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THE CONTROL VARIATIONAL METHOD FOR ELASTIC CONTACT PROBLEMS*

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Abstract

We consider a multivalued equation of the form $Ay + \partial\varphi(y) \ni f$ in a real Hilbert space, where A is a linear operator and $\partial\varphi$ represents the (Clarke) subdifferential of the function φ. We prove existence and uniqueness results of the solution by using the control variational method. The main idea in this method is to minimize the energy functional associated to the nonlinear equation by arguments of optimal control theory. Then we consider a general mathematical model describing the contact between a linearly elastic body and an obstacle which leads to a variational formulation as above, for the displacement field. We apply the abstract existence and uniqueness results to prove the unique weak solvability of the corresponding contact problem. Finally, we present examples of contact and friction laws for which our results work.

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1 Introduction

The control variational method was introduced in [1, 18] in the study of differential equations. A comprehensive presentation of this new variational method, together with various examples and applications, may be found in the monograph [10]. Its use in the study of various models which describe the equilibrium of an Euler-Bernoulli beam in contact with an obstacle was presented in [17]. The main new idea in this method is to perform the minimization of the energy of the system via the optimal control theory, which represents an extension of the arguments via the calculus of variations, used in the classical variational method. This new general framework is very flexible and may offer several different solutions for the same problem, as shown in [20]. Moreover, it is relevant both from the theoretical and the numerical point of view. In particular, in many applications, the control variational method replaces the solution of nonlinear differential equations of order four by the solution of linear equations of lower order and, in addition, it provides regularity results, as shown in [10] and the references therein.

Phenomena of contact between deformable bodies abound in industry and everyday life. Contact of braking pads with wheels, tires with roads, pistons with skirts are just a few simple examples. Common industrial processes such as metal forming and metal extrusion involve contact evolutions. Owing to their inherent complexity, contact phenomena lead to mathematical models expressed in terms of strongly nonlinear elliptic or evolutionary equations.

Considerable progress has been achieved recently in modeling, mathematical analysis and numerical simulations of contact processes and, as a result, a general mathematical theory of Contact Mechanics is currently emerging. It is concerned with the mathematical structures which underlie general contact problems with different constitutive laws, i.e. materials, various geometries and different contact conditions. Its aim is to provide a sound, clear and rigorous background to the constructions of models for contact, proving existence, uniqueness and regularity results, assigning precise meaning to solutions, among others. To this end, it operates with various mathematical concepts which include variational and hemivariational inequalities and multivalued inclusions, as well. The variational analysis of contact problems, including existence and uniqueness results, can be found in the monographs [3, 4, 5, 6, 11, 12, 15, 16]. Computational methods for problems in Contact Mechanics can be found in the works [7, 21, 23] and in the extensive lists of references therein. The state of the art in the field can also be found in the proceedings [8, 13, 22] and in the special issue [14].

The aim of this paper is twofold. The first one is to illustrate the use of the control variational method in the study of nonlinear equations with multivalued operators in a Hilbert space and to obtain existence and uniqueness results of the solution. The second one is to apply these results in the study of mathematical models which describe the frictional or frictionless contact between a linearly elastic body and a foundation. In our examples the contact is either bilateral or is modeled with the Signorini condition or with the normal compliance condition. Friction is modeled with versions of Coulomb's law, including Tresca's law, or with a power-law. In a variational form, the models lead to a nonlinear equation for the displacement field. Thus, we apply the abstract existence and uniqueness results to prove the unique weak solvability of the corresponding contact problems. The abstract results obtained by using of the control variational method and their use in the study of contact problems with linearly elastic materials represent the main trait of novelty of the present paper.

The paper is structured as follows. In Section 2 we present our existence and uniqueness results in the study of nonlinear equations with multivalued operators. In Section 3 we use these results in the study of a general mathematical model which describe the frictional contact of a linearly elastic body with an obstacle. Then, we present examples of contact and friction laws for which our results work.

2 Abstract existence and uniqueness results

In this section we apply the control variational method in the study of multivalued equations in abstract Hilbert spaces. We denote by V and H two real Hilbert spaces with norms $\|\cdot\|_V$ and $\|\cdot\|_H$, respectively. The dual space of V is denoted by V^*, $\langle\cdot,\cdot\rangle_{V^*\times V}$ will represent the duality pairing of between V^* and V and notation 2^{V^*} will be used to denote the set of parts of V^*. Moreover, everywhere in this section we assume that $V \subset H \subset V^*$ with compact and dense embeddings.

Let $A : V \to V^*$ be a linear operator and $\varphi\colon V \to \mathbb{R}$ be a locally Lipschitz function. We recall that the generalized directional derivative of φ at $x \in X$

in the direction $v \in V$, denoted $\varphi^0(x;v)$, is defined by

$$\varphi^0(x;v) = \limsup_{y \to x,\ \lambda \downarrow 0} \frac{\varphi(y+\lambda v) - \varphi(y)}{\lambda}\,, \tag{1}$$

see [2] for details. Also, the generalized gradient of φ at x, denoted $\partial\varphi(x)$, is a subset of a dual space V^* given by

$$\partial\varphi(x) = \{\, \zeta \in V^* \mid \varphi^0(x;v) \geq \langle \zeta, v \rangle_{V^* \times V} \text{ for all } v \in X \,\}, \tag{2}$$

and the application $\partial\varphi : V \to 2^{V^*}$ is called the Clarke subdifferential of φ.

We start with the study of the nonlinear stationary equation

$$Ay + \partial\varphi(y) \ni f \tag{3}$$

where $\partial\varphi$ is the Clarke subdifferential of φ and $f \in V^*$. To this end we assume that

$$\begin{gathered} A \text{ is a linear continuous symmetric and coercive operator i.e.} \\ \text{there exists } m > 0 \text{ such that } \langle Ax, x \rangle_{V^* \times V} \geq m\|x\|_V^2 \quad \forall\, x \in V. \end{gathered} \tag{4}$$

$$\begin{gathered} \text{There exists } c_1 > 0,\ \alpha \in (0,2),\ \beta \in \mathbb{R} \text{ and } c_2 > 0 \text{ such that} \\ \varphi(x) \geq -c_1\,\|x\|_V^\alpha + \beta \quad \forall\, x \in V \text{ which satisfies } \|x\|_V \geq c_2 > 0. \end{gathered} \tag{5}$$

$$\varphi : V \to R \text{ is lower semicontinuous in the topology of } H. \tag{6}$$

Note that the assumptions on φ are very general and a large number of hemivariational inequalities can be cast on the form (3), see for instance [9, 12] and the references therein. Moreover, using (13) it is easy to see that if φ is a convex function, then the multivalued equation (3) is equivalent to the elliptic variational inequality

$$\langle Ay, v - y \rangle_{V^* \times V} + \varphi(v) - \varphi(y) \geq \langle f, v - y \rangle_{V^* \times V} \quad \forall\, v \in V. \tag{7}$$

Note also that the operator A may be nonlinear as well, of monotone type, as the p-Laplacian, for instance. Nevertheless, in this paper we restrict ourselves to the case of linear operators, as stated in (4). Finally, note that it is possible to take $\alpha = 2$ in the subquadratic descent hypothesis (5), if the constant $c_1 > 0$ is dominated by the coercivity constant of A, that is if $m > c_1$.

