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CONCEPTS OF DICHOTOMY FOR SKEW-EVOLUTION SEMIFLOWS IN BANACH SPACES*

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Abstract

In this paper we investigate some dichotomy concepts for skew-evolution semiflows in Banach spaces. Our main objective is to establish relations between these concepts. We motivate our approach by illustrative examples.

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1 Introduction

In the qualitative theory of evolution equations, the exponential dichotomy is one of the most important asymptotic properties, and in the last years it was treated from various perspectives (see [1] –[16]).

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The notion of exponential dichotomy for linear differential equations was introduced by O. Perron in 1930. The classic paper [12] of Perron served as a starting point for many works on the stability theory.

The property of exponential dichotomy for linear differential equations has gained prominence since the appereance of two fundamental monographs due to J.L. Daleckiĭ and M.G. Kreĭn (see [6]) and J.L. Massera and J.J. Schäffer (see [8]).

The notion of linear skew-product semiflow arises naturally when one considers the linearization along an invariant manifold of a dynamical system generated by a nonlinear differential equation (see [14], Chapter 4).

Diverse and important concepts of dichotomy for linear skew-product semiflows were studied by C. Chicone and Y. Latushkin in [4], S.N. Chow and H. Leiva in [5], R.J. Sacker and G.R. Sell in [13].

The particular cases of exponential stability and exponential instability for linear skew-product semiflows have been considered in [9] and [10] .

In this paper we consider the general case of skew-evolution semiflows (introduced in our paper [11]) as a natural generalization of skew-product semiflows. The major difference consists in the fact that a skew-evolution semiflow depends on three variables t, t_0 and x, while the classic concept of skew-product semiflow depends only on t and x, thus justifying a further study of asymptotic behaviors for skew-evolution semiflows in a more general case, the nonuniform setting (relative to the third variable t_0).

The aim of this paper is to define and exemplify various concepts of dichotomies as exponential dichotomy, Barreira-Valls exponential dichotomy, uniform exponential dichotomy, polynomial dichotomy, Barreira-Valls polynomial dichotomy and uniform polynomial dichotomy, and to emphasize connections between them. Thus we consider generalizations of some asymptotic properties for differential equations studied by L. Barreira and C. Valls in [1], [2] and [3].

Some results concerning the properties of stability and instability for skew-evolution semiflows were published by us in [11], in [15] and in [16].

The obtained results clarify the difference between uniform dichotomies and nonuniform dichotomies.

2 Skew-evolution semiflows

Let us consider a metric space (X, d), a Banach space V and $\mathcal{B}(V)$ the space of all bounded linear operators from V into itself. I is the identity operator on V. We denote $Y = X \times V$ and we consider the following sets $\Delta = \{(t, t_0) \in \mathbf{R}_+^2 : \ t \geq t_0\}$ and $T = \{(t, s, t_0) \in \mathbf{R}_+^3 : \ t \geq s \geq t_0 \geq 0\}$.

Definition 1. A mapping $\varphi : \Delta \times X \to X$ is called *evolution semiflow* on X if the following relations hold:

(s_1) $\varphi(t, t, x) = x, \ \forall (t, x) \in \mathbf{R}_+ \times X$;

(s_2) $\varphi(t, s, \varphi(s, t_0, x)) = \varphi(t, t_0, x), \forall (t, s), (s, t_0) \in \Delta, x \in X$.

Definition 2. A mapping $\Phi : \Delta \times X \to \mathcal{B}(V)$ is called *evolution cocycle* over an evolution semiflow φ if:

(c_1) $\Phi(t, t, x) = I, \ \forall (t, x) \in \mathbf{R}_+ \times X$;

(c_2) $\Phi(t, s, \varphi(s, t_0, x))\Phi(s, t_0, x) = \Phi(t, t_0, x), \forall (t, s), (s, t_0) \in \Delta, x \in X$.

Definition 3. The mapping $C : \Delta \times Y \to Y$ defined by the relation

$$C(t, s, x, v) = (\Phi(t, s, x)v, \varphi(t, s, x)),$$

where Φ is an evolution cocycle over an evolution semiflow φ, is called *skew-evolution semiflow* on Y.

Remark 1. The concept of skew-evolution semiflow generalizes the notion of skew-product semiflow, considered and studied by M. Megan, A.L. Sasu and B. Sasu in [9] and [10], where the mappings φ and Φ do not depend on the variables $t \geq 0$ and $x \in X$.

