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In Memoriam Adelina Georgescu

A NEW LOOK AT THE LYAPUNOV INEQUALITY*

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Abstract

Given a Banach space E, it is proved that any function u in $C^2([a,b],E)$ verifies the inequality

$$\max\{\|u(a)\|,\|u(b)\|\}+\frac{b-a}{4}\int_a^b\|u''(t)\|\,dt\geq\sup_{t\in[a,b]}\|u(t)\|.$$

The constant $(b-a)/4$ is sharp. Several applications are included.

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1 Introduction

The well-known Lyapunov inequality states that if $q:[a,b]\to\mathbb{R}$ is a continuous function, then a necessary condition for the boundary value problem

$$\begin{cases} u''+qu=0 \\ u(a)=u(b)=0, \end{cases} \tag{1}$$

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to have nontrivial solutions is that

$$\int_a^b |q(t)|\, dt > \frac{4}{b-a}. \tag{2}$$

See the monograph [11] and the survey [3] (which also includes an excellent account on the history of this result).

The following equivalent version of the Lyapunov inequality was proved by Borg [2] (who attributes it to Beurling): for every twice continuously differentiable function $u : [a, b] \to \mathbb{R}$ such that $u(a) = u(b) = 0$ and $u(t) > 0$ for $t \in (a, b)$, we have

$$\int_a^b \frac{|u''(t)|}{u(t)} dt > \frac{4}{b-a}. \tag{3}$$

The aim of this paper is to embed (3) into a stronger inequality that relates the values of a differentiable function on an interval, the values at the endpoints and the total variation of its derivative:

Theorem 1. *Let $u : [a, b] \to \mathbb{R}^N$ be a function which admits an integrable second derivative. Then*

$$\max\left\{\|u(a)\|, \|u(b)\|\right\} + \frac{b-a}{4}\int_a^b \left\|u''(t)\right\| dt \geq \sup_{t\in[a,b]} \|u(t)\|.$$

As usually, $\mathbb{R}^N$ denotes here the Euclidean N-dimensional space.

The restriction to the case of functions taking values in $\mathbb{R}^N$ is not essential. A similar result works for all functions taking values in an arbitrary Banach space. This will be discussed in Section 4.

Theorem 1 has a very natural kinematic interpretation: Suppose a point moves in the Euclidean space according to the law of motion $u = u(t)$. Then the difference between the maximum deviation from the origin during an interval of time $[a, b]$ and the maximum deviation at the endpoints of this interval does not exceed

$$\frac{1}{4}\,\text{(elapsed time)} \times \text{total variation of speed.}$$

Recall that every differentiable function $v : [a, b] \to \mathbb{R}^N$ with integrable derivative has bounded total variation and this is given by the formula

$$\bigvee_a^b v = \int_a^b \left\|v'(t)\right\| dt.$$

See [1], p. 104.

The proof of Theorem 1 will make clear that we can deal with other boundary conditions and more general second order differential operators. Some important remarks concerning the case of Neumann boundary conditions can be found in [5].

Also, instead of the L^1 norm in the left hand side and the sup norm in the right hand side we may consider other pairs of L^p norms (with $p \in [1,\infty]$). All these questions will be discussed elsewhere.

2 Consequences of the main result

Theorem 1 has a number of interesting consequences even in the 1-dimensional case. We start with the following stronger version of the inequality of Lyapunov:

Corollary 1. (*A. Wintner* [14]). *Let $q = q(t)$ be a real-valued continuous function defined on an interval $[a,b]$. A necessary condition for the equation $u'' + q(t)u = 0$ to have a nontrivial solution possessing (at least) two zeros is that*

$$\int_a^b q^+(t)dt > \frac{4}{b-a}.$$

Here $q^+ = \sup\{q, 0\}$ denotes the positive part of q.

