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# INTERNAL EXACT OBSERVABILITY OF A PERTURBED EULER-BERNOULLI EQUATION*

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**Abstract**

In this work we prove that the exact internal observability for the Euler-Bernoulli equation is robust with respect to a class of linear perturbations. Our results yield, in particular, that for rectangular domains we have the exact observability in an arbitrarily small time and with an arbitrarily small observation region. The usual method of tackling lower order terms, using Carleman estimates, cannot be applied in this context. More precisely, it is not known if Carleman estimates hold for the evolution Euler-Bernoulli equation with arbitrarily small observation region. Therefore we use a method combining frequency domain techniques, a compactness-uniqueness argument and a Carleman estimate for elliptic problems.

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**keywords:** exact observability, Euler-Bernoulli equation, compactness-uniqueness

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# 1 Introduction and main results

The internal exact observability of the Euler-Bernoulli plate equation, modeling the vibrations of elastic plates, is a subject which has been widely tackled in the literature. One of the features differentiating this problem with respect to the corresponding system for the wave equation is that the observability time is arbitrarily small, as it has been first shown in the Appendix 1 of Lions [9]. Much later it has been shown in Miller [10] and Tucsnak and Weiss [13, Section 6.7] that if the wave equation with a given observation region is exactly observable then the Euler-Bernoulli equation with the same region is exactly observable in arbitrarily small time. This holds, in particular, if the observation region satisfies the geometric optics condition of Bardos, Lebeau and Rauch [1] (see Lebeau [8] for a derivation of this result with no explicit reference to the wave equation). However, this condition is not necessary for the exact observability of the Euler-Bernoulli equation. In particular, for rectangular domains, it has been shown in Jaffard [6] and Komornik [7] that the exact observability holds for arbitrary open observation domains and in any time.

The aim of this work is to study the robustness of the above mentioned observability properties with respect to lower order perturbations of the Euler-Bernoulli equation. These perturbations may contain derivatives of order up to two and coefficients depending on the space variable. In the case in which a strong version of the geometric optics condition holds such perturbation can be studied using Carleman estimates for the evolution Euler-Bernoulli equation (see Wang [14]), which are quite appropriate to absorb the lower-order terms. These Carleman estimates are not available for arbitrarily small observation regions so they cannot be used to generalize the results from [6] and [7] to the perturbed plate. Therefore we develop here a general perturbation argument showing that any internal observability result for the Euler-Bernoulli equation is robust with respect to the considered class of perturbations. This implies, in particular, that for rectangular domains, we have exact observability with arbitrarily small observation regions.

Let $\Omega \subset \mathbb{R}^n$ ($n \in \mathbb{N}^*$) be an open and nonempty set with a $C^2$ boundary or let $\Omega$ be a rectangle. We consider the following initial and boundary value problem :

$$\ddot{w}(x,t) + \Delta^2 w(x,t) - a\Delta w(x,t) + b(x)\cdot\nabla w(x,t) + c(x)w(x,t) = 0, \quad \text{for } (x,t) \in \Omega \times (0,\infty) \tag{1}$$

$$w(x,t) = \Delta w(x,t) = 0, \quad \text{for } (x,t) \in \partial\Omega \times (0,\infty) \tag{2}$$

$$w(x,0) = w_0(x), \quad \dot{w}(x,0) = w_1(x), \quad \text{for } x \in \Omega, \tag{3}$$

where $a > 0$, $b \in (L^\infty(\Omega))^n$, $c \in L^\infty(\Omega)$, $w_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $w_1 \in L^2(\Omega)$. We consider the output given by

$$y(t) = \dot{w}(\cdot, t)|_{\mathcal{O}}, \tag{4}$$

where $\mathcal{O}$ is an open and nonempty subset of $\Omega$ and a dot denotes differentiation with respect to the time $t$:

$$\dot{w} = \frac{\partial w}{\partial t}, \quad \ddot{w} = \frac{\partial^2 w}{\partial t^2}.$$

For $n = 2$ the equations (1)-(3) model the vibration of a perturbed Euler-Bernoulli plate with a hinged boundary.

The main result of this work is the following theorem :

**Theorem 1.** *Let $\mathcal{O} \subset \Omega$ be an open and nonempty subset of $\Omega$ such that (1)-(4), with $a = 0$, $b = 0$, $c = 0$, is exactly observable in any time $\tau > 0$. Then, for $a = 0$ and $b = 0$ the system (1)-(4) is exactly observable in arbitrarily small time for every $c \in L^\infty(\Omega)$.*

*Moreover, (1)-(4) is exactly observable for every $a > 0$ and $b$, $c$ real analytic functions.*

Note that, in the case $a > 0$, the above result gives no information on the observability time. For rectangular domains we have the following, more precise, result.