Denote by $g \in V$ the unique solution of $Ag = f$, guaranteed by (4). The control variational method in the study of (3) associates to this nonlinear equation the following optimal control problem:

$$\min_{y \in V} \Big\{ \langle u, y \rangle_{V^* \times V} - 3 \langle u, g \rangle_{V^* \times V} + 2\,\varphi(y) \Big\}, \tag{8}$$

$$Ay = u - f. \tag{9}$$

Note that problem (8)–(9) "decouples" in the cost (8) the nonlinear part from equation (3) and it provides the solution of (3) in a constructive simple way. Moreover, no constraints are imposed in solving problem (8)–(9). In fact, in order to solve this problem, we need only the inverse of the operator A. In practice, the solution of the control problem can be obtained by using an iterative gradient method.

The connection between the optimal control problem (8)–(9) and the nonlinear equation (3) is given by the following result.

Theorem 1. *Assume that* (4)–(6) *hold. Then problem* (8)–(9) *has at least one optimal pair* $[y^*, u^*] \in V \times V^*$ *and* y^* *is a solution of* (3)*. Moreover, if the solution of* (3) *is unique, then there exists a unique optimal pair which solves* (8)–(9)*.*

Proof. We use (9) and substitute $u = Ay + f$ in (8). Then, the optimal control problem becomes

$$\min_{y \in V} \Big\{ \langle Ay, y \rangle_{V^* \times V} + \langle f, y \rangle_{V^* \times V} - 3 \langle Ay, g \rangle_{V^* \times V} - 3 \langle f, g \rangle_{V^* \times V} + 2\,\varphi(y) \Big\}. \tag{10}$$

We use assumptions (4)–(6) to see that any minimizing sequence for the functional in (10) is relatively compact in H and weakly convergent in V. By the lower semicontinuity in H of φ, this implies that (10) has at least one solution $y^* \in V$. Next, the optimal control is obtained by using (9), i.e. $u^* = Ay^* + f$. We conclude from above that (8)–(9) has at least one optimal pair $[y^*, u^*]$. We also note that, since φ is not assumed to be a convex function, the control problem (8)–(9) could have more than one optimal pair.

Consider now admissible variations around $[y^*, u^*]$ of the form $[y^* + \lambda\xi, u^* + \lambda\omega]$ where $\lambda \in \mathbb{R}_+$ and $[\xi, \omega] \in V \times V^*$ satisfies

$$A\xi = \omega. \tag{11}$$

We obtain

$$\begin{aligned}&\langle u^*, y^*\rangle_{V^*\times V} - 3\,\langle u^*, g\rangle_{V^*\times V} + 2\,\varphi(y^*)\\&\qquad \le \langle u^* + \lambda\omega, y^* + \lambda\xi\rangle_{V^*\times V} - 3\,\langle u^* + \lambda\omega, g\rangle_{V^*\times V} + 2\,\varphi(y^* + \lambda\xi).\end{aligned}$$

We divide this inequality by $\lambda > 0$ and take the limit as $\lambda \to 0$ to find that

$$0 \le \langle \omega, y^*\rangle_{V^*\times V} + \langle u^*, \xi\rangle_{V^*\times V} - 3\,\langle \omega, g\rangle_{V^*\times V} + 2\,\varphi^0(y^*; \xi)$$

where the symbol φ^0 denotes the generalized directional derivative, see (1). We use now (9) and (11) to infer

$$0 \le \langle A\xi, y^*\rangle_{V^*\times V} + \langle Ay^* + f, \xi\rangle_{V^*\times V} - 3\,\langle A\xi, g\rangle_{V^*\times V} + 2\,\varphi^0(y^*; \xi).$$

As ξ represents an arbitrary element in V and A is symmetric, the definition of $g \in V^*$ and the definition (2) of the Clarke subdifferential shows that $y^* \in V$ is a solution of (3). This concludes the existence part. The uniqueness part is obvious and is guaranteed by the unique solvability of the equation (3). □

We turn now to the convex case and, to this end, we assume that

$$\begin{aligned}&\varphi : V \to (-\infty, +\infty] \text{ is a convex, proper,}\\&\qquad\qquad \text{lower semicontinuous function.}\end{aligned}\tag{12}$$

We still use the notation $\partial\varphi$ for the subdifferential mapping defined in convex analysis, i.e.

$$\partial\varphi(x) = \{\, \zeta \in V^* \mid \varphi(v) - \varphi(x) \ge \langle \zeta, v - x\rangle_{V^*\times V} \text{ for all } v \in X\,\}.\tag{13}$$

for all $x \in V$. We replace assumptions (5) and (6) by assumption (12) to obtain the following result.

Theorem 2. *Assume that* (4) *and* (12) *hold. Then, both the nonlinear equation* (3) *and the optimal control problem* (8)–(9) *have a unique solution. Moreover,* $y^* \in V$ *is the solution of* (3) *if and only if* $[y^*, Ay^* + f]$ *is the optimal pair of* (8)–(9).

Proof. It follows from (12) that φ is bounded by below by an affine mapping. Then, using arguments of monotonicity and the lower semicontinuity of φ it is easy to see that the nonlinear equation (3) has a unique solution.

We turn now to the equivalence part. We note that one implication is proved by arguments similar to those used in the proof of Theorem 1. Therefore, we have to prove just the converse implication. Thus, we assume in what follows that y^* is the unique solution of (3). Then, there exists $\zeta^* \in \partial\varphi(y^*)$ such that $Ay^* + \zeta^* = f$, i.e.

$$\zeta^* - Ag + Ay^* = 0_{V^*} \tag{14}$$

We take in (14) the inner product with 2ξ, where the pair $[\xi, \omega] \in V \times V^*$ satisfies (11) and, as a result, we obtain

$$\begin{aligned} 0 &= 2\,\langle \zeta^*, \xi\rangle_{V^*\times V} - 2\,\langle Ag, \xi\rangle_{V^*\times V} + 2\,\langle Ay^*, \xi\rangle_{V^*\times V} \\ &= 2\,\langle \zeta^*, \xi\rangle_{V^*\times V} - 2\,\langle \omega, g\rangle_{V^*\times V} + \langle y^*, \omega\rangle_{V^*\times V} \\ &\quad + \langle u^*, \xi\rangle_{V^*\times V} - \langle f, \xi\rangle_{V^*\times V}. \end{aligned} \tag{15}$$

Consider another admissible pair $[y, u] \in V \times V^*$ for the optimal control problem (8)–(9). Note that $\xi = y^* - y$, $\omega = u^* - u$ satisfy (11) and, therefore, the pair $[\xi, \omega]$ may be used as test element in (15). Since φ is convex, (15) and (13) yield

$$\begin{aligned} 0 &= \langle u^*, y^* - y\rangle_{V^*\times V} + \langle u^* - u, y^*\rangle_{V^*\times V} - 3\,\langle u^* - u, g\rangle_{V^*\times V} \\ &\quad + 2\,\langle \zeta^*, y^* - y\rangle_{V^*\times V} \geq \langle u^*, y^* - y\rangle_{V^*\times V} + \langle u^* - u, y^*\rangle_{V^*\times V} \\ &\quad - 3\,\langle u^* - u, g\rangle_{V^*\times V} + 2\varphi(y^*) - 2\varphi(y). \end{aligned} \tag{16}$$

Note also that

$$\begin{aligned} &\langle u^*, y\rangle_{V^*\times V} + \langle u, y^*\rangle_{V^*\times V} - \langle u^*, y^*\rangle_{V^*\times V} - \langle u, y\rangle_{V^*\times V} \\ &= \langle u^* - u, y - y^*\rangle_{V^*\times V} = -\langle \omega, \xi\rangle_{V^*\times V} = -\langle A\xi, \xi\rangle_{V^*\times V} \leq 0. \end{aligned} \tag{17}$$

Combining (16) and (17) we infer that $[y^*, Ay^* + f]$ is the optimal pair of (8)–(9), which concludes the proof. □

Note that Theorem 1 and Theorem 2 provide, implicitly, existence and uniqueness results for the nonlinear multivalued equation (3), comparable with other similar results from the literature, see for instance [9], [10] and the references therein. Moreover, note that the results in Theorem 2 still hold in the case when φ represents the indicator function of a nonempty, closed, convex subset $K \subset X$. In this case, the control problem includes implicitly the constraint $y \in K$.

