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LIPSCHITZ SOLUTIONS OF OPTIMAL CONTROL PROBLEMS WITH STATE CONSTRAINTS OF ARBITRARY ORDER*

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Abstract

In this paper we generalize to an arbitrary order, under minimal hypotheses, some sufficient conditions for Lipschitz continuity of the optimal control. The proof combines the approach by Hager in 1979 for dealing with first-order state constraints, and the high-order alternative formulation of the optimality conditions. It takes into account the restrictive sign conditions taken into account in some recent papers.

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1 Introduction

In this paper we discuss optimal control problems with running state constraints. They are recognized as an important and difficult class of optimal control problems. They were discussed already at the very beginning of the theory (Pontryagin et al. [17]). Alternative optimality systems, motivated by reformulations in which the control enters in (a derivative of) the state

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constraint, appeared in Bryson, Denham and Dreyfus [7], and Jacobson, Lele and Speyer [14]. A clarification of the theory was brought in Maurer [16]. In this references the theory of high order alternative systems is presented, assuming a geometric hypothesis of finite number of arcs over the optimal trajectory. A recent reference along this line is Bonnans and Hermant [3] where also an analysis related to second-order optimality conditions is provided, assuming the hypotheses of linear independence of certain time derivatives of the state constraints, and of strong convexity of the Hamiltonian.

Another path was followed by Hager [12], who introduced a transformation (which actually is a "global" form of the first-order alternative optimality system in [7, 14, 16]) allowing to prove, under suitable hypotheses (well-posed first-order state constraints and strongly convex Hamiltonian), the Lipschitz continuity of the optimal control. This result is limited to first-order constraints, and to some specific form of the optimal control problem, but has no geometric hypothesis.

Proving the Lipschitz continuity of the solution of an optimal control problem is of interest for obtaining error bounds for the discretization, see Dontchev and Hager [10].

There has been a renewed interest on these questions in the recent years. Shvartsman and Vinter [19] considered the case of first-order state constraints combined with control constraints, the latter possibly in an abstract form. Do Rosario de Pinho and Shvartsman [9] extended some of these results to the case when mixed state and control constraints are also present. A standard hypothesis in the field is the one of linear independence of gradients w.r.t. the control of active constraints (more precisely, active mixed constraints and total derivatives of active state constraints). In these two references, this standard hypothesis is weakened by introducing a sign condition on the regularity hypothesis related to the combinations of derivatives of the state constraints (see (34)). Independently, Hermant [13] showed how to extend Hager's result when all state constraints are of second-order.

There are a few generalizations of these techniques for differential systems outside of the field of ODEs. In the case of integral equations, Bonnans and de la Vega [1] extended the Lipschitz property in the case of first-order state constraints. In the case of optimal control of partial differential equations, the first-order optimality system has been introduced by Bonnans and Jaisson [4] in order to prove the continuity of the control and multipliers for a parabolic equation with first-order state constraints. The Lipschitz continuity of the control seems unfortunately out of reach in that case, due to the lack of regularity in time of the solutions of parabolic equations.

In this paper we will provide an extension of Hager's result to the case of state constraints of arbitrary order, combined with mixed state and control constraints. We obtain the Lipschitz continuity of the control and of the multipliers associated with first-order state constraints. Note that in general multipliers associated with higher order state constraints are not continuous, see e.g. the discussion in [13]. We use weaker hypotheses than those in [13]. We provide also variant of the result of continuity of the optimal control with a slightly weaker hypothesis than in [2].

The paper is organized as follows. In section 2 we establish that the solutions of the optimal control problem satisfy the first-order optimality condition. The only assumption here deals with the mixed constraints, and allows to obtain a regularity result for the associated multiplier. We give in section 3 the results on continuity and Lipschitz continuity of the control and multipliers associated with the first-order state constraints. An appendix provides two technical lemmas.

2 First-order extremals

2.1 Statement

Consider state constrained optimal control problems of the following type:

$$\left\{ \begin{array}{ll} \operatorname{Min} \displaystyle\int_0^T \ell(u_t, y_t)\mathrm{d}t + \phi(y_0, y_T); & \\ \text{(i)} \quad \dot{y}_t = f(u_t, y_t); \;\; t \in (0,T); & \\ \text{(ii)} \quad g(y_t) \le 0; \;\; t \in [0,T], & \\ \text{(iii)} \quad c(u_t, y_t) \le 0, \text{ for a.a. } t \in (0,T), & \\ \text{(iv)} \quad \Phi(y_0, y_T) \in K, & \end{array} \right. \tag{1}$$

with $\ell : I\!R^m \times I\!R^n \to I\!R$, $\phi : I\!R^n \times I\!R^n \to I\!R$, $f : I\!R^m \times I\!R^n \to I\!R^n$, $g : I\!R^n \to I\!R^{n_g}$, $n_g \ge 1$, $c : I\!R^m \times I\!R^n \to I\!R^{n_c}$, $\Phi : I\!R^n \times I\!R^n \to I\!R^{n_\Phi}$, and K is a closed and non empty convex subset of $I\!R^{n_\Phi}$. All data f, g, c, ℓ, ϕ, Φ are assumed to be of class C^∞, and f is supposed to be Lipschitz. Set, for $q \in [1, \infty]$

$$\mathcal{U}_q := L^q(0,T,I\!R^m); \quad \mathcal{Y}_q := W^{1,q}(0,T,I\!R^n). \tag{2}$$

The control and state space are $\mathcal{U} := \mathcal{U}_\infty$, $\mathcal{Y} := \mathcal{Y}_\infty$. For given $y_0 \in I\!\!R^n$ and $u \in \mathcal{U}$, the *state equation* (1)(i) has a unique solution in $\mathcal{Y} := \mathcal{Y}_\infty$ denoted $y[u, y_0]$.