Example 1. Let $E : \Delta \to \mathcal{B}(V)$ be an evolution operator on V. If there exists $P : X \to \mathcal{B}(V)$ with the properties

$$P(x)^2 = P(x) \text{ and } P(x)E(t, s) = E(t, s)P(x),$$

for all $(t, s, x) \in \Delta \times X$, then $C = (\Phi, \varphi)$, where

$$\Phi(t, s, x) = P(x)E(t, s), \ \varphi(t, s, x) = x$$

is a linear skew-evolution semiflow.

Example 2. Let us consider a skew-evolution semiflow $C = (\Phi, \varphi)$ and a parameter $\lambda \in \mathbf{R}$. We define the mapping

$$\Phi_\lambda : \Delta \times X \to \mathcal{B}(V),\ \Phi_\lambda(t, t_0, x) = e^{\lambda(t-t_0)}\Phi(t, t_0, x).$$

One can remark that $C_\lambda = (\Phi_\lambda, \varphi)$ also satisfies the conditions of Definition 3, being called λ-*shifted skew-evolution semiflow* on Y.

Let us consider on the Banach space V the Cauchy problem

$$\begin{cases} \dot{v}(t) = Av(t),\ t > 0 \\ v(0) = v_0 \end{cases}$$

where A is an operator which generates a C_0-semigroup $\mathcal{S} = \{S(t)\}_{t\geq 0}$. Then $\Phi(t, s, x)v = S(t-s)v$, where $t \geq s \geq 0$, $(x, v) \in Y$, defines an evolution cocycle. Moreover, the mapping defined by $\Phi_\lambda : \Delta \times X \to \mathcal{B}(V)$, $\Phi_\lambda(t, s, x)v = S_\lambda(t-s)v$, where $\mathcal{S}_\lambda = \{S_\lambda(t)\}_{t\geq 0}$ is generated by the operator $A - \lambda I$, is also an evolution cocycle.

Example 3. Let $f : \mathbf{R}_+ \to \mathbf{R}_+^*$ be a decreasing function with the property that there exists $\lim\limits_{t\to\infty} f(t) = a > 0$. We denote by $\mathcal{C} = \mathcal{C}(\mathbf{R}_+, \mathbf{R}_+)$ the set of all continuous functions $x : \mathbf{R}_+ \to \mathbf{R}_+$, endowed with the topology of uniform convergence on compact subsets of $\mathbf{R}_+$, metrizable by means of the distance

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x, y)}{1 + d_n(x, y)},\ \text{where } d_n(x, y) = \sup_{t\in[0,n]} |x(t) - y(t)|.$$

If $x \in \mathcal{C}$, then, for all $t \in \mathbf{R}_+$, we denote $x_t(s) = x(t+s)$, $x_t \in \mathcal{C}$. Let X be the closure in $\mathcal{C}$ of the set $\{f_t, t \in \mathbf{R}_+\}$. It follows that (X, d) is a metric space. The mapping $\varphi : \Delta \times X \to X$, $\varphi(t, s, x) = x_{t-s}$ is an evolution semiflow on X.

We consider $V = \mathbf{R}^2$, with the norm $\|v\| = |v_1| + |v_2|$, $v = (v_1, v_2) \in V$. If $u : \mathbf{R}_+ \to \mathbf{R}_+^*$, then the mapping $\Phi_u : \Delta \times X \to \mathcal{B}(V)$ defined by

$$\Phi_u(t, s, x)v = \left(\frac{u(s)}{u(t)} e^{-\int_s^t x(\tau-s)d\tau} v_1, \frac{u(t)}{u(s)} e^{\int_s^t x(\tau-s)d\tau} v_2 \right),$$

is an evolution cocycle over φ and $C = (\Phi_u, \varphi)$ is a skew-evolution semiflow.

Example 4. Let X be a metric space, φ an evolution semiflow on X and $A : X \to \mathcal{B}(V)$ a continuous mapping, where V is a Banach space. If $\Phi(t, s, x)$ is the solution of the Cauchy problem

$$\begin{cases} v'(t) = A(\varphi(t, s, x))v(t), & t > s \\ v(s) = x, \end{cases}$$

then $C = (\Phi, \varphi)$ is a linear skew-evolution semiflow.

Other examples of skew-evolution semiflows are given in [15].

3 Exponential dichotomy

In this section we define three concepts of exponential dichotomy for skew-evolution semiflows. We will establish connections between these notions and we will emphasize that they are not equivalent.

Let $C : \Delta \times Y \to Y$, $C(t, s, x, v) = (\Phi(t, s, x)v, \varphi(t, s, x))$ be a skew-evolution semiflow on Y.

We recall that a mapping $P : X \to \mathcal{B}(V)$ with the property

$$P(x)^2 = P(x), \ \forall x \in X$$

is called *projections family* on V.