We provide now some comments on the control variational method described in Theorem 1. First, note that minimization of the cost functional (8) subject to (9) is equivalent with the minimization of the usual energy associated to (3), as the following computation shows :

$$\begin{aligned}
&\langle u, y\rangle_{V^*\times V} - 3\,\langle u, g\rangle_{V^*\times V} + 2\varphi(y)\\
&= \langle Ay + f, y\rangle_{V^*\times V} - 3\,\langle Ay + f, g\rangle_{V^*\times V} + 2\varphi(y)\\
&= \langle Ay, y\rangle_{V^*\times V} + \langle f, y\rangle_{V^*\times V} - 3\,\langle y, Ag\rangle_{V\times V^*} - 3\,\langle f, g\rangle_{V^*\times V} + 2\varphi(y)\\
&= \langle Ay, y\rangle_{V^*\times V} + 2\varphi(y) - 2\,\langle y, f\rangle_{V^*\times V} - 3\,\langle f, g\rangle_{V^*\times V}.
\end{aligned}$$

Note that the last term in the formula above has no importance, since it is a constant. Nevertheless, the advantage of problem (8)–(9) with respect to the classical variational method is that (9) involves just the "good" operator A. Also, the optimal control approach described above has the advantage that it may put into evidence new properties of the solution, see [10, Ch VI] for examples and details. It is flexible and offers a large variety of choices. For this reason, performant numerical algorithms are expected to be associated with the control variational method presented above.

We consider now a version of (3) of the form

$$A_1 y + A_2 y + \partial\varphi(y) \ni f \tag{18}$$

where $\partial\varphi$ represents the Clarke subdifferential of the locally Lipschitz function $\varphi : V \to \mathbb{R}$, $f \in V^*$, and $A = A_1 + A_2$ is a decomposition of the operator A such that

$$A_1 : V \to V^* \text{ is a linear continuous symmetric and coercive operator.} \tag{19}$$

$$A_2 : V \to V^* \text{ is a linear bounded and symmetric operator.} \tag{20}$$

Note that, without additional assumptions on A_2, the sum $A_1 + A_2$ may not be coercive. Also, note that it is not possible to include A_2 in the subdifferential operator $\partial\varphi$ since the term $\langle A_2 y, y\rangle_{V^*\times V}$ may have a quadratic decrease, and this would violate condition (5). Therefore, we conclude from above that the existence and uniqueness result in Theorem 1 cannot be used to solve the nonlinear multivalued equation (18). For this reason, we solve (18) by using a different method that we describe in what follows.

First, we consider a real Hilbert space of controls U with dual U^* and, as usual, we denote by $\langle\cdot,\cdot\rangle_{U^*\times U}$ the duality pairing between U^* and U. Assume that

$$G : V \to U^* \text{ is a linear continuous operator,} \tag{21}$$

and denote by $G^* : U \to V^*$ its adjoint operator. Let $u \in U$ be the control parameter and let g_1 denote the unique solution of the linear equation $A_1 g_1 = f$, guaranteed by (19). We associate to (18) the following optimal control problem:

$$\min_{u\in U} \Big\{ \langle u, Gy\rangle_{U\times U^*} - 3\,\langle u, Gg_1\rangle_{U\times U^*} + \langle A_2 y, y\rangle_{V^*\times V} + 2\varphi(y) \Big\}, \tag{22}$$

$$\langle A_1 y, v\rangle_{V^*\times V} = \langle u, Gv\rangle_{U\times U^*} - \langle f, v\rangle_{V^*\times V} \qquad \forall\, v \in V. \tag{23}$$

One of the main features of the the problem (22)–(23) arises from the fact that the operators involved in the nonlinear equation (18) are decoupled. Also, it is obvious to see that the state equation (23) may be written in the equivalent form

$$A_1 y = G^* u - f,$$

which is more familiar with those working in the optimal control theory. Nevertheless, in what follows we prefer to use the weak formulation (23), as it avoids the computation of the adjoint operator which, in practice, could lead to some difficulties.

Note that the properties (19) of A_1 ensure the existence of a unique solution of the variational equation (23). Moreover, we have the following result.

Theorem 3. *Assume that $\varphi : V \to \mathbb{R}$ is a locally Lipschitz function and* (19)–(21) *hold. If $[y^*, u^*] \in V \times U$ is an optimal pair of the problem* (22)–(23)*, then y^* is a solution of the nonlinear equation* (18)*.*

Proof. We take variations around $[y^*, u^*]$ of the form $[y^* + \lambda\xi, u^* + \lambda\omega]$ where $\lambda \in \mathbb{R}$ and $[\xi, \omega] \in V \times U$ satisfies

$$\langle A_1 \xi, v\rangle_{V^*\times V} = \langle w, Gv\rangle_{U\times U^*} \quad \forall\, v \in V. \tag{24}$$

We use the optimality of $[y^*, u^*]$, divide the corresponding inequality by $\lambda > 0$ and then pass to the limit as $\lambda \to 0$ by using (1). As a result we obtain

$$\begin{aligned}0 \leq \langle u^*, G\xi\rangle_{U\times U^*} + \langle \omega, Gy^*\rangle_{U\times U^*} + 2\,\langle A_2 y^*, \xi\rangle_{V^*\times V}\\ -3\,\langle \omega, Gg_1\rangle_{U\times U^*} + 2\varphi^0(y^*;\xi).\end{aligned}$$

Since $\langle A_1\xi, g_1\rangle_{V^*\times V} = \langle \xi, f\rangle_{V\times V^*}$ and ξ is an arbitrary element of V, the previous inequality combined with the definition (2) and the state equation show that y^* is a solution of the nonlinear equation (18), which concludes the proof. □

We end this section with the remark that both problems (8)–(9) and (22)–(23) solve the nonlinear multivalued equation (3). These two optimal control problems represent distinct versions of the control variational method and, therefore, illustrate the flexibility of this method in the study of nonlinear equations.

3 Applications to elastic contact problems

A large number of contact problems with elastic materials can be cast into the form of a variational inequality as in (7) in which the unknown is the displacement field. In this section we illustrate the use of the control variational method in the study of such problems.