All multipliers (elements of dual spaces) are represented as "horizontal vectors" (possibly depending on time). The dual of $I\!\!R^n$ is denoted $I\!\!R^{n*}$. As in some of the Russian literature e.g. Dmitruk [8], dual variables are seen as parameters of functions and put into brackets. The *generalized Hamiltonian function* $H : [I\!\!R \times I\!\!R^{n*} \times I\!\!R^{n_c*}] \times I\!\!R^m \times I\!\!R^n \to I\!\!R$ is defined resp. as

$$H[\alpha, p, \lambda](u, y) \quad = \alpha\,\ell(u, y) + pf(u, y) + \lambda c(u, y). \tag{3}$$

The *end points Lagrangian*, where $\alpha \in I\!\!R_+$ and $\Psi \in I\!\!R^{n_\Phi *}$, is defined as follows:

$$\Phi[\alpha, \Psi](y_0, y_T) := \alpha\phi(y_0, y_T) + \Psi\Phi(y_0, y_T). \tag{4}$$

By $BV(0, T)^q$ we denote the space of bounded variations functions with value in $I\!\!R^q$ (whose value of elements at time t is an horizontal vector). We denote by $BV_T(0, T)^q$ the functions of $BV(0, T)^q$ vanishing at time $T+$. We may identify $\eta \in BV_T(0, T)^q$ with the corresponding measure $\mathrm{d}\eta$.

Definition 2.1. *We say that* $(\bar{u}, \bar{y}) \in \mathcal{U} \times \mathcal{Y}$ *is a* generalized first-order extremal *if there exists* $\bar{\alpha} \geq 0$, $\bar{\eta} \in BV_T(0, T)^q$, *and* $\bar{\lambda} \in L^\infty(0, T, I\!\!R^{n_c})$ *with* $(\bar{\alpha}, \mathrm{d}\bar{\eta}, \bar{\lambda}) \neq 0$, *and* $p \in BV(0, T)^n$, *such that*

$$\begin{aligned}
\dot{\bar{y}}_t &= f(\bar{u}_t, \bar{y}_t) \quad \textit{a.e. on } [0, T], && (5)\\
-\mathrm{d}\bar{p}_t &= H_y[\bar{\alpha}, \bar{p}_t, \bar{\lambda}_t](\bar{u}_t, \bar{y}_t)\mathrm{d}t + \sum_{i=1}^{n_g} g_i'(\bar{y}_t)\mathrm{d}\bar{\eta}_{i,t} \quad \textit{on } [0, T], && (6)\\
0 &= H_u[\bar{\alpha}, \bar{p}_t, \bar{\lambda}_t](\bar{u}_t, \bar{y}_t), \quad \textit{a.e. on } \ ]0, T[, && (7)
\end{aligned}$$

and in addition

$$\begin{aligned}
g_i(\bar{y}_t) &\leq 0;\ \mathrm{d}\bar{\eta}_{i,t} \geq 0;\ t \in [0, T]; && (8)\\
\int_0^T g_i(\bar{y}_t)\mathrm{d}\bar{\eta}_{i,t} &= 0,\ i = 1, \ldots, q, && (9)\\
c(\bar{u}_t, \bar{y}_t) &\leq 0;\ \bar{\lambda}_t \geq 0;\ \bar{\lambda}_t c(\bar{u}_t, \bar{y}_t) = 0 \ \textit{a.e;} && (10)\\
\Phi(\bar{y}_0, \bar{y}_T) &\in K; \quad \Psi \in N_K(\Phi(\bar{y}_0, \bar{y}_T), && (11)\\
\bar{p}_{0-} &= -\Phi_{y_0}[\alpha, \Psi](\bar{y}_0, \bar{y}_T) && (12)\\
\bar{p}_{T+} &= \Phi_{y_T}[\alpha, \Psi](\bar{y}_0, \bar{y}_T). && (13)
\end{aligned}$$

Definition 2.2. *The set of* $(\bar\alpha, \bar p, \bar\eta, \bar\lambda)$ *satisfying definition 2.1 is called the set of* first-order, or Lagrange multipliers *associated with* $(\bar u, \bar y)$ *and denoted* $M^L(\bar u, \bar y)$. *When* $\alpha = 0$ *(resp.* $\alpha > 0$*) we say that the corresponding multiplier is singular (resp. regular).*

Remark 2.3. *It is well known that, for given* $(\bar\alpha, \bar\eta, \bar\lambda)$ *in the appropriate space, the system made by equations* (6) *and* (13) *has a unique solution* p *in* $BV(0,T)^n$, *and the mapping* $(\bar\alpha, \bar\eta, \bar\lambda) \mapsto p$ *is linear and continuous.*

Remark 2.4. *When* $\bar\alpha > 0$, *dividing* $\bar\eta$ *and* $\bar p$ *by* $\bar\alpha$, *we obtained the* qualified *form of first-order extremal, i.e., with* $\bar\alpha = 1$. *We may then remove* $\bar\alpha$ *from the definition of the Hamiltonian and of the statement of a first-order extremal.*