**Theorem 2.** *Assume that $n = 2$, $\Omega$ is a rectangle and let $\mathcal{O}$ be an open and nonempty subset of $\Omega$. If $b = 0$ then $(1) - (4)$ is exactly observable in any time $\tau > 0$ for every $a > 0$, $c \in L^\infty(\Omega)$. Moreover, if $b \neq 0$ is an analytic function then (1)-(4) is exactly observable in any time $\tau > 0$ for every $a > 0$ and $c$ analytic.*

To prove the above two theorems, we consider an abstract formulation of our exact observability problem. More precisely, in Section 3 we prove an exact observability result for a linear abstract perturbed system.

In Section 5 we prove the Theorem 1 and Theorem 2, applying the abstract results from Section 3. A unique continuation result for the bi-Laplacian is proved in Section 4.

## 2 Background on exact observability

In this section we recall the definition of the exact observability of an infinite dimensional system and we give a perturbation result for the exact observability of a second order infinite dimensional system. In this purpose we need some notation.

Let $X$ and $Y$ be two complex Hilbert spaces which are identified with their duals, and let $\mathbb{T} = (\mathbb{T}_t)_{t\geq 0}$ be a strongly continuous semigroup on $X$, with the generator $A : \mathcal{D}(A) \to X$.

We consider the following infinite dimensional system

$$\dot{z}(t) = Az(t), \quad z(0) = z_0, \tag{5}$$

$$y(t) = Cz(t), \tag{6}$$

where $C \in \mathcal{L}(X, Y)$ is a bounded linear observation operator.

We recall the classical definition of the exact observability.

**Definition 1.** *The pair $(A, C)$ is exactly observable in time $\tau > 0$ if there exists a constant $k_\tau > 0$ such that any solution of (5)-(6) satisfies*

$$\int_0^\tau \|Cz(t)\|_Y^2 \; dt \geq k_\tau^2 \|z_0\|_X^2, \qquad (z_0 \in X). \tag{7}$$

Let $H$ be a Hilbert space equipped with the norm $\|\cdot\|_H$, let $A_0 : \mathcal{D}(A_0) \to H$ be a self-adjoint, positive and boundedly invertible operator, with compact resolvents and let $C_0 \in \mathcal{L}(H, Y)$ be a bounded linear operator. For such an operator $A_0$ we denote $H_\alpha$ the Hilbert space defined by $H_\alpha = \mathcal{D}(A_0^\alpha)$ for any $\alpha \geq 0$ and $H_{-\alpha}$ is the dual space of $H_\alpha$ with respect to the pivot space $H$.

We consider the following second-order abstract system :

$$\ddot{w}(t) + A_0^2 w(t) = 0, \tag{8}$$

$$w(0) = w_0, \quad \dot{w}(0) = w_1, \tag{9}$$

with the output function

$$y(t) = C_0 \dot{w}(t). \tag{10}$$

The system (8)-(10) can be described by a first order system. Indeed, if we denotes $X = H_1 \times H$, $\mathcal{D}(A) = H_2 \times H_1$ and

$$A : \mathcal{D}(A) \to X, \quad A \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ -A_0^2 f \end{bmatrix} \qquad \left( \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A) \right), \tag{11}$$

we can write (8)-(9) as

$$\dot{z}(t) = Az(t), \qquad z(0) = z_0,$$

where $z(t) = \begin{bmatrix} w(t) \\ \dot{w}(t) \end{bmatrix}$, $z_0 = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}$. The operator $A$ defined above is a skew-adjoint operator and, therefore, generates a strongly continuous semigroup $(\mathbb{T}_t)_t$ on $X$.

Let $C \in \mathcal{L}(X, Y)$ be the operator defined by $C = [0 \;\; C_0]$. We say that (8)-(10) is exactly observable if the pair $(A, C)$ is exactly observable in the sense of Definition 1.

In our recent work [3], we have shown that if (8)-(10) is exactly observable then the following initial value problem

$$\ddot{w}(t) + A_0^2 w(t) + aA_0 w(t) = 0 \tag{12}$$

$$w(0) = w_0, \quad \dot{w}(0) = w_1, \tag{13}$$

is exactly observable with respect to the same output, for every $a > 0$. More precisely, we proved the following result :

**Theorem 3.** *Assume that the system (8)-(10) is exactly observable. Then the system (12)-(13) is exactly observable in rapport with the observation (10), i.e., there exist a time $\tau > 0$ and a constant $k_\tau > 0$ such that every solution $w$ of (12)-(13) satisfies*

$$\int_0^\tau \|C_0 \dot{w}(t)\|_Y^2 \, dt \geq k_\tau^2 \left( \|w_0\|_{H_1}^2 + \|w_1\|_H^2 \right), \quad \left( \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \in H_2 \times H_1 \right). \tag{14}$$

*Moreover, if $\Omega$ is a rectangle the observability time $\tau > 0$ can be arbitrarily small.*