The physical setting is the following. An elastic body occupies, in the reference configuration, an open bounded connected set $\Omega \subset \mathbb{R}^3$ with Lipschitz continuous boundary Γ, decomposed into three parts $\overline{\Gamma}_1$, $\overline{\Gamma}_2$ and $\overline{\Gamma}_3$, with Γ_1, Γ_2 and Γ_3 being relatively open and mutually disjoint. The body is clamped on Γ_1 and we assume that $\mathrm{meas}\,(\Gamma_1) > 0$. Surface traction of density $\boldsymbol{f}_2$ act on Γ_2 and volume forces of density $\boldsymbol{f}_0$ act in Ω. Here and in the rest of the paper we use bold face letters for vectors and tensors. The body is in contact on Γ_3 with an obstacle, the so-called foundation.

We are interested to describe the mathematical model of the equilibrium of the elastic body in the physical setting above. To this end we use the notation $\boldsymbol{x} = (x_i)$ for a typical point in Ω and we denote by $\boldsymbol{\nu} = (\nu_i)$ the outward unit normal at Γ_3. Here and below the indices i, j, k, l run between 1 and 3 and, unless stated otherwise, the summation convention over repeated indices is used. Also, the index that follows a coma indicates a partial derivative with the corresponding component of the spatial variable $\boldsymbol{x}$. We denote by $\boldsymbol{y} = (y_i)$, $\boldsymbol{\sigma} = (\sigma_{ij})$, and $\boldsymbol{e}(\boldsymbol{y}) = (e_{ij}(\boldsymbol{y}))$ the displacement vector, the stress tensor, and the linearized strain tensor, respectively. We note that sometimes we do not indicate the dependence of the variables on the spacial variable $\boldsymbol{x}$ and we recall that the components of the linearized strain tensor $\boldsymbol{e}(\boldsymbol{y})$ are given by

$$e_{ij}(\boldsymbol{y}) = \frac{1}{2}\,(y_{i,j} + y_{j,i}) \tag{25}$$

where $y_{i,j} = \partial y_i/\partial x_j$. The state of the system is completely determined by $(\boldsymbol{y}, \boldsymbol{\sigma})$, in other words, the displacement field $\boldsymbol{y}$ and the stress field $\boldsymbol{\sigma}$ will play the role as the unknowns in elastic contact problems.

We denote by $\mathbb{R}^d$ the d-dimensional real linear space and the symbol $\mathbb{S}^d$ stands for the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. The canonical inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i, \quad \|\boldsymbol{v}\| = (\boldsymbol{v}\cdot\boldsymbol{v})^{1/2} \quad \forall\,\boldsymbol{u} = (u_i),\, \boldsymbol{v} = (v_i) \in \mathbb{R}^d, \tag{26}$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{\theta} = \sigma_{ij}\theta_{ij}, \quad \|\boldsymbol{\theta}\| = (\boldsymbol{\theta}\cdot\boldsymbol{\theta})^{1/2} \quad \forall\,\boldsymbol{\sigma} = (\sigma_{ij}),\, \boldsymbol{\theta} = (\theta_{ij}) \in \mathbb{S}^d, \tag{27}$$

respectively, and note that below we use the space $\mathbb{R}^d$ for $d = 3$ and $d = 9$ and the space $\mathbb{S}^d$ for $d = 3$. Finally, we use standard notation for the spaces L^p spaces and Sobolev spaces associated to Ω and Γ.

The classical formulation of the problem we consider in this section is the following: find a displacement field $\boldsymbol{y} : \Omega \to \mathbb{R}^3$ and a stress field $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^3$ such that

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{e}(\boldsymbol{y}) \qquad \text{in } \Omega, \tag{28}$$

$$\operatorname{Div}\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \qquad \text{in } \Omega, \tag{29}$$

$$\boldsymbol{y} = \boldsymbol{0} \qquad \text{on } \Gamma_1, \tag{30}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \qquad \text{on } \Gamma_2, \tag{31}$$

$$\boldsymbol{y} \in K, \ -\boldsymbol{\sigma\nu}\cdot(\boldsymbol{v} - \boldsymbol{y}) \le j(\boldsymbol{v}) - j(\boldsymbol{y}) \quad \forall\,\boldsymbol{v} \in K \qquad \text{on } \Gamma_3. \tag{32}$$

Here (28) represents the linear elastic constitutive law constitutive law, i.e.

$$\sigma_{ij} = a_{ijkl}\, e_{kl}(\boldsymbol{y})$$

with $\mathcal{A} = (a_{ijkl})$ being the elasticity tensor. Equation (29) represents the equation of equilibrium and we use it since we assume that the process is static. Conditions (30) and (31) represent the displacement and traction boundary conditions, respectively. Finally, inequality (32) is the contact condition in which K represents the set of admissible displacement fields and $j : \Gamma_3 \times \mathbb{R}^3 \to \mathbb{R}$ is a given function to be precised in the sequel. Examples and detailed explanations of inequality problems in Contact Mechanics which lead to boundary conditions of this subdifferential form will be presented at the end of this section.

We turn now to the variational formulation of the contact problem (28)–(32). To this end we consider the closed space of the space $H^1(\Omega)^3$ given by

$$V(\Omega) = \{\ \boldsymbol{v} \in H^1(\Omega)^3\ :\ \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_1\ \}.$$

Since $\mathrm{meas}(\Gamma_1) > 0$, Korn's inequality implies that $V(\Omega)$ is a Hilbert space with the inner product

$$(\boldsymbol{y}, \boldsymbol{v})_{V(\Omega)} = \int_\Omega \boldsymbol{e}(\boldsymbol{y}) \cdot \boldsymbol{e}(\boldsymbol{v})\, dx$$

and the associated norm $\|\cdot\|_{V(\Omega)}$. As usual, we denote by $V(\Omega)^*$ the dual of the space $V(\Omega)$ and let $\langle\cdot,\cdot\rangle_{V(\Omega)^*\times V(\Omega)}$ be the duality pairing between $V(\Omega)^*$ and $V(\Omega)$. We note that we have $V(\Omega) \subset L^2(\Omega)^3 \subset V(\Omega)^*$ with compact and dense embeddings.

We assume in what follows that the components a_{ijkl} of the elasticity tensor are bounded and satisfy the usual properties of symmetry and ellipticity, that is

$$\begin{cases} \text{(a) } a_{ijkl} \in L^\infty(\Omega). \\ \text{(b) } a_{ijkl} = a_{jikl} = a_{klij}. \\ \text{(c) There exists } m > 0 \text{ such that} \\ \qquad a_{ijkl}\xi_{ij}\xi_{kl} \geq m\|\boldsymbol{\xi}\|^2 \qquad \forall\, \boldsymbol{\xi} = (\xi_{ij}) \in \mathbb{S}^3, \text{ a.e. in } \Omega. \end{cases} \tag{33}$$

Then, we consider the elasticity operator $A : V(\Omega) \to V(\Omega)^*$ defined by

$$\begin{aligned} \langle A\boldsymbol{y}, \boldsymbol{v}\rangle_{V(\Omega)^*\times V(\Omega)} &= \int_\Omega \mathcal{A}\boldsymbol{e}(\boldsymbol{y}) \cdot \boldsymbol{e}(\boldsymbol{v})\, dx \qquad (34) \\ &= \int_\Omega a_{ijkl}\, e_{ij}(\boldsymbol{y}) e_{kl}(\boldsymbol{v})\, dx \quad \forall\, \boldsymbol{y},\, \boldsymbol{v} \in V(\Omega). \end{aligned}$$