2.2 Proof of the first-order optimality conditions

Consider the *linearization of the state equation*

$$\dot z_t = f'(\bar u_t, \bar y_t)(v_t, z_t); \quad t \in (0,T), \tag{14}$$

whose unique solution in $\mathcal{Y}$ (for given initial condition z_0 and $v \in \mathcal{V}$) will be denoted $z[v, z_0]$. For any $q \in [1, \infty]$, with any $(z_0, v) \in I\!R^n \times \mathcal{U}_q$ is associated a unique solution of (14) denoted $z[v, z_0]$. We define a mapping $J : I\!R^n \times \mathcal{U} \to I\!R$ by

$$J(u, y_0) := \int_0^T \ell(u_t, y_t[u, y_0])\mathrm{d}t + \phi(y_0, y_T[u, y_0]). \tag{15}$$

We now give a short proof of the existence of a generalized Lagrange multiplier, without assumption on the convex set K, and with the following "qualification like" condition, involving the mixed state and control constraint only:

$$\begin{array}{l} \text{There exists } \hat v \in \mathcal{U} \text{ and } \hat\beta > 0 \text{ such that} \\ c(\bar u_t, \bar y_t) + c_u(\bar u_t, \bar y_t)\hat v_t \le -\hat\beta \quad \text{for a.a. } t \in (0,T). \end{array} \tag{16}$$

Theorem 2.5. *Let* $(\bar u, \bar y) \in \mathcal{U} \times \mathcal{Y}$ *be a local solution of* (1) *and the associated state. If* (16) *holds, then* $(\bar u, \bar y)$ *is a generalized first-order extremal.*

Proof. (i) An equivalent optimal control problem, obtained by elimination of the state variable from the state equation, is

$$\left\{\begin{array}{l} \mathrm{Min}_{u,y_0}\, J(u,y_0); \\ g(y_t[u,y_0]) \leq 0, \quad t \in [0,T], \\ c(u_t, y_t[u,y_0]) \leq 0, \ \text{ for a.a. } t \in (0,T), \\ \Phi(y_0, y_T[u,y_0]) \in K. \end{array}\right. \tag{17}$$

Obviously $(\bar{u}, \bar{y}_0)$ is solution of this problem. Set

$$\mathcal{K} := C([0,T])_-^{n_g} \times L^\infty(0,T)_-^{n_c} \times K, \tag{18}$$

and let $G : I\!\!R^n \times \mathcal{U} \to C([0,T])^{n_g} \times L^\infty(0,T)^{n_c}$ be defined by

$$G(u,y_0) := \left(g(y[u,y_0]);\ \ c(u, y[u,y_0]);\ \ \Phi(y_0, y_T[u,y_0])\right). \tag{19}$$

We can rewrite problem (17) under the standard form

$$\operatorname*{Min}_{u,y_0} J(u,y_0); \quad G(u,y_0) \in \mathcal{K}. \tag{20}$$

We claim that the set of associated generalized Lagrange multipliers is non empty. All mappings are continuously differentiable. In view of [6, Prop. 3.16], the conclusion will hold if we prove that the set $E := \mathcal{R}(G'(\bar{y}_0, \bar{u})) - \mathcal{K}$ has a non empty relative interior (where $\mathcal{R}(G'(\bar{y}_0, \bar{u}))$ denotes the range of the linear mapping $G'(\bar{y}_0, \bar{u})$). For this we apply lemma A.2 to the set $\mathcal{K}$. The set C of that lemma corresponds to $C([0,T])_-^{n_g} \times L^\infty(0,T)_-^{n_c}$. The claim follows.

(ii) We next relate an associated generalized Lagrange multiplier for problem (17) to the notion of first-order extremal for the original problem (1). Denote the Lagrangian function of problem (17) by

$$\left\{\begin{array}{ll} L[\alpha,\eta,\lambda,\Psi](u,y_0) := & \alpha J(u,y_0) + \langle \eta, g(y[u,y_0])\rangle \\ & +\langle \lambda, c(u, y[u,y_0])\rangle + \Psi\Phi(y_0, y_T[u,y_0]). \end{array}\right. \tag{21}$$

Note that here $\lambda \in L^\infty(0,T,\mathbb{R}^{n_c})^*$. The first-order optimality conditions (e.g. [18, 20], or [6, Section 3.1]) are that multipliers belong to normal cones to the corresponding constraints, with sign condition on α and non zero generalized multiplier. In our case this boils down to

$$\left\{\begin{array}{l} \bar{\alpha} \geq 0; \quad (\bar{\alpha}, \mathrm{d}\bar{\eta}, \bar{\lambda}) \neq 0; \quad \bar{\lambda} \in N_{L^\infty(0,T,I\!\!R^{n_c})}(c(\bar{u},\bar{y})); \\ \text{(8)-(9) and (11) holds,} \end{array}\right. \tag{22}$$

and the condition that $(\bar{u}, \bar{y})$ is a stationary point of the Lagrangian, which means that, for an arbitrary $(v, z_0) \in \mathcal{U} \times \mathbb{R}^n$, denoting by $z = z(v, z_0)$ the solution of the linearized equation (14), the following holds:

$$\begin{array}{l} \bar{\alpha} \displaystyle\int_0^T \ell'(\bar{u}_t, \bar{y}_t)(v_t, z_t)\mathrm{d}t + \sum_{i=1}^{n_g} \int_0^T g_i'(\bar{y}_t) z_t \mathrm{d}\bar{\eta}_{i,t} \\ + \langle \bar{\lambda}, c'(\bar{u}, \bar{y})(v, z)\rangle + \Phi'[\alpha, \Psi](\bar{y}_0, \bar{y}_T)(z_0, z_T) = 0. \end{array} \tag{23}$$