The mapping $Q : X \to \mathcal{B}(V)$ defined by $Q(x) = I - P(x)$ is a projections family, which is called the *complementary* of P.

Definition 4. A projections family $P : X \to \mathcal{B}(V)$ is said to be *compatible* with the skew-evolution semiflow $C = (\Phi, \varphi)$ iff:

$$\Phi(t, s, x)P(x) = P(\varphi(t, s, x))\Phi(t, s, x),$$

for all $(t, s, x) \in \Delta \times X$.

In what follows, if P is a given projections family, we will denote

$$\Phi_P(t, s, x) = \Phi(t, s, x)P(x),$$

for every $(t, s, x) \in \Delta \times X$.

We remark that

(*i*) $\Phi_P(t, t, x) = P(x)$, for all $(t, x) \in \mathbf{R}_+ \times X$;

(*ii*) $\Phi_P(t, s, \varphi(s, t_0, x))\Phi_P(s, t_0, x) = \Phi_P(t, t_0, x)$, for all $(t, s, t_0, x) \in T \times X$.

Definition 5. The skew-evolution semiflow $C = (\Phi, \varphi)$ is *exponentially dichotomic* relative to the projections family $P : X \to \mathcal{B}(V)$ (and we denote P.e.d.) iff there exist a constant $\alpha > 0$ and a nondecreasing mapping $N : \mathbf{R}_+ \to [1, \infty)$ such that:

(ed_1) $e^{\alpha(t-s)} \|\Phi_P(t, t_0, x_0)v_0\| \leq N(s) \|\Phi_P(s, t_0, x_0)v_0\|$;

(ed_2) $e^{\alpha(t-s)} \|\Phi_Q(s, t_0, x_0)v_0\| \leq N(t) \|\Phi_Q(t, t_0, x_0)v_0\|$,

for all $(t, s, t_0, x_0, v_0) \in T \times Y$, where Q is the complementary of P.

Remark 2. The skew-evolution semiflow $C = (\Phi, \varphi)$ is P.e.d. if and only if there exist a constant $\alpha > 0$ and a nondecreasing mapping $N : \mathbf{R}_+ \to [1, \infty)$ such that:

(ed'_1) $e^{\alpha(t-s)} \|\Phi_P(t, s, x)v\| \leq N(s) \|P(x)v\|$;

(ed'_2) $e^{\alpha(t-s)} \|Q(x)v\| \leq N(t) \|\Phi_Q(t, s, x)v\|$,

for all for all $(t, s, x, v) \in \Delta \times Y$.

A particular case of P.e.d. is given by

Definition 6. The skew-evolution semiflow $C = (\varphi, \Phi)$ is called *Barreira-Valls exponentially dichotomic* relative to the projections family $P : X \to \mathcal{B}(V)$ (and we denote P.B.V.e.d.) iff there exist $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that:

$(BVed_1)$ $e^{\alpha(t-s)} \|\Phi_P(t, t_0, x_0)v_0\| \leq Ne^{\beta s} \|\Phi_P(s, t_0, x_0)v_0\|$;

$(BVed_2)$ $e^{\alpha(t-s)} \|\Phi_Q(s, t_0, x_0)v_0\| \leq Ne^{\beta t} \|\Phi_Q(t, t_0, x_0)v_0\|$,

for all $(t, s, t_0, x_0, v_0) \in T \times Y$.

Remark 3. The skew-evolution semiflow $C = (\Phi, \varphi)$ is P.B.V.e.d. if and only if there exist $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that:

$(BVed'_1)$ $e^{\alpha(t-s)} \|\Phi_P(t, s, x)v\| \leq Ne^{\beta s} \|P(x)v\|$

$(BVed'_2)$ $e^{\alpha(t-s)} \|Q(x)v\| \leq Ne^{\beta t} \|\Phi_Q(t, s, x)v\|$,

for all for all $(t, s, x, v) \in \Delta \times Y$.

Remark 4. It is obvious that if C is P.B.V.e.d., then it is P.e.d.

The converse is not true, fact illustrated by

Example 5. We consider the metric space (X, d), the Banach space V and the evolution semiflow φ defined as in Example 3. Let us consider the complementary projections families $P, Q : X \to \mathcal{B}(V)$, $P(x)v = (v_1, 0)$, $Q(x)v = (0, v_2)$, for all $x \in X$ and all $v = (v_1, v_2) \in V$, compatible with C.