We also assume that the set of admissible displacements fields satisfies

$$K \text{ is a nonempty, closed, convex subset of } V(\Omega). \tag{35}$$

Let $\varphi : V(\Omega) \to (-\infty, +\infty]$ be the functional

$$\varphi(\boldsymbol{v}) = \begin{cases} \displaystyle\int_{\Gamma_3} j(\boldsymbol{v})\, da & \text{if } \boldsymbol{v} \in K, \\ +\infty & \text{otherwise} \end{cases} \tag{36}$$

where $j(.)$ is a real functional to be defined in the sequel and such that

$$\begin{gathered}\varphi : K \to \mathbb{R} \text{ is a convex, proper, lower semicontinuous} \\ \text{function such that } \varphi(\boldsymbol{v}) < +\infty \quad \forall\, \boldsymbol{v} \in K.\end{gathered} \tag{37}$$

Finally, for the body force and surface tractions we assume

$$\boldsymbol{f}_0 \in L^2(\Omega)^3, \qquad \boldsymbol{f}_2 \in L^2(\Gamma_2)^3, \tag{38}$$

and we denote by $\boldsymbol{f}$ the element of $V(\Omega)^*$ given by

$$\langle \boldsymbol{f}, \boldsymbol{v} \rangle_{V(\Omega)^* \times V(\Omega)} = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{v}\, da \quad \forall\, \boldsymbol{v} \in V(\Omega). \tag{39}$$

It is straightforward to show that if $(\boldsymbol{y}, \boldsymbol{\sigma})$ is a pair of sufficiently regular functions satisfying (29)–(32) then $\boldsymbol{y} \in K$ and, moreover,

$$\begin{aligned}&\int_\Omega \boldsymbol{\sigma} \cdot (\boldsymbol{e}(\boldsymbol{v}) - \boldsymbol{e}(\boldsymbol{y}))\, dx + \int_{\Gamma_3} j(\boldsymbol{v})\, da - \int_{\Gamma_3} j(\boldsymbol{y})\, da \\ &\qquad \geq \int_\Omega \boldsymbol{f}_0 \cdot (\boldsymbol{v} - \boldsymbol{y})\, dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot (\boldsymbol{v} - \boldsymbol{y})\, da \quad \forall\, \boldsymbol{v} \in K.\end{aligned}$$

Combining this inequality with the constitutive law (28) and using the notation (34), (36), (39) we obtain the following variational formulation of the problem (28)–(32) with the displacement as the unknown: find a displacement field $\boldsymbol{y} \in V(\Omega)$ such that

$$\langle A\boldsymbol{y}, \boldsymbol{v} - \boldsymbol{y} \rangle_{V(\Omega)^* \times V(\Omega)} + \varphi(\boldsymbol{v}) - \varphi(\boldsymbol{y}) \geq \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{y} \rangle_{V(\Omega)} \quad \forall\, \boldsymbol{v} \in V(\Omega). \tag{40}$$

Note that inequality (40) includes implicitly the constraint $\boldsymbol{y} \in K$, see (36), (37).

In what follows we use the abstract results in Section 2 with the choice $V = V(\Omega)$. To this end we consider the optimal control problem.

$$\min_{\boldsymbol{y} \in V(\Omega)} \Big\{ \langle \boldsymbol{u}, \boldsymbol{y} \rangle_{V(\Omega)^* \times V(\Omega)} - 3\, \langle \boldsymbol{u}, \boldsymbol{g} \rangle_{V(\Omega)^* \times V(\Omega)} + 2\, \varphi(\boldsymbol{y}) \Big\}, \tag{41}$$

$$A\boldsymbol{y} = \boldsymbol{u} - \boldsymbol{f}. \tag{42}$$

Here $\boldsymbol{u} = (u_i) \in V(\Omega)^*$ is the control parameter, $A : V(\Omega) \to V(\Omega)^*$ is the linear elasticity differential operator, (34), and $\boldsymbol{g} \in V(\Omega)$ is the solution of the linear equation $A\boldsymbol{g} = \boldsymbol{f}$. We note that the advantage of the transformation of the variational inequality (40) into the optimal control problem (41)–(42) arises from the fact that it "decouples" the nonlinear part corresponding to $\varphi(\cdot)$, which has just to be computed, after solving (42).

Using assumptions (33) it is easy to see that the operator A satisfies assumption (4) and, using (35)–(37) it follows that (12) holds, too. Therefore, we may apply Theorem 2 to see that both the variational inequality (40) and the optimal control problem (41)–(42) have a unique solution. Moreover, $\boldsymbol{y}^* \in V(\Omega)$ is the solution of (40) if and only if $[\boldsymbol{y}^*, A\boldsymbol{y}^* + \boldsymbol{f}]$ is the optimal pair of (41)–(42). And, again, the constraint $\boldsymbol{y}^* \in K$ is implicitly included.

Next, we consider the case of linear isotropic materials. It is well know that in this case the elasticity tensor is characterized by only two constants, the Lamé coefficients, denoted λ and μ, which satisfy the inequalities $\lambda > 0$ and $\mu > 0$. The constitutive law is given by

$$\boldsymbol{\sigma} = 2\,\mu\,\boldsymbol{e}(\boldsymbol{y}) \,+ \lambda\, tr(\boldsymbol{e}(\boldsymbol{y}))\,\boldsymbol{I}$$

where $tr(\boldsymbol{e}(\boldsymbol{y}))$ denotes the trace of the tensor $\boldsymbol{e}(\boldsymbol{y})$ defined by $tr(\boldsymbol{e}(\boldsymbol{y})) = e_{ii}(\boldsymbol{y})$, and $\boldsymbol{I}$ denotes the identity tensor on $\mathbb{R}^3$. It follows from above that the elasticity operator is

$$\mathcal{A}(\boldsymbol{e}) = 2\,\mu\,\boldsymbol{e} \,+ \lambda\, tr(\boldsymbol{e})\,\boldsymbol{I}.$$

In components, we have

$$\sigma_{ij} = 2\,\mu\, e_{ij}(\boldsymbol{y}) \,+ \lambda\, e_{kk}(\boldsymbol{y})\,\delta_{ij}$$

where δ_{ij} is the Kronecker symbol, i.e., δ_{ij} are the components of the unit matrix 3×3. As a consequence, the operator (34) becomes

$$\langle A\boldsymbol{y}, \boldsymbol{v}\rangle_{V(\Omega)^*\times V(\Omega)} = \int_\Omega \Big[\lambda e_{ii}(\boldsymbol{y})e_{jj}(\boldsymbol{v}) + 2\mu e_{ij}(\boldsymbol{y})(e_{ij}(\boldsymbol{v})\Big]\,dx \quad \forall\,\boldsymbol{y},\,\boldsymbol{v} \in V(\Omega)$$

and, therefore, the variational inequality (40) reads

$$\int_\Omega \Big[\lambda e_{ii}(\boldsymbol{y})(e_{jj}(\boldsymbol{v}) - e_{jj}(\boldsymbol{y})) + 2\mu e_{ij}(\boldsymbol{y})(e_{ij}(\boldsymbol{v}) - e_{ij}(\boldsymbol{y})\Big]\,dx \tag{43}$$

$$+\varphi(\boldsymbol{v}) - \varphi(\boldsymbol{y}) \geq \int_\Omega \boldsymbol{f}_0 \cdot (\boldsymbol{v} - \boldsymbol{y})\,dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot (\boldsymbol{v} - \boldsymbol{y})\,da \quad \forall\,\boldsymbol{v} \in V(\Omega).$$

