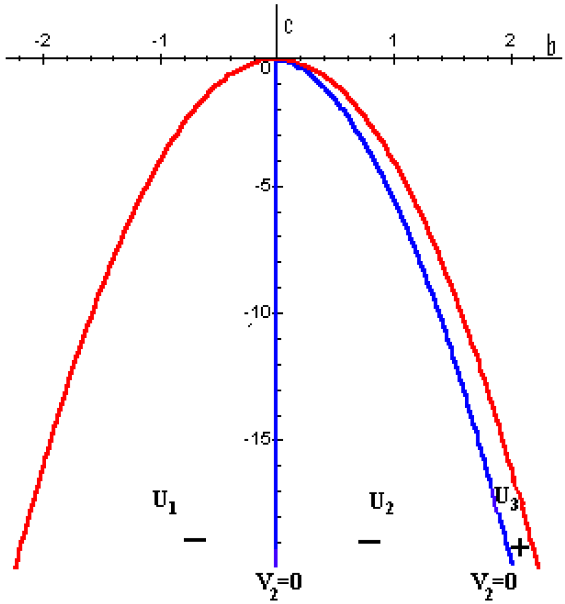

Figure 3: The sign of V_2 in $\left\{\alpha_2 = b, c < -4b^2\right\}$.

Theorem 2.2.1 *The point of equilibrium* $\bar{\mathbf{u}}_3$ *is locally a Bautin bifurcation with the Liapunov coefficient* $V_2 < 0$ *for* $(\alpha_2, b, c) \in U_1 \cup U_2$*, a Bautin bifurcation with the Liapunov coefficient* $V_2 > 0$ *for* $(\alpha_2, b, c) \in U_3$*, and a degenerated Hopf bifurcation of order* $k \geq 3$ *for a point of* γ_3 *or of the negative half-axis* $c < 0$.

2.3 Conclusions

From economic point of view, the variation of capital K and labor L over time it can be observed, starting from the initial significant data corresponding to some points in the parameter space. Therefore, there are situations when the system considered enable a periodic solution appropriate to a cyclical economic evolution. Negative phenomena such as production shortage and increase of unemployment rate and also the positive ones, featured by refurbishment of production capacities that could induce the growth of demand for consumption goods and determination of employment level, can be relieved.

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