Let $g : \mathbf{R}_+ \to [1, \infty)$ be a continuous function with

$$g(n) = e^{n \cdot 2^{2n}} \text{ and } g\left(n + \frac{1}{2^{2n}}\right) = e^4, \text{ for all } n \in \mathbf{N}.$$

The mapping $\Phi : \Delta \times X \to \mathcal{B}(V)$, defined by

$$\Phi(t, s, x)v = \left(\frac{g(s)}{g(t)} e^{-(t-s)-\int_s^t x(\tau - s)d\tau} v_1, \frac{g(s)}{g(t)} e^{t-s+\int_s^t x(\tau-s)d\tau} v_2\right)$$

is an evolution cocycle over the evolution semiflow φ.

We observe that for $\alpha = 1 + a$ we have that

$$e^{\alpha(t-s)} \|\Phi_P(t, s, x)v\| \leq g(s) \|P(x)v\|$$

and

$$e^{\alpha(t-s)} \|Q(x)v\| \leq g(s)e^{\alpha(t-s)} \|Q(x)v\| \leq g(t) \|\Phi_Q(t, s, x)v\| ,$$

for all $(t, s, x, v) \in \Delta \times Y$. Thus, conditions (ed_1') and (ed_2') are satisfied for

$$\alpha = 1 + a \text{ and } N(t) = \sup_{s \in [0,t]} g(s)$$

and, hence, $C = (\Phi, \varphi)$ is P.e.d.

If we suppose that C is P.B.V.e.d., then there exist $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that

$$g(s)e^{\alpha t} \leq N g(t) e^{\beta s + t - s + \int_s^t x(\tau - s)d\tau},$$

for all $(t, s, x) \in \Delta \times X$.

From here, for $t = n + \dfrac{1}{2^{2n}}$ and $s = n$, it follows that

$$e^{n(2^{2n} + \alpha - \beta)} \leq 81 N e^{\frac{1 - \alpha + f(0)}{2^{2n}}},$$

which, for $n \to \infty$, implies a contradiction.

Another particular case of P.e.d. is introduced by

Definition 7. The skew-evolution semiflow $C = (\Phi, \varphi)$ is *uniformly exponentially dichotomic* relative to the projections family $P : X \to \mathcal{B}(V)$ (and we denote P.u.e.d.) iff there exist some constants $N \geq 1$ and $\alpha > 0$ such that:

(ued_1) $e^{\alpha(t-s)} \|\Phi_P(t, t_0, x_0)v_0\| \leq N \|\Phi_P(s, t_0, x_0)v_0\|$;
(ued_2) $e^{\alpha(t-s)} \|\Phi_Q(s, t_0, x_0)v_0\| \leq N \|\Phi_Q(t, t_0, x_0)v_0\|$,
for all $(t, s, t_0, x_0, v_0) \in T \times Y$.

Remark 5. The skew-evolution semiflow $C = (\Phi, \varphi)$ is P.u.e.d. if and only if there exist some constants $N \geq 1$ and $\alpha > 0$ such that:

(ued_1') $e^{\alpha(t-s)} \left\| \Phi_P(t,s,x)v \right\| \leq N \left\| P(x)v \right\|$;

(ued_2') $e^{\alpha(t-s)} \left\| Q(x)v \right\| \leq N \left\| \Phi_Q(t,s,x)v \right\|$,

for all for all $(t,s,x,v) \in \Delta \times Y$.

Remark 6. It is obvious that if C is P.u.e.d., then it is P.B.V.e.d.

The following example shows that the converse implication is not valid.

Example 6. We consider the metric space (X,d), the Banach space V and the evolution semiflow φ defined as in Example 3. Let us consider the complementary projections families $P, Q : X \to \mathcal{B}(V)$, $P(x)v = (v_1, 0)$, $Q(x)v = (0, v_2)$, for all $x \in X$ and all $v = (v_1, v_2) \in V$, compatible with C.

The mapping $\Phi : \Delta \times X \to \mathcal{B}(V)$, defined by

$$\Phi(t,s,x)v =$$

$$= \left(v_1 e^{t \sin t - s \sin s - 2(t-s) - \int_s^t x(\tau - s) d\tau}, v_2 e^{3(t-s) - 2t\cos t + 2s \cos s + \int_s^t x(\tau-s)d\tau} \right)$$

is an evolution cocycle over the evolution semiflow φ.