We turn now to a formulation of inequality (43) which will allow the use of Theorem 3 with the linear continuous operator $G : V(\Omega) \to L^2(\Omega)^9$ which associates to each vector its Jacobian matrix, that is

$$G(\boldsymbol{v}) = \nabla \boldsymbol{v}.$$

We consider the following optimal control problem:

$$\min_{\boldsymbol{y}\in V(\Omega)} \Big\{ \int_\Omega \boldsymbol{w} \cdot \nabla \boldsymbol{y}\, dx - 3 \int_\Omega \boldsymbol{w} \cdot \nabla \boldsymbol{g}_1\, dx + \lambda \int_\Omega (\operatorname{div} \boldsymbol{y})^2\, dx \tag{44}$$

$$+\mu \int_\Omega \Big[\Big(\frac{\partial y_1}{\partial x_1}\Big)^2 + \Big(\frac{\partial y_2}{\partial x_2}\Big)^2 + \Big(\frac{\partial y_3}{\partial x_3}\Big)^2 \Big]\, dx$$

$$+2\,\mu \int_\Omega \Big(\frac{\partial y_1}{\partial x_2}\frac{\partial y_2}{\partial x_1} + \frac{\partial y_1}{\partial x_3}\frac{\partial y_3}{\partial x_1} + \frac{\partial y_2}{\partial x_3}\frac{\partial y_3}{\partial x_2} \Big)\, dx + 2\varphi(\boldsymbol{y}) \Big\},$$

$$\mu \int_\Omega \nabla \boldsymbol{y} \cdot \nabla \boldsymbol{v}\, dx = \int_\Omega \boldsymbol{w} \cdot \nabla \boldsymbol{v}\, dx - \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\, dx \tag{45}$$

$$- \int_{\Gamma_3} \boldsymbol{f}_2 \cdot \boldsymbol{v}\, da \quad \forall\, \boldsymbol{v} \in V(\Omega).$$

Here $\boldsymbol{w} \in L^2(\Omega)^9$ is the control parameter and note that no constraints are imposed in the problem (44)–(45). Also, the function $\boldsymbol{g}_1$ in (44) is defined by

$$\mu \int_\Omega \nabla \boldsymbol{g}_1 \cdot \nabla \boldsymbol{v}\, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx \qquad \forall\, \boldsymbol{v} \in V(\Omega). \tag{46}$$

Relations (45) or (46) are equivalent with the solution of three independent Laplace equations that define the operator A_1 appearing in Theorem 3. The operator $A_2 : V(\Omega) \to V(\Omega)^*$ is defined by the equality

$$\langle A_2 \boldsymbol{y}, \boldsymbol{v} \rangle_{V(\Omega)\times V(\Omega)^*} = \lambda \int_\Omega (\operatorname{div} \boldsymbol{y})\, (\operatorname{div} \boldsymbol{v})\, dx \tag{47}$$

$$+\mu \int_\Omega \Big(\frac{\partial y_1}{\partial x_1}\frac{\partial v_1}{\partial x_1} + \frac{\partial y_2}{\partial x_2}\frac{\partial v_2}{\partial x_2} + \frac{\partial y_3}{\partial x_3}\frac{\partial v_3}{\partial x_3} \Big)\, dx$$

$$+2\mu \int_\Omega \Big(\frac{\partial y_1}{\partial x_2}\frac{\partial v_2}{\partial x_1} + \frac{\partial y_1}{\partial x_3}\frac{\partial v_3}{\partial x_1} + \frac{\partial y_2}{\partial x_3}\frac{\partial v_3}{\partial x_2} \Big)\, dx \quad \forall\, \boldsymbol{y},\, \boldsymbol{v} \in V(\Omega).$$

We have all the ingredients to state and prove the following result.

Theorem 4. *Assume that* $[\boldsymbol{y}^*, \boldsymbol{w}^*] \in V(\Omega) \times L^2(\Omega)^9$ *is an optimal pair of the problem* (44)–(45)*. Then* $\boldsymbol{y}^*$ *is a solution of the variational inequality* (43)*.*

To prove Theorem 4 we use computations and arguments similar to those from the proofs of Theorems 1–3. Nevertheless, for the convenience of the reader we present below a sketch of the proof.

Proof. Let $[\boldsymbol{y}^*, \boldsymbol{w}^*] \in V(\Omega) \times L^2(\Omega)^9$ be an optimal pair of (44)–(45) and take variation of the form $[\boldsymbol{y}^* + \theta\boldsymbol{\xi}, \boldsymbol{w}^* + \theta\boldsymbol{\omega}]$ where $\theta \in \mathbb{R}$ and $[\boldsymbol{\xi}, \boldsymbol{\omega}] \in V(\Omega) \times L^2(\Omega)^9$ satisfies the homogeneous variant of (45). After same lengthly but standard computation we get

$$
\begin{aligned}
0 = &\int_\Omega \boldsymbol{\omega} \cdot \nabla y^* \, dx + \int_\Omega \boldsymbol{w}^* \cdot \nabla \boldsymbol{\xi} \, dx - 3 \int_\Omega \boldsymbol{\omega} : \nabla g_1 \, dx \qquad (48)\\
&+2\lambda \int_\Omega (\operatorname{div} \boldsymbol{y})\,(\operatorname{div} \xi)\, dx + 2\mu \int_\Omega \Big(\frac{\partial y_1^*}{\partial x_1}\frac{\partial \xi_1}{\partial x_1} + \frac{\partial y_2^*}{\partial x_2}\frac{\partial \xi_2}{\partial x_2} + \frac{\partial y_3^*}{\partial x_3}\frac{\partial \xi_3}{\partial x_3} \Big)\, dx\\
&+2\mu \int_\Omega \Big(\frac{\partial y_1^*}{\partial x_2}\frac{\partial \xi_2}{\partial x_1} + \frac{\partial \xi_1}{\partial x_2}\frac{\partial y_2^*}{\partial x_1} + \frac{\partial y_1^*}{\partial x_3}\frac{\partial \xi_3}{\partial x_1}\\
&+ \frac{\partial \xi_1}{\partial x_3}\frac{\partial y_3^*}{\partial x_1} + \frac{\partial y_2^*}{\partial x_3}\frac{\partial \xi_3}{\partial x_2} + \frac{\partial \xi_2}{\partial x_3}\frac{\partial y_3}{\partial x_2} \Big)\, dx\\
&+2\, \langle \boldsymbol{\zeta}^*, \boldsymbol{\xi} \rangle_{V(\Omega)^* \times V(\Omega)}
\end{aligned}
$$

where $\boldsymbol{\zeta}^* \in \partial\varphi(\boldsymbol{y}^*)$. We eliminate now $\boldsymbol{w}$, $\boldsymbol{\omega}$ and $\boldsymbol{g}_1$ from (48) by using (45) with $\boldsymbol{v} = \boldsymbol{\xi}$, the corresponding equation in variations and the definition of $\boldsymbol{g}_1$, respectively. As a result we obtain

$$
\begin{aligned}
0 = 2 &\int_\Omega \Big[\lambda e_{ii}(\boldsymbol{y}^*) e_{jj}(\boldsymbol{\xi}) + 2\mu e_{ij}(\boldsymbol{y}^*) e_{ij}(\boldsymbol{\xi}) \Big]\, dx\\
&-2 \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{\xi}\, dx - 2 \int_{\Gamma_3} \boldsymbol{f}_2 \cdot \boldsymbol{\xi}\, da + 2\, \langle \boldsymbol{\zeta}^*, \boldsymbol{\xi} \rangle_{V(\Omega)^* \times V(\Omega)}.
\end{aligned}
$$