Taking $z_0 = 0$ and v arbitrarily in $\mathcal{U}$, we deduce from (23) the existence of $\gamma > 0$ such that $|\langle \bar{\lambda}, c_u(\bar{u}_t, \bar{y}_t)v_t\rangle| \leq \gamma \|v\|_1$ (since all other linear forms are continuous w.r.t. the L^1 norm) and hence $c_u(\bar{u}, \bar{y})^\top \bar{\lambda}$ may be identified to an element of $L^\infty(0, T, I\!R^{n_c})$. Combining with (16), we deduce with lemma A.1 that $\bar{\lambda} \in L^\infty(0, T, \mathbb{R}^{n_c *})$.

Next, according to remark 2.3, define the costate $\bar{p}$ as the solution in $BV(0, T)^n$ of the equation (6) with final condition (12). Then (23) reduces to

$$\begin{array}{l} -\displaystyle\int_0^T \mathrm{d}\bar{p}_t z_t + \int_0^T (-\bar{p}_t f_y(\bar{u}_t, \bar{y}_t) + \bar{\alpha}\ell_u'(\bar{u}_t, \bar{y}_t) + \bar{\lambda}_t c_y(\bar{u}_t, \bar{y}_t))v_t \mathrm{d}t \\ +\Phi[\bar{\alpha}, \Psi](\bar{y}_0, \bar{y}_T)(z_0, z_T) = 0. \end{array} \tag{24}$$

The integration by parts formula can be extended to the product of an absolutely continuous function and of a bounded variation function (similar to [11, Vol. I, ch. 3, Theorem 22, p. 154], but here integrating over $[0, T]$ and not $(0, T)$; see also the discussion in [4]). Applying this result to the term $\int_0^T \mathrm{d}\bar{p}_t z_t$ and using the final condition on the costate, we obtain

$$\textstyle\int_0^T H_u[\bar{\alpha}, \bar{p}_t, \bar{\eta}_t, \bar{\lambda}_t](\bar{u}_t, \bar{y}_t)v_t \mathrm{d}t + (\bar{p}_{0-} + D_{y_0}\Phi[\bar{\alpha}, \Psi](\bar{y}_0, \bar{y}_T))\, z_0 = 0. \tag{25}$$

That this is zero for any v and z_0 is equivalent to conditions (7) and (12). Therefore $(\bar{u}, \bar{y})$ is a generalized extremal, as was to be proved. ∎

For the statement of the qualification condition we remind that, K being a convex subset of an Euclidean space it has a nonempty relative interior (possibly reduced to one point). Therefore we may represent it after an affine change of cordinates as

$$K = \{0\}_{I\!R^{n_{\Phi,1}}} \times K_2, \tag{26}$$

with $K_2 \subset I\!R^{n_{\Phi,1}}$ of nonempty interior. We partition accordingly the mapping $\Phi(\cdot,\cdot)$ into two blocks:

$$\Phi(\cdot,\cdot) = \begin{pmatrix} \Phi_1(\cdot,\cdot) \\ \Phi_2(\cdot,\cdot) \end{pmatrix}. \tag{27}$$

Consider the following qualification condition, where z_T stands for $z_T[v, z_0]$:

$$\begin{array}{ll} (v, z_0) \mapsto \Phi_1'(\bar{y}_0, \bar{y}_T)(z_0, z_T) \text{ is onto,} & \\ \text{For some } \beta > 0 \text{ and } (\bar{v}, \bar{z}) \in \mathcal{U} \times \mathcal{Y}, \text{ solution of (14):} & \\ \qquad \Phi_1'(\bar{y}_0, \bar{y}_T)(\bar{z}_0, \bar{z}_T) & = 0 \\ \qquad \Phi_2'(\bar{y}_0, \bar{y}_T)(\bar{z}_0, \bar{z}_T) & \in \operatorname{int}(K_2), \\ \qquad g(\bar{y}_t) + g'(\bar{y}_t)\bar{z}_t & < 0, \text{for all } t \in [0, T]; \\ \qquad c(\bar{u}_t, \bar{y}_t) + c'(\bar{u}_t, \bar{y}_t)(\bar{v}_t, \bar{z}_t) & \leq -\beta, \text{for a.a. } t \in [0, T]. \end{array} \tag{28}$$

Theorem 2.6. *Let $\bar{u} \in L^\infty(0, T, U)$ be a local solution of* (1) *and $\bar{y}$ be the associated state. Then the following conditions are equivalent:* (i) *the qualification condition* (28) *holds,* (ii) *the set of singular Lagrange multipliers is empty,* (iii) *the set of Lagrange multipliers (with $\alpha = 1$) is non empty and bounded.*