## 3 An exact observability result for second-order perturbed systems

In this section we study the exact observability of (12)-(13) with the output (10), perturbed with a term of the form $P_0 w$, where $P_0 \in \mathcal{L}(H_{1-\varepsilon}, H)$ and $\varepsilon \in (0, 1]$. More precisely, we consider the following second-order system :

$$\ddot{v}(t) + A_a^2 v(t) + P_0 v(t) = 0, \qquad t > 0 \tag{15}$$

$$v(0) = v_0, \qquad \dot{v}(0) = v_1, \tag{16}$$

with the output function

$$y(t) = C_0\dot{v}(t), \tag{17}$$

seen as a perturbation of (12)-(13), where we denote $A_a : H_1 \to H$ the operator defined by $A_a = (A_0^2 + aA_0)^{\frac{1}{2}}$. It is easy to see that $A_a$ is a self-adjoint, strictly positive, boundedly invertible operator, with compact resolvents. Remark that (12) can be written, using this notation, as

$$\ddot{w}(t) + A_a^2 w(t) = 0, \qquad t > 0.$$

Let $\widetilde{A_a} : H_2 \times H_1 \to H_1 \times H$ be the operator defined by

$$\widetilde{A_a} = \begin{bmatrix} 0 & I \\ -A_a^2 & 0 \end{bmatrix}$$

and denote $P = \begin{bmatrix} 0 & 0 \\ -P_0 & 0 \end{bmatrix}$, $C = [0\ C_0] \in \mathcal{L}(H_1 \times H, Y)$. We can consider $P \in \mathcal{L}(H_1 \times H)$. Then $A_P : H_2 \times H_1 \to H_1 \times H$, with $A_P = \widetilde{A_a} + P$, is well defined. Hence, according to Theorem 1.1 from Pazy [11, p.76], $A_P$ is the generator of a strongly continuous semigroup in $H_1 \times H$, denoted $\left(\mathbb{T}_t^P\right)_{t\geq 0}$. We denote

$$\mathcal{N}(T) = \left\{ W_0 = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \in H_1 \times H \mid C\mathbb{T}_t^P W_0 = 0, \quad \text{for any } t \in [0,T] \right\}. \tag{18}$$

The aim of this section is to prove that the exact observability of (12)-(13) with the observation (10) implies the exact observability of (15)-(17). The main result of this section is the following theorem :

**Theorem 4.** *With the above notations, we assume that (12)-(13), with the observation (10), is exactly observable in time $\tau > 0$. We assume, moreover, that $\mathcal{N}(T) = \{0\}$. If $C_0\phi \neq 0$ for every eigenvector $\phi$ of $A_a^2 + P_0$ then (15)-(17) is exactly observable in any time $T > \tau$, i.e., there exists a constant $k_T > 0$ such that any solution $v$ of (15)-(16) satisfies*

$$\int_0^T \|C_0\dot{v}(t)\|_Y^2 \, dt \geq k_T^2 \left(\|v_0\|_{H_1}^2 + \|v_1\|_H^2\right), \qquad \left(\begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \in H_1 \times H\right).$$

**Lemma 1.** *Let $G \in \mathcal{L}(H_{1-\varepsilon} \times H_{-\varepsilon}, H_1 \times H)$ be the compact operator defined by*

$$G = \begin{bmatrix} A_a^{-\varepsilon} & 0 \\ 0 & A_a^{-\varepsilon} \end{bmatrix}. \tag{19}$$

*Then, with the assumptions of Theorem 4, there exists a positive constant $C_T$ such that*

$$\int_0^T \|C_0\dot{\psi}(t)\|_Y^2 \; dt \le C_T^2 \left\| G \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_{H_1\times H}^2, \qquad \left( \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \in H_1 \times H \right), \tag{20}$$

*where $\psi$ is the solution of*

$$\ddot{\psi}(t) + A_a^2\psi(t) + P_0\psi(t) = -P_0 w(t), \qquad t \in (0,\infty) \tag{21}$$

$$\psi(0) = \dot{\psi}(0) = 0 \tag{22}$$

*and $w$ is the solution of (12)-(13).*

*Proof.* Since $C_0 \in \mathcal{L}(H, Y)$ we have the following estimate:

$$\|C_0\dot{\psi}\|_{C([0,T];Y)} \le \|C_0\|_{\mathcal{L}(H,Y)}\|\dot{\psi}\|_{C([0,T];H)} \le \|C_0\|_{\mathcal{L}(H,Y)}\|\Psi\|_{C([0,T];H_1\times H)},$$

where $\Psi(t) = \begin{bmatrix} \psi(t) \\ \dot{\psi}(t) \end{bmatrix}$ is the solution of the following initial value problem

$$\dot{\Psi}(t) = A_P\Psi(t) + P \begin{bmatrix} w(t) \\ \dot{w}(t) \end{bmatrix}, \qquad \Psi(0) = 0. \tag{23}$$

Then, there exists a constant $\tilde{C}_T > 0$ such that

$$\|\Psi\|_{C([0,T];H_1\times H)} \le \tilde{C}_T \left\| P \begin{bmatrix} w \\ \dot{w} \end{bmatrix} \right\|_{L^1([0,T];H_1\times H)} = \tilde{C}_T \|P_0 w\|_{L^1([0,T];H)}.$$