The choice $\boldsymbol{\xi} = \boldsymbol{y}^* - \boldsymbol{v}$, with $\boldsymbol{v}$ being an arbitrary element in $\in V(\Omega)$, is admissible in the above equation. Therefore, by using the definition (13) we deduce that $\boldsymbol{y}^*$ satisfies (43) which concludes the proof. □

We recall that the control variational method was used in [19] in the study of a simpler displacement-traction boundary value problem with linearly elastic materials. There, a direct existence argument was used in solving an optimal control problem of the form (44)–(45) and the coercivity property of the operator A_2 defined in (47) is open. Theorem 4 shows that the solution of the elasticity system is reduced to solving iteratively several independent Laplace equations. The nonlinearity and the "unconvenient" part of the linear operator may be moved in the cost functional. It has just to be computed and is not involved in the solution of the linear state equation associated to the control problem. This is the advantage of the control variational method presented above.

Examples of subdifferential boundary conditions. We turn now to present examples of frictionless or frictional contact conditions which lead to an inequality of the form (32) such that assumptions (35) and (37) hold. We conclude that the control variational method described above can be applied in the study of the contact problems for each of the examples below. Everywhere in this section we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and the tangential components of a vector field $\boldsymbol{v} \in H^1(\Omega)^3$, respectively, defined by

$$v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}, \quad \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}.$$

In particular, we use the notation y_ν and $\boldsymbol{y}_\tau$ for the normal and the tangential components of the displacement field $\boldsymbol{y}$. We also denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the normal and tangential components of the stress field $\boldsymbol{\sigma}$ on the boundary, that is

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}.$$

The list of examples below is far from being exhaustive since many other examples can be considered by combining the different contact conditions and friction laws. Details and mechanical interpretations on the boundary conditions described below can be found in [4, 15] and the references therein.

Example 1. (*Signorini frictionless condition.*) This contact condition is of the form

$$y_\nu \le 0, \quad \sigma_\nu \le 0, \quad \sigma_\nu y_\nu = 0, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{0} \quad \text{on } \Gamma_3. \tag{49}$$

The equality $\boldsymbol{\sigma}_\tau = \boldsymbol{0}$ in (50) shows that the friction force vanishes, i.e. the contact is frictionless. This represents an idealization of the process, since even completely lubricated surfaces generate shear resistance to tangential motions. The rest of conditions in (49) represent the well know Signorini conditions which describe the contact with a rigid obstacle.

The set of admissible test functions K consists of those elements of $V(\Omega)$ whose normal component is negative on Γ_3, i.e.

$$K = \{\, \boldsymbol{v} \in V(\Omega) : \, v_\nu \le 0 \text{ on } \Gamma_3 \,\}.$$

It is straightforward to show that if $(\boldsymbol{y}, \boldsymbol{\sigma})$ is a pair of regular functions satisfying (49) then

$$\boldsymbol{\sigma}\,\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{y}) \ge 0 \qquad \forall\, \boldsymbol{v} \in K,$$

a.e. on Γ_3. Thus, the contact condition (32) holds with $j(\boldsymbol{v}) = 0$ and, therefore, the functional φ of (36) is the indicator function of the set K, i.e.

$$\varphi(\boldsymbol{v}) = \begin{cases} 0 & \text{if } \boldsymbol{v} \in K, \\ +\infty & \text{otherwise.} \end{cases}$$

It is easy to see that in this case conditions (35) and (37) hold.

Example 2. (*Bilateral contact with Tresca's friction law.*) This contact condition is of the form

$$\left.\begin{array}{l} y_\nu = 0, \quad \|\boldsymbol{\sigma}_\tau\| \le g, \\ \|\boldsymbol{\sigma}_\tau\| < g \;\Rightarrow\; \boldsymbol{y}_\tau = \boldsymbol{0}, \\ \|\boldsymbol{\sigma}_\tau\| = g \;\Rightarrow\; \exists\, \xi \ge 0 \text{ such that } \boldsymbol{\sigma}_\tau = -\xi \boldsymbol{y}_\tau \end{array}\right\} \quad \text{on } \Gamma_3. \tag{50}$$

The equality $y_\nu = 0$ in (50) shows that contact is bilateral, i.e., there is no loss of contact during the process. The rest of condition represents the Tresca friction law in which $g \in L^\infty(\Gamma_3)$ is a positive function which represents the friction bound, i.e., the magnitude of the limiting friction traction at which slip begins.

The set of admissible test functions K consists of those elements of $V(\Omega)$ whose normal component vanishes on Γ_3, i.e.

$$K = \{\, \boldsymbol{v} \in V(\Omega) : \, v_\nu = 0 \text{ on } \Gamma_3 \,\}. \tag{51}$$

It is straightforward to show that if $(\boldsymbol{y}, \boldsymbol{\sigma})$ is a pair of regular functions satisfying (50) then

$$\boldsymbol{\sigma}\,\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{y}) \ge g\,\|\boldsymbol{y}_\tau\| - g\,\|\boldsymbol{v}_\tau\| \qquad \forall\, \boldsymbol{v} \in K,$$

a.e. on Γ_3. Thus, the frictional contact condition (32) holds with $j(\boldsymbol{v}) = g\,\|\boldsymbol{v}_\tau\|$ and, therefore, the functional φ defined by (36) satisfies

$$\varphi(\boldsymbol{v}) = \int_{\Gamma_3} g\,\|\boldsymbol{v}_\tau\|\,da \qquad \forall\,\boldsymbol{v} \in K.$$

It is easy to see that in this case conditions (35) and (37) hold, too.

Example 3. (*Bilateral contact with regularized friction.*) We consider the boundary conditions

$$y_\nu = 0, \quad \boldsymbol{\sigma}_\tau = -g\,\frac{\boldsymbol{y}_\tau}{\sqrt{\|\boldsymbol{y}_\tau\|^2 + \rho^2}} \quad \text{on } \Gamma_3, \tag{52}$$

where $\rho > 0$ is a regularization parameter and, again, $g \in L^\infty(\Gamma_3)$ is a positive function. The frictional contact condition in (52) represents a regularization of the Tresca friction law in Example 2 and is used in the literature mainly for numerical reasons. Note that, formally, we can recover (50) from (52) in the limit as $\rho \to 0$.

It is straightforward to show that if $(\boldsymbol{u}, \boldsymbol{\sigma})$ is a pair of regular functions satisfying (52), then the contact condition (32) holds with K given by (51) and j is the convex function given by

$$j(\boldsymbol{v}) = g\,\frac{\boldsymbol{v}_\tau}{\sqrt{\|\boldsymbol{v}_\tau\|^2 + \rho^2}}\,.$$

The corresponding contact functional φ satisfies

$$\varphi(\boldsymbol{v}) = \int_{\Gamma_3} g\,\frac{\boldsymbol{v}_\tau}{\sqrt{\|\boldsymbol{v}_\tau\|^2 + \rho^2}}\,da \qquad \forall\,\boldsymbol{v} \in K.$$

Clearly, conditions (35) and (37) hold in this case.