Proof. We have checked in the proof of theorem 2.5, see relations (18)-(19), that with the notations of that proof, the set $E := \mathcal{R}(G'(\bar{y}_0, \bar{u})) - \mathcal{K}$ has a non empty relative interior. By [6, Prop. 3.16], condition (ii) is equivalent to Robinson's qualification condition. Since the convex set has the product form of "zero" times a convex set with nonempty interior, the latter is equivalent to (i) by [6, Corollary 2.101]. It is known that (i) implies (iii). On the other hand, (iii) implies (ii) since the set of singular multipliers is the asymptotic cone of the set of Lagrange multipliers (it the set of the latter is non empty), and so (iii) implies (ii). The conclusion follows. ■

3 Continuity and Lipschitz properties of the control

3.1 Continuity of the control

We need to introduce the concepts below. The total derivative of the function $g(y)$ is

$$g^{(1)}(u, y) := g'(y) f(u, y). \tag{29}$$

By *trajectory*, we mean a solution of the state equation. Along a trajectory, we have that $g^{(1)}(u_t, y_t) = \frac{\mathrm{d}}{\mathrm{d}t} g(y_t)$. In a similar way we can define upper order total derivatives. These formal expressions are the sum of all partial derivatives multiplied by the corresponding derivative of the variable, except for y whose derivative is replaced by $f(u, y)$. They involve time derivatives of u.

Definition 3.1. (i) *For $1 \leq i \leq n_g$, the order of the state constraint $g_i(y)$ is the smallest positive integer q_i such that $g_{i,u}^{(k)}(u, y) = 0$, for all $0 \leq k < q_i$ (and indeed then $g_{i,u}^{(k)}(u, y)$ does not depend on the derivatives of u for $k \leq q_i$).* (ii) *Let $(\bar{u}, \bar{y})$ be a trajectory with a continuous control. We say that the state constraint i is regular along $(\bar{u}, \bar{y})$ if*

$$g_{i,u}^{(q_i)}(\bar{u}_t, \bar{y}_t) \neq 0, \quad \textit{for all } \; t \in [0, T]. \tag{30}$$

For a state constraint g_i of order q, and $k < q$, we may write $g_i^{(k)}(y)$, and we have

$$g_i^{(k+1)}(u, y) = g_{i,y}^{(k)}(y) f(u, y); \quad g_{i,u}^{(k+1)}(u, y) = g_{i,y}^{(k)}(y) f_u(u, y). \tag{31}$$

Define the set of state constraints of order κ, and those active at time t along the trajectory $(\bar{u}, \bar{y})$:

$$I_\kappa := \{1 \leq i \leq n_g;\ q_i = \kappa\}; \quad I_\kappa(t) := \{i \in I_\kappa;\ g_i(\bar{y}_t) = 0\}. \tag{32}$$

By $I_0(t)$ we denote the set of active mixed constraints at time $t \in (0, T)$. We need the following hypothesis of *positive linear independence of derivatives w.r.t. the control of active mixed constraints*

$$\sum_{i \in I_0(t)} \alpha_i c_{i,u}(\bar{u}_t, \bar{y}_t) = 0 \;\text{ with } \alpha \geq 0 \text{ implies } \alpha = 0. \tag{33}$$

This is equivalent to the Mangasarian-Fromovitz hypothesis [15] (w.r.t. the variable u) for the mixed constraints. It implies that $c_{ju}(\bar{u}_t, \bar{y}_t) \neq 0$, for $j \in I_0(t)$. Consequently we say that the constraint $c(u, y) \leq 0$ is of zero order. The (stronger) hypothesis of *joint qualification of zero and first-order state constraints* is as follows:

$$\left\{ \begin{array}{l} \sum_{j \in I_0(t)} \alpha_i c_{i,u}(\bar{u}_t, \bar{y}_t) + \sum_{i \in I_1(t)} \beta_j g_{ju}^{(1)}(\bar{u}_t, \bar{y}_t) = 0 \\ \text{with } \beta \geq 0 \text{ implies } (\alpha, \beta) = 0. \end{array} \right. \tag{34}$$

Note that this is the Mangasarian-Fromovitz hypothesis (w.r.t. the variable u) for the system

$$g_i^{(1)}(\bar{u}_t, \bar{y}_t) \leq 0,\ i \in I_1(t); \quad c_j(\bar{u}_t, \bar{y}_t) = 0,\ j \in I_0(t). \tag{35}$$

When the control $\bar{u}$ has left and right limits denoted by $\bar{u}_t^{\pm}$, for $\sigma \in [0,1]$, we denote $\bar{u}_t^{\sigma} := \sigma \bar{u}^+ + (1-\sigma)\bar{u}^-$ and we adopt the same convention for other functions of time such as the costate and Lagrange multiplier λ. The hypothesis of positivity of the Hessian of the Hamiltonian w.r.t. the control is

$$0 < H^0_{uu}[\bar{p}_t^{\sigma}, \bar{\lambda}_t^{\sigma}](\bar{u}_t^{\sigma}, \bar{y}_t)([\bar{u}_t],[\bar{u}_t]), \ \text{ for all } \sigma \in [0,1],\ t \in [0,T]. \tag{36}$$

This hypothesis holds, of course, if $H[\bar{p}_t^{\sigma}, \bar{\lambda}_t^{\sigma}](\cdot, \bar{y}_t)$ is a strongly convex function of the control variable, as was assumed e.g. in [3]. The next theorem is a slight improvement of Prop. 4.8 of that reference, due to the weaker hypothesis (36) and also since hypothesis (34) is weaker than the corresponding one used in this reference. In that theorem we make an assumption only on the first-order state constraints, but state constraints of higher order may also be present.