Recall that $P_0 \in \mathcal{L}(H_{1-\varepsilon}, H)$. Combining this fact with the above two inequalities, we obtain :

$$\|C_0\dot{\psi}\|_{C([0,T];Y)} \le \tilde{C}_T\|C_0\|_{\mathcal{L}(H,Y)}\|P_0\|_{\mathcal{L}(H_{1-\varepsilon},H)}\|w\|_{L^1([0,T];H_{1-\varepsilon})}. \tag{24}$$

We recall that $w$ is the solution of (12)-(13) and, therefore, we can bound its $L^1$ norm by the norm of the initial data. We have that

$$\|w\|_{L^1([0,T];H_{1-\varepsilon})} \le C_T \left\| \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_{H_{1-\varepsilon}\times H_{-\varepsilon}}. \tag{25}$$

Using the fact that the operator $G$ is an isomorphism from $H_{1-\varepsilon} \times H_{-\varepsilon}$ onto $H_1 \times H$, we can conclude that there exists a constant $C_T > 0$ such that

$$\|C_0\dot{\psi}\|_{C([0,T];Y)} \leq C_T \left\| G \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_{H_1 \times H}$$

and the proof of the lemma is complete. $\square$

*Proof of Theorem 4.* The solution $v$ of (15)-(16) can be written as $v = w + \psi$, where $w$ is the solution of (12)-(13) and $\psi$ is the solution of (21)-(22). Then we have

$$\int_0^T \|C_0\dot{v}(t)\|_Y^2 \,\mathrm{d}t + \int_0^T \|C_0\dot{\psi}(t)\|_Y^2 \,\mathrm{d}t \geq \frac{1}{2}\int_0^T \|C_0\dot{w}(t)\|_Y^2 \,\mathrm{d}t. \qquad (26)$$

Using the exact observability of (12)-(13), we obtain from (26) that there exists a constant $k_T > 0$ such that

$$\int_0^T \|C_0\dot{v}(t)\|_Y^2 \,\mathrm{d}t + \int_0^T \|C_0\dot{\psi}(t)\|_Y^2 \,\mathrm{d}t \geq \frac{k_T^2}{2}(\|w_0\|_{H_1}^2 + \|w_1\|_H^2). \qquad (27)$$

From (27) and Lemma 1 we obtain

$$\begin{aligned} \textstyle\int_0^T \|C_0\dot{v}(t)\|_Y^2 \,\mathrm{d}t + C_T^2 \left\| G \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_{H_1 \times H}^2 &\geq \tfrac{1}{2}k_T^2(\|w_0\|_{H_1}^2 + \|w_1\|_H^2), \\ &\text{for any } \left( \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \in H_1 \times H \right), \end{aligned} \qquad (28)$$

where $G$ is the operator defined by (19).

The idea of the proof is to show that in (28), from Lemma 1, we can remove the term $C_T\|GW_0\|_{H_1\times H}^2$ and thus we obtain the requested observability inequality

$$\int_0^T \|C\mathbb{T}_t^P W_0\|_Y^2 \,\mathrm{d}t \geq \frac{1}{2}k_T^2\|W_0\|_{H_1\times H}^2, \qquad (W_0 \in H_1 \times H). \qquad (29)$$

We assume that there exists a sequence $(W_0^n)_n \in H_1 \times H$ such that

$$\|W_0^n\|_{H_1\times H} = 1$$

and

$$\int_0^T \|C\mathbb{T}_t^P W_0^n\|_Y^2 \,\mathrm{d}t \to 0, \quad \text{when } n \to \infty,$$

which contradicts (29). Since the operator $G$ provided by Lemma 1, is compact, we can extract a subsequence of $(W_0^n)_n$, denoted with the same notation, such that

$$GW_0^n \to W_0 \in H_1 \times H, \text{ when } n \to \infty,$$

Passing to the limit in (28), we obtain

$$C_T \|W_0\|^2_{H_1 \times H} \geq \frac{1}{2} k_T^2$$

that is

$$\|W_0\|^2_{H_1 \times H} \geq \frac{k_T^2}{2C_T} > 0,$$

and so, $W_0 \neq 0$. Recall that

$$\int_0^T \|C\mathbb{T}_t^P W_0\|_Y^2 \; \mathrm{d}t = 0,$$

which implies

$$C\mathbb{T}_t^P W_0 = 0, \qquad (t \in (0,T)).$$

Also, we proved that $W_0 \in \mathcal{N}(T)$ and $W_0 \neq 0$. This is in contradiction with the the assumption of the theorem and so the observability inequality (29) is true. □

In the case $\varepsilon = 1$ (i.e. $P_0 \in \mathcal{L}(H)$), using a compactness and uniqueness argument, we can prove that $\mathcal{N}(T) = 0$. More precisely, we obtained the following result.

**Theorem 5.** *With the notations from the beginning of this section, we assume that (12)-(13), with the observation (10), is exactly observable in time $\tau > 0$. If $C_0\phi \neq 0$ for every eigenvector $\phi$ of $A_a^2 + P_0$ then (15)-(17) is exactly observable in any time $T > \tau$, i.e., there exists a constant $k_T > 0$ such that any solution $v$ of (15)-(16) satisfies*

$$\int_0^T \|C_0 \dot{v}(t)\|_Y^2 \; dt \geq k_T^2 \left(\|v_0\|^2_{H_1} + \|v_1\|^2_H\right), \qquad \left(\begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \in H_1 \times H\right).$$

*Proof.* To prove this theorem is enough to show that $\mathcal{N}(T) = \{0\}$ and to apply Theorem 3. In this purpose, let $W_0 = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}$ be an element of $\mathcal{N}(T)$.