Example 4. (*Bilateral contact with power-law friction.*) We consider now the boundary conditions

$$y_\nu = 0, \quad \boldsymbol{\sigma}_\tau = -\mu\,\|\boldsymbol{y}_\tau\|^{p-1}\boldsymbol{y}_\tau \quad \text{on } \Gamma_3, \tag{53}$$

where μ is the coefficient of friction and $0 < p \leq 1$. Here, the tangential shear is proportional to the power p of the tangential displacement. Such a boundary condition arises when the contact surface is lubricated with a thin layer of non-Newtonian fluid.

Assume that $\mu \in L^\infty(\Gamma_3)$ is a positive function. Then, it is straightforward to show that if $(\boldsymbol{y}, \boldsymbol{\sigma})$ is a pair of regular functions satisfying (53), then the condition (32) holds with K given by (51) and

$$j(\boldsymbol{v}) = \frac{\mu}{p+1}\,\|\boldsymbol{v}_\tau\|^{p+1}.$$

We deduce from the definition (36) of the functional φ that

$$\varphi(\boldsymbol{v}) = \frac{1}{p+1}\int_{\Gamma_3} \mu\,\|\boldsymbol{v}_\tau\|^{p+1} da \qquad \forall\, \boldsymbol{v} \in K.$$

We note that, again, conditions (35) and (37) are satisfied in this case.

Example 5. (*Contact with imposed normal stress and Coulomb's friction.*) We consider the boundary conditions

$$\left.\begin{array}{l} -\sigma_\nu = F, \quad \|\boldsymbol{\sigma}_\tau\| \le \mu\,|\sigma_\nu|, \\ \|\boldsymbol{\sigma}_\tau\| < \mu\,|\sigma_\nu| \;\Rightarrow\; \boldsymbol{y}_\tau = \boldsymbol{0}, \\ \|\boldsymbol{\sigma}_\tau\| = \mu\,|\sigma_\nu| \;\Rightarrow\; \exists\,\xi \ge 0 \text{ such that } \boldsymbol{\sigma}_\tau = -\xi \boldsymbol{y}_\tau \end{array}\right\} \quad \text{on } \Gamma_3. \tag{54}$$

Here F and μ are given positive functions which belong to $L^2(\Gamma_3)$ and $L^\infty(\Gamma_3)$, respectively. The first equality in (54) shows that the normal stress is imposed on the contact surface and the rest of relations represent the classical Coulomb's law of dry friction in which μ denotes the coefficient of friction.

It is straightforward to show that if $(\boldsymbol{y}, \boldsymbol{\sigma})$ is a pair of regular functions satisfying (54), then the contact condition (32) holds with $K = V(\Omega)$ and

$$j(\boldsymbol{v}) = F\,v_\nu + \mu\,F\,\|\boldsymbol{v}_\tau\|.$$

Moreover, from the definition (36) we deduce that

$$\varphi(\boldsymbol{v}) = \int_{\Gamma_3} (F\,v_\nu + \mu\,F\|\boldsymbol{v}_\tau\|)\,da \quad \forall\, \boldsymbol{v} \in V(\Omega).$$

We note that, again, conditions (35) and (37) are satisfied in this case.

Example 6. (*Normal compliance frictionless condition.*) We consider the boundary conditions

$$-\sigma_\nu = \kappa\,(y_\nu^+)^q, \quad \boldsymbol{\sigma}_\tau = 0 \quad \text{on } \Gamma_3. \tag{55}$$

in which $\kappa \in L^\infty(\Gamma_3)$ is a positive function, $0 < q \leq 1$ and r^+ denotes the positive part of r, i.e. $r^+ = max\,\{r, 0\}$. The first equality in (55) represents the so called normal compliance contact condition, in which κ denotes the stiffness coefficient of the surface and q is the normal exponent. It assigns a reactive normal pressure that depends on a power of the penetration of the asperities on the body's surface and on the foundation, which vanish when there is separation, i.e. when $y_\nu < 0$. The second equality in (55) represents, again, the frictionless condition.

Using the inequality

$$(u^+)^q(v-u) \leq \frac{1}{q+1}(v^+)^{q+1} - \frac{1}{q+1}(u^+)^{q+1} \quad \forall\, u,\, v \in \mathbb{R}$$

it is easy to see that, if $(\boldsymbol{y}, \boldsymbol{\sigma})$ is a pair of regular functions satisfying (55), then almost everywhere on Γ_3 the following inequality holds:

$$-\boldsymbol{\sigma}\,\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{y}) \leq \frac{\kappa}{q+1}\,(v_\nu^+)^{q+1} - \frac{\kappa}{q+1}\,(y_\nu^+)^{q+1} \quad \forall\, \boldsymbol{v} \in V(\Omega).$$

So, the contact condition (32) holds with $K = V(\Omega)$ and $j(\boldsymbol{v}) = \frac{\kappa}{q+1}\,(v_\nu^+)^{q+1}$. Moreover, from the definition (36) we deduce that

$$\varphi(\boldsymbol{v}) = \int_{\Gamma_3} \frac{\kappa}{q+1}\,(v_\nu^+)^{q+1}\,da \qquad \forall\, \boldsymbol{v} \in V(\Omega).$$

It is easy to see that in this case conditions (35) and (37) hold.

Example 7. (*Elastic contact with power-law friction.*) In this example the normal stress is proportional to a power of the normal displacement, while the tangential shear is proportional to another power of the tangential displacement. Thus, the boundary conditions are the following:

$$-\sigma_\nu = \kappa\,|y_\nu|^{q-1} y_\nu, \quad \boldsymbol{\sigma}_\tau = -\mu\,\|\boldsymbol{y}_\tau\|^{p-1}\boldsymbol{y}_\tau \quad \text{on } \Gamma_3. \tag{56}$$

Here $\mu \in L^\infty(\Gamma_3)$ and $\kappa \in L^\infty(\Gamma_3)$ are positive functions and the exponents p, q, are such that $0 < p, q \leq 1$.

We choose $K = V(\Omega)$ and

$$j(\boldsymbol{v}) = \frac{\kappa}{q+1}|v_\nu|^{q+1} + \frac{\mu}{p+1}\,\|\boldsymbol{v}_\tau\|^{p+1}.$$

Then, the contact condition (32) holds and, from the definition (36) we deduce that

$$\varphi(\boldsymbol{v}) = \int_{\Gamma_3} \frac{\kappa}{q+1}|v_\nu|^{q+1} + \frac{\mu}{p+1}\,\|\boldsymbol{v}_\tau\|^{p+1}\,da \qquad \forall\,\boldsymbol{v} \in V(\Omega).$$

It is easy to see that in this case conditions (35) and (37) hold, too.

We end this section with the remark that in Examples 2–7 above the set of admissible displacement fields is a linear subspace of the space $V(\Omega)$. This is dictated by the structure of the contact conditions which do not involve unilateral restriction on the normal displacement field. Unlike this situation, the Signorini contact problem presented in Example 1 involves unilateral conditions for the normal displacement. And, therefore, in this case the set K is a convex subset of $V(\Omega)$ which is not a subspace of $V(\Omega)$.

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