Theorem 3.2. *Let $(\bar{u}, \bar{y})$ be a first-order extremal for (P).*
(i) *Assume that* (36) *and* (33) *hold. If $\bar{u}$ has left and right limits at time $t \in]0,T[$, then it is continuous at time t.*
(ii) *Assume that the control is continuous and that* (34) *hold. Then the multiplier λ associated with the mixed control-state constraints and the components of η associated with first-order state constraints are continuous.*

Proof. (i) In view of (34) and (7), the multiplier $\bar{\lambda}$ being uniformly bounded, it has at time t (non necessarily unique) limit points on the left and right side. We denote by $\bar{\lambda}_t^{\pm}$ some of these limit points, and set $[\bar{\lambda}_t] := \bar{\lambda}^+ - \bar{\lambda}^-$. By the costate equation (6), the jump of $\bar{p}$ is such that

$$[\bar{p}] = \bar{p}^+ - \bar{p}^- = -\sum_{i=1}^{n_g} \nu_i g_{i,y}(\bar{y}_t), \qquad \text{with } \nu_i := [\bar{\eta}_{i,t}] \geq 0. \tag{37}$$

We have that

$$\begin{aligned} 0 \ &= \left[H^0_u[\bar{p}_t, \bar{\lambda}_t](\bar{u}_t, \bar{y}_t)\right] \\ &= \int_0^1 \{H_{uu}[\bar{p}_t^{\sigma}, \bar{\lambda}_t^{\sigma}](\bar{u}_t^{\sigma}, \bar{y}_t)[\bar{u}_t] + [\bar{p}_t] f_u(\bar{u}_t^{\sigma}, \bar{y}_t) + [\bar{\lambda}_t] c_u(\bar{u}_t^{\sigma}, \bar{y}_t)\} \mathrm{d}\sigma. \end{aligned} \tag{38}$$

Using (37) and observing that, by definition of the order of the state constraint, $g_{i,y} f_u = g^{(1)}_{i,u}$ equals zero if $q_i > 1$, we obtain that

$$\int_0^1 H_{uu}[\bar{p}^\sigma_t](\bar{u}^\sigma_t, \bar{y}_t)[\bar{u}_t]\mathrm{d}\sigma \;=\; \sum_{i:q_i=1} \nu_i \int_0^1 g^{(1)}_{i,u}(\bar{u}^\sigma_t, \bar{y}_t)\mathrm{d}\sigma - [\bar{\lambda}_t] \int_0^1 c_u(\bar{u}^\sigma_t, \bar{y}_t)\mathrm{d}\sigma. \tag{39}$$

We compute the scalar product of both sides of (39) by $[\bar{u}_t]$, using

$$\int_0^1 g^{(1)}_{i,u}(\bar{u}^\sigma_t, \bar{y}_t)[\bar{u}_t]\mathrm{d}\sigma = [g^{(1)}_i(\bar{u}_t, \bar{y}_t)]; \quad \int_0^1 c_u(\bar{u}^\sigma_t, \bar{y}_t)[\bar{u}_t]\mathrm{d}\sigma = [c(\bar{u}_t, \bar{y}_t)] \tag{40}$$

and observing that the first-integral is equal to zero for state constraints of order greater than 1, and that $[\bar{\lambda}_t][c(\bar{u}_t, \bar{y}_t)] \geq 0$ (in view of the complementarity relations between $\bar{\lambda}^\pm_t$ and $c(\bar{u}_{t^\pm}, \bar{y}_t)$). Using hypothesis (36), we deduce that

$$\alpha |[\bar{u}_t]|^2 \;\leq\; \sum_{i:q_i=1} \nu_i [g^{(1)}_i(\bar{u}_t, \bar{y}_t)]. \tag{41}$$

If $\nu_i > 0$, then $g_i(\bar{y}_t) = 0$, and hence $[g^{(1)}_i(\bar{u}_t, \bar{y}_t)] \leq 0$ since t is a local maximum of $g_i(\bar{y}_t)$. Therefore, the right-hand side in (41) is nonpositive, implying $[\bar{u}_t] = 0$. Point (i) follows.
(ii) Since $[\bar{u}_t] = 0$, the right-hand side of (39) equals zero, which since $\bar{u}$ is continuous and $\nu \geq 0$ means by (34) that ν and $[\bar{\lambda}_t]$ are equal to zero, as was to be proved. Point (ii) follows. □

Remark 3.3. *The hypothesis of existence of left and right limits for the control is assumed for instance if the Hamiltonian attains its minimum at a unique point equal to $\bar{u}_t$, for a.a. t; see e.g. the analysis of [5, Lemma 2.7].*

3.2 Hager's lemma

In this section we recall Hager's lemma [12] and provide a slightly simplified proof (that however, is based as the original proof on the concept of compatible pairs introduced in [12]). This lemma is instrumental for proving the Lipschitz continuity of the control in the next section. Let X be a Banach space, and x be a continuous function $[0, T] \to X$. Let $I : [0, T] \to \{1, \ldots, n\}$ be upper continuous, i.e.,

$$\text{If } t_n \to t \in [0, T], \text{ and } i \in I(t_n), \text{ then } i \in I(t). \tag{42}$$