Proof: By Sturm's Separation Theorem, since $q^+ \geq q$, the equation $v'' + q^+(t)v = 0$ is a Sturm majorant for the equation $u'' + q(t)u = 0$, and hence has a nontrivial solution v with two zeros $\alpha < \beta$ in $[a,b]$. See [8], Corollary 3.1, p. 335. Lyapunov's result follows now from Theorem 1, applied to the restriction of v to $[\alpha,\beta]$. In fact,

$$\begin{aligned}\sup_{t\in[\alpha,\beta]} |v(t)| &< \frac{\beta-\alpha}{4}\int_\alpha^\beta q^+(t)\,|v(t)|\,dt \\ &\leq \frac{b-a}{4}\left(\sup_{t\in[\alpha,\beta]} |v(t)|\right)\int_a^b q^+(t)dt,\end{aligned}$$

and it remains to simplify both sides by $\sup_{t\in[\alpha,\beta]} |v(t)|$. ∎

Using a change of variable due to Hille [9], one can extend easily Corollary 1 to all second-order differential equations of the form

$$u'' + g(t)u' + f(t)u = 0,$$

where f is continuous and g is continuously differentiable. In fact, the corresponding equation for $v = u \exp\left(\frac{1}{2}\int_a^t g(s)ds\right)$ is in normal form,

$$v'' + q(t)v = 0,$$

where $q(t) = f(t) - \frac{1}{4}g^2(t) - \frac{1}{2}g'(t)$.

Theorem 1 imposes an obstruction on the nonzero eigenvalues of the operator $Du = -u'' + qu$ with Dirichlet boundary conditions:

Corollary 2. *Suppose that $q : [a,b] \to \mathbb{R}$ is a continuous function, and $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function which admits an estimate of the form $|f(t,u)| \leq \varphi(t)\,|u|$ for a suitable $\varphi \in C([a,b],\mathbb{R})$ with $\varphi > 0$ on (a,b). Then every eigenvalue of the regular Sturm-Liouville problem,*

$$\begin{cases} -u'' + qu = \lambda f(t,u) \\ u(a) = u(b) = 0, \end{cases} \tag{4}$$

admits an estimate of the form

$$|\lambda| \geq \left(\frac{4}{b-a} - \int_a^b |q|\,dt\right)\left(\int_a^b \varphi dt\right)^{-1}.$$

The linear case of the Sturm-Liouville problem (4) (that is, when $f(t,u) = \varphi(t)u$) is presented in many books, for example in [8] and [13]. In this case the spectrum $-u'' + qu$ consists of an increasing sequence of positive eigenvalues λ_n with $\lambda_n \to \infty$.

Notice that Corollary 2 also works in the vector case (when u and f take values in $\mathbb{R}^N$).

Theorem 1 provides useful to establish Weierstrass type criteria of convergence:

Corollary 3. *Let $(u_n)_n$ be a sequence of real-valued twice differentiable functions defined on an interval $[a,b]$. If:*

i) this sequence is convergent at the endpoints; and

ii) the derivatives of second order u_n'' are integrable and

$$\lim_{m,n\to\infty} \int_a^b \left|u_m''(t) - u_n''(t)\right| dt = 0,$$

then the sequence $(u_n)_n$ is uniformly convergent.

Moreover, if u is the limit of the sequence $(u_n)_n$, and all derivatives u_n'' are bounded, then u is differentiable and

$$u' = \lim_{n\to\infty} u_n' \quad \textit{uniformly.}$$

Proof: The first part is a direct consequence of Theorem 1. The second part follows from an old result due to Hadamard [7] (see also [12]): Let I be an interval and let $f : I \to \mathbb{R}$ be a twice differentiable bounded function, with bounded second derivative. Then f' is also bounded and

$$\|f'\|_\infty \leq \begin{cases} \dfrac{2\,\|f\|_\infty}{m(I)} + \dfrac{m(I)}{2}\,\|f''\|_\infty\,, & \text{if } m(I) \leq 2\sqrt{\|f\|_{L^\infty} / \|f''\|_\infty} \\ 2\sqrt{\|f\|_\infty \cdot \|f''\|_\infty}, & \text{if } m(I) \geq 2\sqrt{\|f\|_{L^\infty} / \|f''\|_\infty} \text{ and } I \neq \mathbb{R} \\ \sqrt{2\,\|f\|_\infty \cdot \|f''\|_\infty}, & \text{if } I = \mathbb{R}. \end{cases}$$

Here $m(I)$ denotes the length of I. ■

3 The scalar case of Theorem 1

The scalar case of Theorem 1 is a consequence of the following more general result:

Theorem 2. *Let $u : [a, b] \to \mathbb{R}$ be a real-valued differentiable function whose derivative has bounded variation. Then*

$$\max\{|u(a)|, |u(b)|\} + \frac{b-a}{4} \bigvee_a^b u' > \sup_{t\in[a,b]} |u(t)|,$$

except for the affine functions, where equality holds true.