We observe that for $\alpha = 1 + a$ we have that

$$e^{\alpha(t-s)} \left\| \Phi_P(t,s,x)v \right\| \leq e^{\alpha(t-s)} e^{-(1+a)t} e^{(3+a)s} \left\| P(x)v \right\| \leq e^{2s} \left\| P(x)v \right\|,$$

for all $(t,s,x,v) \in \Delta \times Y$. Similarly,

$$e^{\alpha(t-s)} \left\| Q(x)v \right\| \leq e^{\alpha(t-s)} e^{-3t+3s+2t\cos t - 2s\cos s - \int_s^t x(\tau-s)d\tau} \left\| \Phi_Q(t,s,x)v \right\|$$

$$\leq \left\| \Phi_Q(t,s,x)v \right\|,$$

for all $(t,s,x,v) \in \Delta \times Y$. Thus, conditions $(BVed_1')$ and $(BVed_2')$ are satisfied for

$$\alpha = 1 + a, \;\; N = 1 \text{ and } \beta = \min\{0, 2\}.$$

This shows that $C = (\Phi, \varphi)$ is P.B.V.e.d.

If we suppose that C is P.u.e.d., then there exist $N > 1$ and $\alpha > 0$ such that

$$e^{\alpha(t-s)} e^{t \sin t - s \sin s - 2t + 2s - \int_s^t x(\tau-s)d\tau} \leq N,$$

for all $(t, s) \in \Delta$. In particular, for $t = 2n\pi + \frac{\pi}{2}$ and $s = 2n\pi$, we obtain

$$2n\pi + (\alpha - 1)\frac{\pi}{2} \leq \ln N \int_{2n\pi}^{2n\pi+\frac{\pi}{2}} x(\tau - 2n\pi)d\tau =$$

$$= \ln N \int_{0}^{\frac{\pi}{2}} x(u)du \leq f\left(\frac{\pi}{2}\right) \ln N,$$

which, for $n \to \infty$, leads to a contradiction.

4 Polynomial dichotomy

Let $C : \Delta \times Y \to Y$, $C(t, s, x, v) = (\Phi(t, s, x)v, \varphi(t, s, x))$ be a skew-evolution semiflow on Y and let $P : X \to \mathcal{B}(V)$ be a projections family on V, compatible with C, and $Q : X \to \mathcal{B}(V)$ the complementary projections family of P.

Definition 8. The skew-evolution semiflow $C = (\Phi, \varphi)$ is *polynomially dichotomic* with respect to P (and we denote P.p.d.) iff there exist $\alpha > 0$, $t_1 > 0$ and a nondecreasing function $N : \mathbf{R}_+ \to [1, \infty)$ such that:

(pd_1) $t^\alpha \|\Phi_P(t, t_0, x_0)v_0\| \leq N(s)s^\alpha \|\Phi_P(s, t_0, x_0)v_0\|$;

(pd_2) $t^\alpha \|\Phi_Q(s, t_0, x_0)v_0\| \leq N(t)s^\alpha \|\Phi_Q(t, t_0, x_0)v_0\|$,

for all $(t, s, t_0, x_0, v_0) \in T \times Y$ with $t_0 \geq t_1$.

Remark 7. The skew-evolution semiflow $C = (\Phi, \varphi)$ is P.p.d. if and only if there exist $\alpha > 0$, $t_0 > 0$ and a nondecreasing function $N : \mathbf{R}_+ \to [1, \infty)$ such that:

(pd'_1) $t^\alpha \|\Phi_P(t, s, x)v\| \leq N(s)s^\alpha \|P(x)v\|$;

(pd'_2) $t^\alpha \|Q(x)v\| \leq N(t)s^\alpha \|\Phi_Q(t, s, x)v\|$,

for all for all $(t, s, x, v) \in \Delta \times Y$ with $s \geq t_0$.

Example 7. We consider the metric space (X, d), the Banach space V and the evolution semiflow φ defined as in Example 3. Let us consider the complementary projections families $P, Q : X \to \mathcal{B}(V)$, $P(x)v = (v_1, 0)$, $Q(x)v = (0, v_2)$, for all $x \in X$ and all $v = (v_1, v_2) \in V$, compatible with C.

The mapping $\Phi : \Delta \times X \to \mathcal{B}(V)$, defined by

$$\Phi(t, s, x)v = \left(\frac{s+1}{t+1}e^{-\int_s^t x(\tau - s)d\tau}v_1, \frac{t+1}{s+1}e^{\int_s^t x(\tau - s)d\tau}v_2\right)$$

is an evolution cocycle with

$$t^a \|\Phi_P(t,s,x)v\| \leq \frac{t^a(s+1)e^{-a(t-s)}}{t+1} \|P(x)v\| \leq s^a \|P(x)v\|$$

and

$$t^a \|Q(x)v\| \leq s^a e^{a(t-s)} \|Q(x)v\| \leq \frac{s^a(t+1)}{s+1} e^{\int_s^t x(\tau-s)d\tau} \leq s^a \|\Phi_Q(t,s,x)v\|,$$

for all $t \geq s \geq 1$ and all $(x,v) \in Y$. It follows that $C = (\Phi, \varphi)$ is P.p.d.