We will speak of $I(t)$ as a set of active constraints since this is the case in our application. We say that the pair (a, b) in $[0, T]^2$ is *compatible* if

$$a < b; \quad I(a) = I(b); \quad I(t) \subset I(a), \quad \text{for all } \ t \in (a, b), \tag{43}$$

i.e., the same constraints are active at times a and b, and no other constraint is active for $t \in (a, b)$. We say that $L > 0$ is a Lipschitz constant for x over $E \subset [0, T]^2$ if

$$\|x(a) - x(b)\| \leq L|b - a| \quad \text{whenever } (a, b) \in E. \tag{44}$$

Lemma 3.4. *Assume that $x \in C([0, T], X)$ and that I is upper continuous. Let $L > 0$ be a Lipschitz constant for x over the set of compatible pairs. Then L is a Lipschitz constant for x i.e., we have that*

$$\|x(a) - x(b)\| \leq L|b - a|, \quad \textit{for all } \ (a, b) \in [0, T]^2. \tag{45}$$

Proof. We make an induction over the following sets, for $m = 0$ to n:

$$\begin{aligned} T_m := \quad & \{(t, t') \in [0, T]^2; \ t \leq t'; \ \text{ there exists } \ J \subset \{1, \dots, n\}; \\ & \qquad\qquad\qquad |J| \leq m; \ I(t) \subset J, \ \text{for all } \ t \in (t, t')\}. \end{aligned} \tag{46}$$

Since each pair $(t, t') \in [0, T]^2$ such that $t < t'$ belongs to T_n, it suffices to prove that (45) holds on each T_m, by induction on m. Since $L > 0$ is a Lipschitz constant for x over the set of compatible pairs, (46) holds for $m = 0$ (with in that case $I(t) = I(t') = \emptyset$). So, assuming that (44) holds for $E = T_{m-1}$, for $1 \leq m \leq n$, it suffices to prove that it holds on T_m.

Let $(a, b) \in T_m$ with associated set J in (46). Consider two cases:
Case 1: the set below is not empty:

$$F := \{t \in [a, b]; \ I(t) = J\}. \tag{47}$$

In view of (42) and the definition of T_m, F is a closed set; let a' and b' be its minimum and maximum, resp. Then (a', b') is a compatible pair, and hence, $\|x(b') - x(a')\| \leq L(b' - a')$. Since

$$\|x(b) - x(a)\| \leq \|x(a') - x(a)\| + \|x(b') - x(a')\| + \|x(b) - x(b')\|, \tag{48}$$

we see that it suffices to prove that

$$\text{(i)} \ \ \|x(a') - x(a)\| \leq L(a' - a); \quad \text{(ii)} \ \ \|x(b) - x(b')\| \leq L(b - b'). \tag{49}$$

Obviously, if $a' = a$ (resp. $b' = b$) then (49)(i) (resp. (49)(ii)) holds. Since $x(t)$ is continuous, if $a' > a$, for proving (49)(i), it suffices to check that

$$\|x(t_0'') - x(a)\| \leq L(t_0'' - a), \qquad \text{for all } \; t_0'' \in (a, a'). \tag{50}$$

A similar statement holds for (49)(ii). So we have reduced case 1 to
Case 2: the pair (a, b) is such that

$$|I(t)| < m, \quad \text{for all} \quad t \in [a, b]. \tag{51}$$

By (42), for any $t \in [a, b]$, there exists a neigborhood V_t of t in $[a, b]$ such that $I(t') \subset I(t)$, for all $t' \in V_t$. Since $[a, b]$ is compact, there exists a finite sequence $a = t_0 < t_1 < \cdots < t_p = b$ such that $V_{t_i} \cap V_{t_{i-1}} \neq \emptyset$, for $i = 1$ to p. Let $\tau_i \in V_{t_i} \cap V_{t_{i-1}}$, for $i = 1$ to p. We have that

$$I(t) \subset I(t_{i-1}), \; t \in (t_{i-1}, \tau_i); \quad I(t) \subset I(t_i), \; t \in (\tau_i, t_i); \tag{52}$$

By (51) and (52) we have that (t_{i-1}, τ_i) and (τ_i, t_i) belong to T_{m-1}, $i = 1$ to p. We conclude with the triangle inequality

$$\|x(a - x(b)\| \leq \sum_{i=1}^{p} \left(\|x(t_{i-1}) - x(\tau_i)\| + \|x(\tau_i) - x(t_i)\| \right). \tag{53}$$

$\square$

3.3 Main result: Lipschitz continuity of the control

We recall that q_i denotes the order of the ith state constraint, set $q := (q_1, \ldots, q_{n_g})$ and $n_G := n_g + n_c$, and define $G^q(u, y) : I\!R^m \times I\!R^n \to I\!R^{n_G}$ by

$$G_i^q(u, y) := \begin{cases} g_i^{(q_i)}(u, y), & i = 1, \ldots, n_g, \\ c_{i-n_g}(u, y), & i = n_g + 1, \ldots, n_G. \end{cases} \tag{54}$$

Our main result generalizes the ones in Hager's [12], restricted to the first-order, and of Hermant [13] (restricted to the second-order, and with a stronger second-order condition).