Proof: Step 1. We first consider the case where

$$u(a) = u(b) = 0. \tag{5}$$

In this case (by replacing u by $-u$, if necessary) we may assume that $|u|$ attains its maximum at a point $c \in (a, b)$ and

$$\sup_{t\in[a,b]} |u(t)| = u(c).$$

Then by the Lagrange mean value theorem there are points $t_1 \in (a, c)$ and $t_2 \in (c, b)$ such that

$$u(c) = u(c) - u(a) = u'(t_1)(c - a)$$

and

$$u(c) = u(c) - u(b) = -u'(t_2)(b - c).$$

Therefore

$$\begin{aligned} \bigvee_a^b u' &\geq \sup_{a<s_1<c<s_2<b} \left|u'(s_1) - u'(s_2)\right| \\ &\geq u'(t_1) - u'(t_2) \\ &= \left(\frac{1}{c-a} + \frac{1}{b-c}\right) u(c) \\ &\geq \frac{4}{b-a} \sup_{t\in[a,b]} |u(t)|, \end{aligned} \tag{6}$$

the last step being a consequence of the arithmetic mean - harmonic mean inequality.

Step 2. We prove next (under the condition (5)) that the equality

$$\frac{b-a}{4} \bigvee_a^b u' = \sup_{t\in[a,b]} |u(t)| \tag{7}$$

occurs only for the function u identically zero. In fact, it suffices to show that $u|_{[a,c]}$ equals the affine function g joining $(a, 0)$ and $(c, u(c))$ and $u|_{[c,b]}$ equals the affine function h joining $(c, u(c))$ and $(b, 0)$. These equalities yield

$$g'(c) = u'_-(c) = u'_+(c) = h'(c)$$

whence $\frac{u(c)}{c-a} = -\frac{u(c)}{b-c}$. Therefore $u(c) = 0$ and this forces $u \equiv 0$.

The equality $u|_{[a,c]} = g$ (as well as the equality $u|_{[c,b]} = h$) can be proved by reductio ad absurdum. For example, if $u(d) < g(d)$ for some point $d \in (a, c)$, then by the Lagrange mean value theorem there is a $t' \in (d, c)$ such that

$$\begin{aligned} u'(t') &= \frac{u(c) - u(d)}{c - d} > \frac{u(c) - g(d)}{c - d} \\ &= \frac{g(c) - g(d)}{c - d} = \frac{u(c)}{c - a} = g'(t_1) = u'(t_1). \end{aligned}$$

This yields to a contradiction since

$$\begin{aligned}\bigvee_a^b u' &= u'(t_1) - u'(t_2) < u'(t') - u'(t_2)\\ &= \left|u'(t') - u'(t_2)\right| \leq \bigvee_a^b u';\end{aligned}$$

the first equality is a consequence of (6) and (7).

The case where $u(d) > g(d)$ for some point $d \in (a, c)$ can be treated similarly.

Step 3. In the general case we have to represent u as

$$u = (u - \varphi) + \varphi,$$

where φ is the affine function joining the points $(a, u(a))$ and $(b, u(b))$. Then $u - \varphi$ vanishes at the endpoints and the result established at Step 1 applies. Therefore

$$\begin{aligned}\sup_{t\in[a,b]} |u(t)| &\leq \sup_{t\in[a,b]} |(u - \varphi)(t)| + \sup_{t\in[a,b]} |\varphi(t)|\\ &\leq \frac{b-a}{4} \bigvee_a^b (u - \varphi)' + \max\{|u(a)|, |u(b)|\}\\ &= \frac{b-a}{4} \bigvee_a^b u' + \max\{|u(a)|, |u(b)|\},\end{aligned}$$

the equality being possible only when $u - \varphi \equiv 0$. ■

4 The case of vector-valued functions

The proof of Theorem 1 can be reduced to the scalar case by *linearization*, taking into account that

$$\left(\sum_{k=1}^N u_k^2\right)^{1/2} = \sup\left\{\sum_{k=1}^N \alpha_k u_k : \sum_{k=1}^N \alpha_k^2 \leq 1\right\}.$$