Proposition 1. *If $C=(\Phi,\varphi)$ is a P–exponentially dichotomic skew-evolution semiflow, then it is P–polynomially dichotomic.*

Proof. If C is P.e.d., then there exist $\alpha > 0$ and $N : \mathbf{R}_+ \to [1,\infty)$ such that conditions (ed_1') and (ed_2') are satisfied.

We observe that the function

$$u : [1,\infty) \to (0,\infty),\ u(t) = \frac{e^t}{t}$$

is nondecreasing on $[1,\infty)$ and, hence,

$$\frac{t^\alpha}{s^\alpha} \|\Phi_P(t,s,x)v\| \leq e^{\alpha(t-s)} \|\Phi_P(t,s,x)v\| \leq N(s) \|P(x)v\|$$

and

$$\frac{t^\alpha}{s^\alpha} \|Q(x)v\| \leq e^{\alpha(t-s)} \|Q(x)v\| \leq N(t) \|\Phi_Q(t,s,x)v\|,$$

for all $t \geq s \geq t_0 \geq 1$ and all $(x,v) \in Y$.

Finally, it results that conditions (pd_1') and (pd_2') are satisfied, which proves that C is P.p.d.

The converse of the preceding proposition is not valid. This fact is illustrated by

Example 8. Let $X = \mathbf{R}_+$ and $V = \mathbf{R}^2$. The mapping $\varphi : \Delta \times X \to X$, defined by $\varphi(t,s,x) = x$ is an evolution semiflow on $\mathbf{R}_+$.

We define the evolution cocycle $\Phi : \Delta \times X \to \mathcal{B}(V)$ by

$$\Phi(t,s,x)(v_1,v_2) = \left(\frac{s+1}{t+1}v_1, \frac{t+1}{s+1}v_2\right),$$

with $(t, s, x, v) \in \Delta \times Y$. Then $P : X \to \mathcal{B}(V)$, $P(x)(v_1, v_2) = (v_1, 0)$ is a projections family which is compatible with the skew-evolution semiflow $C = (\Phi, \varphi)$. Q denotes the complementary projections family of P. Furthermore

$$t \|\Phi_P(t, s, x)v\| \leq s^2 \|P(x)v\|$$

and

$$t \|Q(x)v\| \leq ts \|\Phi_Q(t, s, x)v\|$$

for all $(t, s, x, v) \in \Delta \times Y$.

Hence, the conditions (pd_1') and (pd_2') are satisfied for

$$\alpha = 1,\ t_0 = 1 \text{ and } N(t) = t.$$

Thus, C is P.p.d.

If we suppose that C is P.e.d., then there exist $\alpha > 0$ and a mapping $N : \mathbf{R}_+ \to [1, \infty)$ such that

$$(s + 1)e^{\alpha(t-s)} \leq (t + 1)N(s),$$

for all $t \geq s \geq 0$. From here, for s fixed and $t \to \infty$, we obtain a contradiction.

A particular case of polynomial dichotomy is introduced by

Definition 9. The skew-evolution semiflow $C = (\Phi, \varphi)$ is *polynomially dichotomic in the sense Barreira-Valls* with respect to the projections family $P : X \to \mathcal{B}(V)$ (and we denote P.B.V.p.d.) iff there exist $N \geq 1$, $t_1 > 0$, $\alpha > 0$ and $\beta \geq 0$ such that:

$(BVpd_1)$ $t^\alpha \|\Phi_P(t, t_0, x_0)v_0\| \leq N s^{\alpha+\beta} \|\Phi_P(s, t_0, x_0)v_0\|$;

$(BVpd_2)$ $t^\alpha \|\Phi_Q(s, t_0, x_0)v_0\| \leq N s^\alpha t^\beta \|\Phi_Q(t, t_0, x_0)v_0\|$,

for all $(t, s, t_0, x_0, v_0) \in T \times Y$ with $t_0 \geq t_1$.

Remark 8. The skew-evolution semiflow $C = (\Phi, \varphi)$ is P.B.V.p.d. if and only if there exist $N \geq 1$, $t_0 > 0$, $\alpha > 0$ and $\beta \geq 0$ such that:

$(BVpd_1')$ $t^\alpha \|\Phi_P(t, s, x)v\| \leq N s^{\alpha+\beta} \|P(x)v\|$;

$(BVpd_2')$ $t^\alpha \|Q(x)v\| \leq N s^\alpha t^\beta \|\Phi_Q(t, s, x)v\|$,

for all for all $(t, s, x, v) \in \Delta \times Y$ with $s \geq t_0$.