Let $G = (\beta I - \widetilde{A_a})^{-1} \in \mathcal{L}(H_1 \times H, H_2 \times H_1)$ be a compact operator for a fixed $\beta \in \rho(\widetilde{A_a}) \cap \rho(A_P)$. The proof of Lemma 1 remains the same for this $G$, hence

$$C_T \|GW_0\|^2_{H_2 \times H_1} \geq \frac{1}{2} k_T^2 \|W_0\|^2_{H_1 \times H}. \tag{30}$$

In a first step, we prove that $\mathcal{N}(T)$ is a finite dimensional space, i.e., the unit ball of the space $(\mathcal{N}(T), \|\cdot\|_{H_1 \times H})$ is compact. In this purpose, let $\left(\begin{bmatrix} w_{0n} \\ w_{1n} \end{bmatrix}\right)_n$ be a bounded sequence in $(\mathcal{N}(T), \|\cdot\|_{H_1 \times H})$. Applying the inequality (30), we obtain that $\left(\begin{bmatrix} w_{0n} \\ w_{1n} \end{bmatrix}\right)_n$ is bounded in $H_2 \times H_1$. We recall that $H_2 \times H_1 \subset H_1 \times H$ with compact embedding. Therefore, there exists $\begin{bmatrix} f \\ g \end{bmatrix} \in H_2 \times H_1$ such that

$$\begin{bmatrix} w_{0n} \\ w_{1n} \end{bmatrix} \xrightarrow{w} \begin{bmatrix} f \\ g \end{bmatrix} \text{ in } H_2 \times H_1 \text{ and} \tag{31}$$

$$\begin{bmatrix} w_{0n} \\ w_{1n} \end{bmatrix} \longrightarrow \begin{bmatrix} f \\ g \end{bmatrix} \text{ in } H_1 \times H. \tag{32}$$

From $C\mathbb{T}_t \in \mathcal{L}(H_1 \times H, Y)$ and (31) we obtain that

$$C\mathbb{T}_t \begin{bmatrix} w_{0n} \\ w_{1n} \end{bmatrix} \xrightarrow{w} C\mathbb{T}_t \begin{bmatrix} f \\ g \end{bmatrix} \text{ in } H_1 \times H.$$

Therefore, $\begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{N}(T)$ and, so, $\mathcal{N}(T)$ is a finite dimensional space.

In a second step, we show that $\mathcal{N}(T)$ is an invariant subspace of $A_P$. Indeed, it is clear that for $\delta \in (0, T)$, if $W_0 \in \mathcal{N}(T)$ then $\mathbb{T}_t^P W_0 \in \mathcal{N}(T-\delta)$ for each $0 < t < \delta$. Since $A_P$ commutes with $(\beta I - A_P)^{-1} \in \mathcal{L}(H_1 \times H, H_2 \times H_1)$, we have

$$(\beta I - A_P)^{-1} \frac{\mathbb{T}_t^P - I}{t} W_0 \to A_P (\beta I - A_P)^{-1} W_0,$$

when $t \to 0$. Therefore, $\left(\frac{\mathbb{T}_t^P - I}{t} W_0\right)_t$ is a Cauchy family for the norm $W \mapsto \|(\beta I - A_P)^{-1} W\|$ in $\mathcal{N}(T-\delta)$. From the Remark 2.11.3 in [13] it follows that the norms $\|((\beta I - A_P)^{-1} W)\|$ and $\|(\beta I - \widetilde{A_a})^{-1} W\|$ are equivalent. Hence, we obtain that $\mathcal{N}(T) \subset \mathcal{D}(A_P)$. Moreover,

$$C\mathbb{T}_t^P W_0 = 0, \qquad (t \in [0, T]).$$

After a differentiation with respect to $t$, the relation above becomes

$$C\mathbb{T}_t^P A_P W_0 = 0, \qquad (t \in [0, T]),$$

and, therefore, $\mathcal{N}(T)$ is $A_P$-stable.