Indeed, by assuming that Theorem 1 works in the case of scalar functions, for every $x \in [a, b]$ and every family $(\alpha_k)_{k=1}^N$ of real numbers such that

$\sum_{k=1}^{N} \alpha_k^2 \leq 1$ we have

$$\begin{aligned}\left|\sum_{k=1}^{N} \alpha_k u_k(x)\right| &\leq \frac{b-a}{4} \int_a^b \left(\sum_{k=1}^{N} |\alpha_k| \left|u_k''(t)\right|\right) dt \\ &\quad + \max\left\{\sum_{k=1}^{N} |\alpha_k| \left|u_k(a)\right|, \sum_{k=1}^{N} |\alpha_k| \left|u_k(b)\right|\right\} \\ &\leq \frac{b-a}{4} \int_a^b \left\|u''(t)\right\| dt + \max\left\{\|u(a)\|, \|u(b)\|\right\},\end{aligned}$$

that yields the conclusion of Theorem 1 in the Euclidean case.

It is worth to mention that Theorem 1 actually works in the general framework of Banach spaces.

Theorem 3. *Given a Banach space E, every twice differentiable function $u : [a, b] \to E$ whose second derivative is (Bochner) integrable verifies the inequality*

$$\max\left\{\|u(a)\|, \|u(b)\|\right\} + \frac{b-a}{4} \int_a^b \left\|u''(t)\right\| dt \geq \sup_{t\in[a,b]} \|u(t)\|.$$

The constant $(b - a)/4$ is sharp.

Proof: In fact, according to a classical result due Weierstrass, there exists a point $t_0 \in [a, b]$ such that

$$\|u(t_0)\| = \sup_{t\in[a,b]} \|u(t)\|.$$

Then, by Theorem 1, for every norm-1 linear functional x' in the dual space E' we have

$$\begin{aligned}\left|x'(u(t_0))\right| &\leq \max\left\{\left|x'\left(u(a)\right)\right|, \left|x'\left(u(b)\right)\right|\right\} + \frac{b-a}{4} \int_a^b \left|x'\left(u''(t)\right)\right| dt \\ &\leq \max\left\{\|u(a)\|, \|u(b)\|\right\} + \frac{b-a}{4} \int_a^b \left\|u''(t)\right\| dt.\end{aligned}$$

The proof ends by taking the least upper bound in both sides over all $x' \in E'$ with $\|x'\| = 1$, and using the following well-known consequence of the Hahn-Banach extension theorem:

$$\sup_{x'\in E',\ \|x'\|=1} \left|x'(u(t_0))\right| = \|u(t_0)\|.$$

See [15], Corollary 2, p. 108. ■

5 Some open questions

The literature concerning the analogues of Lyapunov inequality for partial differential equations already counts some important contributions. See for example [4], [5] and [6]. It is thus natural to ask whether Theorem 1 admits an extension to the case of functions of several variables.

Suppose that Ω is a bounded open subset Ω of $\mathbb{R}^N$. Does there exist a second order differential operator A (which in the case of functions of one real variable coincide with the second derivative) and a positive constant $c(\Omega)$ (that depends only on the geometry of the domain Ω) such that every real-valued continuous function $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ with Au integrable verify the inequality

$$\max_{x \in \partial\Omega} |u(x)| + c\,(\Omega) \int_{\Omega} \|Au(x)\|\, dx \geq \sup_{x \in \bar{\Omega}} |u(x)|? \tag{8}$$

Adrian Tudorascu (oral communication) provided a simple counterexample showing that the natural candidate for A, the Laplacian of u,

$$\Delta u = \sum_{k=1}^{N} \frac{\partial^2 u}{\partial x_k^2},$$

fails even in the case where Ω is the unit ball in $\mathbb{R}^2$. However, the status of (8) is open for $Au = \operatorname{Hess} u$, where

$$\operatorname{Hess} u = \left(\frac{\partial^2 u}{\partial x_j \partial x_k} \right)_{j,k=1}^{N}$$

represents the Hessian matrix of u. Adrian Tudorascu and I have found some consequences that make plausible a positive answer.

A final open question comes in connection with Corollary 3 above. We do not know if the hypothesis regarding the boundedness of the derivatives of second order is essential or not.

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