Remark 9. It is obvious that if C is P.B.V.p.d. then it is P.p.d.

The following example shows that the converse is not true.

Example 9. We consider the skew-evolution semiflow $C = (\Phi, \varphi)$ given in Example 3 and the complementary projections families $P, Q : X \to \mathcal{B}(V)$, $P(x)v = (v_1, 0)$, $Q(x)v = (0, v_2)$, for all $x \in X$ and all $v = (v_1, v_2) \in V$, compatible with C. Because C is P.e.d., then it is also P.p.d.

If we suppose that C is P.B.V.p.d., then there exist $N \geq 1$, $t_0 > 0$, $\alpha > 0$ and $\beta \geq 0$ such that

$$t^{\alpha} g(s) \leq N g(t) s^{\alpha+\beta} e^{t-s+\int_0^{t-s} x(u)du},$$

for all $t \geq s \geq t_0$. From here, for $t = n + \dfrac{1}{2^{2n}}$ and $s = n \to \infty$, we obtain a contradiction.

Another particular case of polynomial dichotomy is given by

Definition 10. The skew-evolution semiflow $C = (\Phi, \varphi)$ is *uniformly polynomially dichotomic* in rapport with the projections family $P : X \to \mathcal{B}(V)$ (and we denote P.u.p.d.) iff there exist $N \geq 1$, $\alpha > 0$ and $t_1 > 0$ such that:
(upd_1) $t^{\alpha} \left\| \Phi_P(t, t_0, x_0) v_0 \right\| \leq N s^{\alpha} \left\| \Phi_P(s, t_0, x_0) v_0 \right\|;$
(upd_2) $t^{\alpha} \left\| \Phi_Q(s, t_0, x_0) v_0 \right\| \leq N s^{\alpha} \left\| \Phi_Q(t, t_0, x_0) v_0 \right\|,$
for all $(t, s, t_0, x_0, v_0) \in T \times Y$ with $t_0 \geq t_1$.

Remark 10. The skew-evolution semiflow $C = (\Phi, \varphi)$ is P.u.p.d. if and only if there exist $N \geq 1$, $\alpha > 0$ and $t_0 > 0$ such that:
(upd_1') $t^{\alpha} \left\| \Phi_P(t, s, x) v \right\| \leq N s^{\alpha} \left\| P(x) v \right\|;$
(upd_2') $t^{\alpha} \left\| Q(x) v \right\| \leq N s^{\alpha} \left\| \Phi_Q(t, s, x) v \right\|,$
for all for all $(t, s, x, v) \in \Delta \times Y$ with $s \geq t_0$.

Remark 11. If C is P.u.p.d. then it is P.B.V.p.d.

The reciprocal is not valid, fact illustrated by

Example 10. We consider the metric space (X, d), the Banach space V and the evolution semiflow φ defined as in Example 3. Let us consider the complementary projections families $P, Q : X \to \mathcal{B}(V)$, $P(x)v = (v_1, 0)$, $Q(x)v = (0, v_2)$, for all $x \in X$ and all $v = (v_1, v_2) \in V$, compatible with C.

We consider the function

$$g : \mathbf{R}_+ \to \mathbf{R},\ g(t) = \frac{(t+1)^3}{(t+1)^{\sin \ln(t+1)}}$$

and the evolution cocycle $\Phi : \Delta \times X \to \mathcal{B}(V)$ over φ defined by

$$\Phi(t,s,x)v = \left(\frac{g(s)}{g(t)} v_1, \frac{g(t)}{g(s)} v_2 \right).$$

Then

$$t \left\| \Phi_P(t,s,x)v \right\| \leq \frac{t(s+1)^4}{(t+1)^2} \left\| P(x)v \right\| \leq s(s+1)^2 \left\| P(x)v \right\| \leq 4s^3 \left\| P(x)v \right\|$$

and

$$t \left\| Q(x)v \right\| \leq t(t+1)^2 \left\| Q(x)v \right\| \leq \frac{s(t+1)^4}{(s+1)^2} \left\| Q(x)v \right\|$$
$$\leq s \left\| \Phi_Q(t,s,x)v \right\| \leq 4st^2 \left\| \Phi_Q(t,s,x)v \right\|,$$

for all $t \geq s \geq 1$ and all $(x,v) \in Y$. Thus, the conditions $(BVpd_1')$ and $(BVpd_2')$ are satisfied for

$$\alpha = 1,\ \beta = 2,\ N = 4 \text{ and } t_0 = 1.$$

If we suppose that C is P.u.p.d., then there are $N \geq 1$, $\alpha > 0$ and $t_0 > 0$ such that

$$t^\alpha (s+1)^3 (t+1)^{\sin \ln(t+1)} \leq N s^\alpha (t+1)^3 (s+1)^{\sin \ln(s+1)},$$

for all $t \geq s \geq t_0$. From here, for $t = e^{2n\pi + \frac{\pi}{2}} - 1$ and $s = e^{2n\pi - \frac{\pi}{2}} - 1$ and $n \to \infty$, we obtain a contradiction.