Finally, we prove that $\mathcal{N}(T) = \{0\}$. Assume that $\mathcal{N}(T) \neq \{0\}$. Since $\mathcal{N}(T)$ is finite dimensional and $A_P$-stable, then it contains an eigenvector of $A_P$. Let $W_0 = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \in \mathcal{N}(T)$ be a eigenvector of $A_P$. Then exists $\lambda \neq 0$ such that

$$A_P W_0 = \lambda W_0,$$

and, so, $w_1$ is an eigenvector of $A_a^2 + P_0$. From the definition of $\mathcal{N}(T)$ we obtain that $C_0 w_1 = 0$ which contradicts the assumption of Theorem 4 that $C_0 \phi \neq 0$ for every eigenvector $\phi$ of $A_a^2 + P_0$. □

**Remark 1.** *In [2] the authors proved a result similar with Theorem 4, using a spectral method completely different to the one presented in this section.*

# 4 A unique continuation result for bi-Laplacian

The aim of this section is to prove the following theorem :

**Theorem 6.** *Let $a \in (0, \infty)$, $b \in (L^\infty(\Omega))^n$, $c \in L^\infty(\Omega)$, $\mu \in \mathbb{R}$ and let $u \in H^4(\Omega)$ be a function such that*

$$\Delta^2 u - a\Delta u + b \cdot \nabla u + cu = \mu^2 u \qquad \text{in } \Omega \tag{33}$$

$$u = \Delta u = 0 \qquad \text{on } \partial\Omega \tag{34}$$

*and*

$$u = 0 \quad \text{in } \mathcal{O}. \tag{35}$$

*Then $u = 0$ in $\Omega$.*

The key of the proof of Theorem 6 is a global Carleman estimate for bi-Laplacian (Proposition 1), which we obtained applying two times a particular case of the global Carleman estimate proved by Imanuvilov and Puel in [5].

Let $\Omega$ be an nonempty open set with a $C^2$ boundary or a rectangle. Let $y \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of the problem

$$\Delta y = f, \quad \text{in } \Omega \tag{36}$$

$$y = 0, \quad \text{on } \partial\Omega, \tag{37}$$

where $f \in L^2(\Omega)$. We use the following classic lemma stated in [5], and proved in Fursikov-Imanuvilov [4].

**Lemma 2.** *Let $\mathcal{O}$ be an open and nonempty subset of $\Omega$. Then there exists a function $\psi \in C^2(\overline{\Omega})$ such that*

$$\psi(x) = 0, \qquad x \in \partial\Omega \tag{38}$$

$$\psi(x) > 0, \qquad x \in \Omega \tag{39}$$

$$|\nabla\psi(x)| > 0, \quad x \in \overline{\Omega \setminus \mathcal{O}}. \tag{40}$$

We consider the following weight function

$$\varphi(x) = e^{\lambda\psi(x)}, \tag{41}$$

where $\lambda \geq 1$ will be chosen later.

Using the definition of the function $\varphi$, and the properties of function $\psi$ given by Lemma 2, we have

$$\frac{1}{\varphi(x)} = \frac{1}{e^{\lambda\psi(x)}} = e^{-\lambda\psi(x)} \leq 1 \leq e^{2\lambda\psi(x)} = \varphi^2(x), \tag{42}$$

for all $\lambda \geq 1$.

Theorem 7 is a particular case of the Carleman estimate proved in [5] for general elliptic operators.

**Theorem 7.** *Assume that (38)-(41) are verified and let $y \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of (36)-(37). Then there exists a constant $C > 0$ independent of $s$ and $\lambda$, and parameters $\widehat{\lambda} > 1$ and $\widehat{s} > 1$ such that for all $\lambda \geq \widehat{\lambda}$ and for all $s > \widehat{s}$ we have*

$$\int_\Omega |\nabla y|^2 e^{2s\varphi} \mathrm{d}x + s^2\lambda^2 \int_\Omega |y|^2 \varphi^2 e^{2s\varphi} \mathrm{d}x \leq$$

$$C\left(\frac{1}{s\lambda^2}\int_\Omega \frac{|f|^2}{\varphi} e^{2s\varphi}\mathrm{d}x + \int_{\mathcal{O}} \left(|\nabla y|^2 + s^2\lambda^2\varphi^2|y|^2\right) e^{2s\varphi}\mathrm{d}x\right). \tag{43}$$

Let $u \in H^4(\Omega)$ be the solution of the problem

$$\Delta^2 u - a\Delta u = g, \qquad \text{in } \Omega \tag{44}$$

$$u = \Delta u = 0, \quad \text{on } \partial\Omega, \tag{45}$$

where $g \in L^2(\Omega)$.

**Proposition 1.** *Let $\psi \in C^2(\overline{\Omega})$ be a function such that (38)-(40) are verified and let $\varphi$ given by (41). Let $u \in H^4(\Omega)$ be a solution of (44)-(45). Then there exist $\widehat{s} > 1$, $\widehat{\lambda} > 1$ and a constant $C > 0$ independent of $s \geq \widehat{s}$ and $\lambda \geq \widehat{\lambda}$, such that*

$$s\lambda^2 \int_\Omega \left(|\nabla(\Delta u)|^2 + s^3\lambda^4|\nabla u|^2 + s^5\lambda^6|u|^2\varphi^2\right) e^{2s\varphi} \leq C \left( \int_\Omega \frac{|g|^2}{\varphi} e^{2s\varphi} \right.$$

$$\left. + s\lambda^2 \int_{\mathcal{O}} (|\nabla(\Delta u)|^2 + s^2\lambda^2\varphi^2|\Delta u|^2 + s^3\lambda^4|\nabla u|^2 + s^5\lambda^6\varphi^2|u|^2)e^{2s\varphi} \right). \quad (46)$$

*Proof.* We denote $y = \Delta u$ and $g_1 = g + a\Delta u$. Then (44) and the last part of (45) can be written as

$$\Delta y = g_1, \quad \text{in } \Omega \quad (47)$$

$$y = 0, \quad \text{on } \partial\Omega \quad (48)$$

We can apply Theorem 7. Therefore, there exist $s_1 > 1$, $\lambda_1 > 1$ and a constant $C_1 > 0$ independent of $s$ and $\lambda$ such that for all $s \geq s_1$, $\lambda \geq \lambda_1$ the following estimate is satisfied

$$s\lambda^2 \int_\Omega |\nabla y|^2 e^{2s\varphi} \mathrm{d}x + s^3\lambda^4 \int_\Omega |y|^2\varphi^2 e^{2s\varphi} \mathrm{d}x \leq$$

$$C_1 \left( \int_\Omega |g_1|^2\varphi^{-1}e^{2s\varphi}\mathrm{d}x + \int_{\mathcal{O}} \left(s\lambda^2|\nabla y|^2 + s^3\lambda^4\varphi^2|y|^2\right) e^{2s\varphi} \right) \mathrm{d}x \leq$$

$$C_1 \left( 2 \int_\Omega (|g|^2\varphi^{-1} + a^2|\Delta u|^2\varphi^2)e^{2s\varphi}\mathrm{d}x + \right.$$

$$\left. \int_{\mathcal{O}} \left(s\lambda^2|\nabla y|^2 + s^3\lambda^4\varphi^2|y|^2\right) e^{2s\varphi}\mathrm{d}x \right)$$

where bellow we used (42). Replacing $y$ with $\Delta u$ in the previous estimate, we obtain

$$s\lambda^2 \int_\Omega |\nabla(\Delta u)|^2 e^{2s\varphi}\mathrm{d}x + (s^3\lambda^4 - 2a^2C_1) \int_\Omega |\Delta u|^2\varphi^2 e^{2s\varphi}\mathrm{d}x \leq$$

$$C_1 \left( 2 \int_\Omega |g|^2\varphi^{-1}e^{2s\varphi}\mathrm{d}x + \int_{\mathcal{O}} \left(s\lambda^2|\nabla(\Delta u)|^2 + s^3\lambda^4\varphi^2|\Delta u|^2\right) e^{2s\varphi}\mathrm{d}x \right). \quad (49)$$

Now consider the problem

$$\Delta u = y, \quad \text{in } \Omega \quad (50)$$

$$u = 0, \quad \text{on } \partial\Omega, \quad (51)$$

and apply Theorem 7. Then there exists a constant $C_2 > 0$, $s_2 > 1$, $\lambda_2 > 1$ such that for $s \geq s_2$ and $\lambda \geq \lambda_2$ we have

$$\begin{aligned} s\lambda^2 \int_\Omega |\nabla u|^2 e^{2s\varphi} \mathrm{d}x + s^3\lambda^4 \int_\Omega |u|^2\varphi^2 e^{2s\varphi} \mathrm{d}x \leq \\ C_2 \left( \int_\Omega |\Delta u|^2 \varphi^{-1} e^{2s\varphi} \mathrm{d}x + \int_{\mathcal{O}} \left( s\lambda^2 |\nabla u|^2 + s^3\lambda^4\varphi^2|u|^2 \right) e^{2s\varphi} \mathrm{d}x \right) \leq \\ C_2 \left( \int_\Omega |\Delta u|^2 \varphi^{2} e^{2s\varphi} \mathrm{d}x + \int_{\mathcal{O}} \left( s\lambda^2 |\nabla u|^2 + s^3\lambda^4\varphi^2|u|^2 \right) e^{2s\varphi} \mathrm{d}x \right), \end{aligned} \tag{52}$$

where for the last part of the inequality bellow we used (42).

We denote $\widehat{\lambda} = \max\{\lambda_1, \lambda_2\}$ and $\widehat{s} = \max\{s_1, s_2\}$. For every $s \geq \widehat{s}$, $\lambda \geq \widehat{\lambda}$, combining (49) and (52) we have

$$\begin{aligned} s\lambda^2 \int_\Omega |\nabla(\Delta u)|^2 e^{2s\varphi} dx \\ + \frac{s^3\lambda^4 - 2a^2C_1}{C_2} s\lambda^2 \int_\Omega (|\nabla u|^2 + s^2\lambda^2|u|^2\varphi^2) e^{2s\varphi} dx \\ -(s^3\lambda^4 - a^2C_1) \left( \int_{\mathcal{O}} \left( s\lambda^2|\nabla u|^2 + s^3\lambda^4\varphi^2|u|^2 \right) e^{2s\varphi} dx \right) \leq \\ C_1 \left( 2\int_\Omega |g|^2\varphi^{-1} e^{2s\varphi} dx + \int_{\mathcal{O}} \left( s\lambda^2 |\nabla(\Delta u)|^2 + s^3\lambda^4\varphi^2|\Delta u|^2 \right) e^{2s\varphi} dx \right). \end{aligned}$$