Proposition 2. *If the skew-evolution semiflow $C = (\Phi, \varphi)$ is uniformly exponentially dichotomic with respect to the projections family $P : X \to \mathcal{B}(V)$, then C is uniformly polynomially dichotomic with respect to P.*

Proof. If $C = (\Phi, \varphi)$ is P.u.e.d., then there are $N \geq 1$ and $\alpha > 0$ such that the conditions (ued_1) and (ued_2) are satisfied. Using the inequalities

$$t+1 \leq e^t,\ \frac{e^s}{s} \leq \frac{e^t}{t} \text{ and } \frac{t}{s} \leq t-s+1, \text{ for } t \geq s \geq 1,$$

we obtain

$$t^\alpha \left\| \Phi_P(t,s,x)v \right\| \leq N t^\alpha e^{-\alpha(t-s)} \left\| P(x)v \right\| \leq \frac{N t^\alpha \left\| P(x)v \right\|}{(1+t-s)^\alpha} \leq N s^\alpha \left\| P(x)v \right\|$$

and

$$t^{\alpha\|Q(x)v\|} \leq Nt^{\alpha}e^{-\alpha(t-s)}\|\Phi_Q(t,s,x)v\| \leq Ns^{\alpha}\|\Phi_Q(t,s,x)v\|,$$

for all $(t,s,x,v) \in \Delta \times Y$ with $s \geq t_0 = 1$.

Finally, we obtain that C is P.u.p.d.

Now, we give an example which shows that the converse of the preceding result is not valid.

Example 11. We consider the metric space (X,d), the Banach space V and the evolution semiflow φ defined as in Example 3. Let us consider the complementary projections families $P,Q : X \to \mathcal{B}(V)$, $P(x)v = (v_1,0)$, $Q(x)v = (0,v_2)$, for all $x \in X$ and all $v = (v_1,v_2) \in V$, compatible with C.

We consider the evolution cocycle $\Phi : \Delta \times X \to \mathcal{B}(V)$, defined by

$$\Phi(t,s,x)v = \left(\frac{s^2+1}{t^2+1}e^{-\int_s^t x(\tau-s)d\tau}v_1, \frac{t^2+1}{s^2+1}e^{\int_s^t x(\tau-s)d\tau}v_2\right),$$

for $(t,s,x) \in \Delta \times X$ and $v = (v_1,v_2) \in V = \mathbf{R}^2$. Using the inequalities

$$\frac{s^2+1}{t^2+1} \leq \frac{s}{t} \text{ and } \frac{e^s}{e^t} \leq \frac{s}{t}, \text{ for } t \geq s \geq 1,$$

we obtain

$$\begin{aligned} t^{\alpha}\|\Phi_P(t,s,x)v\| &\leq \frac{t^{\alpha}(s^2+1)}{t^2+1}e^{-a(t-s)}\|P(x)v\| \\ &\leq \frac{t^{\alpha}\cdot s}{t}\left(\frac{s}{t}\right)^a = s^{\alpha}\|P(x)v\|, \end{aligned}$$

for all $t \geq s \geq t_0 = 1$ and $(x,v) \in Y$, where $\alpha = 1+a$.

Similarly,

$$\begin{aligned} t^{\alpha}\|Q(x)v\| = t\cdot t^a &\leq ts^a e^{at}e^{-as}\|Q(x)v\| \leq \frac{t}{s}s^{\alpha}e^{a(t-s)}\|Q(x)v\| \\ &\leq \frac{s^{\alpha}(t^2+1)}{s^2+1}e^{a(t-s)}\|Q(x)v\| \leq s^{\alpha}\|\Phi_Q(t,s,x)v\|, \end{aligned}$$

for all $t \geq s \geq t_0 = 1$ and $(x,v) \in Y$, with $\alpha = 1+a$. Thus, C is P.u.p.d.

If we suppose that C is P.u.e.d., then there exist $N \geq 1$, $\alpha > 0$ and $t_0 > 0$ such that

$$(s^2+1)e^{\alpha(t-s)} \leq N(t^2+1)e^{-a(t-s)},$$

for all $t \geq s \geq t_0$. Then, for $s = t_0$ and $t \to \infty$, we obtain a contradiction, which can be eliminated only if C is not P.u.e.d.

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