Fixing $\lambda$ in above inequality, is easy to see that there exists a constant $C > 0$ such that (46) is verified. Therefore, the proof of the proposition is complete. □

*Proof of Theorem 6.* The proof is a direct consequence of Theorem 1. Let us denote $g = (\mu^2 - a)u - b \cdot \nabla u \in L^2(\Omega)$. Applying Theorem 1 to (33)-(34) and using (35), we obtain

$$s\lambda^2 \int_\Omega (|\nabla(\Delta u)|^2 + s^3\lambda^4|\nabla u|^2 + s^5\lambda^6|u|^2\varphi^2) e^{2s\varphi} \mathrm{d}x \leq C \int_\Omega \frac{|g|^2}{\varphi} e^{2s\varphi} \mathrm{d}x.$$

We can easily verify that

$$|g(x)|^2 \leq 2\left(\mu^4 + \|a\|^2_{L^\infty(\Omega)}\right)|u(x)|^2 + 2\|b\|^2_{(L^\infty(\Omega))^n}|\nabla u(x)|^2, \qquad (x \in \Omega).$$

Combining the above inequality with (42), we obtain

$$\int_{\Omega} \left(s\lambda^2|\nabla(\Delta u)|^2 + s^4\lambda^6|\nabla u|^2 + s^6\lambda^8|u|^2\varphi^2\right) e^{2s\varphi}\mathrm{d}x \leq$$
$$2C\left(\left(\mu^4 + \|a\|^2_{L^\infty(\Omega)}\right)\int_{\Omega}|u|^2\varphi^2 e^{2s\varphi}\mathrm{d}x + \|b\|^2_{(L^\infty(\Omega))^n}\int_{\Omega}|\nabla u|^2 e^{2s\varphi}\mathrm{d}x\right).$$

Taking $s \to \infty$ in previous inequality, we easily obtain that $u = 0$ in $\Omega$. □

# 5 Proof of main results

The idea of the proofs of Theorem 1 and Theorem 2 is to apply the abstract results proven in Section 2. In order to apply Theorem 4 or Theorem 5, we use the unique continuation result for the bi-Laplacian obtained in the previous section.

In the remaining part of this section, $A_a : H_1 \to H$ denotes the following operator

$$H_1 = H^2(\Omega) \cap H_0^1(\Omega), \quad H = L^2(\Omega),$$

$$A_a\varphi = (\Delta^2 - a\Delta)^{\frac{1}{2}}\varphi, \qquad (a > 0, \quad \varphi \in H_1)$$

and $P_0 \in \mathcal{L}(H_{\frac{1}{2}}, H)$

$$P_0\varphi = b \cdot \nabla\varphi + c\varphi, \qquad (\varphi \in H_{\frac{1}{2}}),$$

where $a, b, c$ are as in Theorem 1 and $H_\alpha$ are as in Section 2. Therefore (1)-(3) can be written as

$$\ddot{w}(t) + A_a^2 w(t) + P_0 w(t) = 0, \qquad t > 0 \tag{53}$$

$$w(0) = w_0, \qquad \dot{w}(0) = w_1. \tag{54}$$

Let $Y = L^2(\mathcal{O})$ and let $C_0 \in \mathcal{L}(H, Y)$ be the bounded linear operator given by

$$C_0 w(t) = w(\cdot, t)|_{\mathcal{O}}.$$

We consider the following output function

$$y(t) = C_0\dot{w}(t) \tag{55}$$

To prove Theorem 1 or Theorem 2 is nothing else than to prove the exact observability of (53)-(55), the only difference between the two theorems being the exact observability time.

*Proof of Theorem 1.* If we translate the result of Proposition 6 in the operator notation introduced above, we obtain that

$$\begin{cases} A_0^2\psi + P_0\psi = \mu^2\psi \\ C_0\psi = 0 \end{cases} \quad \text{implies } \psi = 0.$$

In other words, for every eigenvector $\psi$ of $A_0^2+P_0$ we have $C_0\psi \neq 0$. Therefore we can apply Theorem 4.

If $b = 0$ we can consider $P_0 \in \mathcal{L}(H)$ and then, applying Theorem 5 in the case $a = 0$, we obtain the exact observability of (53)-(55) in a time arbitrarily small for every $c \in L^\infty(\Omega)$. In the case $a > 0$, applying Theorem 3 and Theorem 4 we obtain the exact observability of (53)-(55) with no information about the observability time.

If $b$ and $c$ are analytic functions we obtain the same results as in the case $b = 0$, using John's global Holmgren theorem (see for instance Rauch [12, Theorem 1, p.42]) to deduce that $\mathcal{N}(T) = \{0\}$ for Theorem 4. □

*Proof of Theorem 2.* The only difference between this proof and the previous one is that we apply here Proposition 5.1 in [3] which give us the observability of the pair $(\widetilde{A_a}, C)$ in any time $\tau > 0$ and, applying Theorem 4, we obtain that (53)-(55) is exactly observable in any time $\tau > 0$. □

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