Denote by $I_{\{0,1\}}(t) := I_0(t) \cup I_1(t)$ the set of active constraints of order not greater than one. We need two conditions. The first is the qualification condition below, stronger than (34), but still not involving state constraints of order higher than 1:

$$\text{The family } \left\{ G^q_{i,u}(\bar{u}_t, \bar{y}_t);\ i \in I_{\{0,1\}}(t) \right\} \text{ is linearly independant.} \tag{55}$$

The second condition is of strong Legendre-Clebsch type, but reduced to a subspace:

$$\begin{array}{l} \text{For some } \alpha_H > 0: \quad \alpha_H |\upsilon|^2 \leq H_{uu}[\bar{\alpha}, \bar{p}_t, \bar{\lambda}](\bar{u}_t, \bar{y}_t)(\upsilon, \upsilon), \\ \text{whenever } G^{(q)}_{i,u}(\bar{u}_t, \bar{y}_t)\upsilon = 0, \text{ for all } \ i \in I_{\{0,1\}}(t),\ t \in [0, T]. \end{array} \tag{56}$$

Theorem 3.5. *Let* $(\bar{u}, \bar{y}, \bar{p}, \bar{\eta}, \bar{\lambda})$ *be a first-order extremal and associated multipliers, with* $\bar{u}$ *continuous. If* (55)-(56) *hold, then* $\bar{u}$, $\bar{\lambda}$ *and the components of* $\bar{\eta}$ *associated with first order state constraints are Lipschitz function of time.*

The proof is based on the alternative optimality system, defined in Maurer [16] as follows. The first-order alternative multiplier is $\eta^1 := -\bar{\eta}$. For $k \geq 2$, define the higher-order alternative multipliers by

$$\eta^k_t := \int_t^T \eta^{k-1}_t \mathrm{d}t,\ k = 2, \ldots; \quad \eta^q_i := \eta^{q_i}_i, \ \ i = 1, \ldots, n_g, \tag{57}$$

and

$$\eta^q_i := \bar{\lambda}_i, \quad i = n_g + 1, \ldots, n_g + n_c. \tag{58}$$

The *alternative costate* (of order q) is defined as

$$p^q_t \ := \bar{p}_t - \sum_{i=1}^{n_g} \sum_{j=1}^{q_i} \eta^j_{i,t} g^{(j-1)}_{i,y}(\bar{y}_t). \tag{59}$$

For instance, if all constraints are of first-order, then

$$p^q_t = \bar{p}_t - \sum_{i=1}^{n_g} \eta^1_{i,t} g'_i(\bar{y}_t), \tag{60}$$

and if all constraints are of second order, then

$$p^q_t = \bar{p}_t - \sum_{i=1}^{n_g} \left(\eta^1_t g'(\bar{y}_t) + \eta^2_t g^{(1)}_y(\bar{y}_t) \right). \tag{61}$$

The corresponding alternative Hamiltonian is defined as

$$H^q[\alpha, p^q, \eta^q](u, y) := \alpha\ell(u, y) + p^q f(u, y) + \eta^q G^{(q)}(u, y). \tag{62}$$

The following result is classical [16] and shows that the alternative optimality system has the same Hamiltonian form as the original one; we provide a proof for the convenience of the reader.

Lemma 3.6. *The alternative costate and multiplier satisfy the alternative costate equation*

$$\begin{aligned} -\dot p^q_t &= H^q_y[\bar\alpha, p^q_t, \eta^q_t](\bar u_t, \bar y_t), \quad t \in (0, T), \\ p^q_{T+} &= \Phi_{y_T}[\bar\alpha, \Psi](\bar y_0, \bar y_T), \end{aligned} \tag{63}$$

as well as the property of invariance w.r.t. the control up to a constant

$$H^q[\bar\alpha, p^q_t, \eta^q_t](u, \bar y_t) = \quad H[\bar\alpha, \bar p_t, \bar\lambda_t](u, \bar y_t) + \textit{term not depending on } u. \tag{64}$$

Proof. Since all multipliers have zero value at time $T+$, the final condition in (63) obviously holds. Next, using $\dot\eta^k_t = -\eta^{k-1}_t$ when $k > 1$, obtain with (59)

$$\begin{aligned} \mathrm{d}p^q_t \quad = \quad & \mathrm{d}p_t - \textstyle\sum_{i=1}^{n_g} \mathrm{d}\eta^1_{i,t} g^{(j-1)}_{i,y}(\bar y_t) - \sum_{i=1}^{n_g} \eta^{q_i}_i g^{(q_i-1)}_{i,yy}(\bar y_t) f(\bar u_t, \bar y_t)\mathrm{d}t \\ & - \textstyle\sum_{i=1}^{n_g} \sum_{j=1}^{q_i-1} \eta^j_{i,t} \left(-g^{(j)}_{i,y}(\bar y_t) + g^{(j-1)}_{i,yy}(\bar y_t) f(\bar u_t, \bar y_t)\right) \mathrm{d}t. \end{aligned} \tag{65}$$

Using $\mathrm{d}\eta^1 = -\mathrm{d}\bar\eta$ and the costate equation, we see that the contribution of $\mathrm{d}\eta$ on the first row vanishes, so that p^q is absolutely continuous, and eliminating $\bar p$ from (59), we get

$$\begin{aligned} -\dot p^q_t \quad = \quad & p^q_t f(\bar u_t, \bar y_t) + \textstyle\sum_{i=1}^{n_c} \bar\lambda_i c_{i,y}(\bar u_t, \bar y_t) + \sum_{i=1}^{n_g} \sum_{j=1}^{q_i-1} \eta^j_{i,t} \Delta_i \\ & + \textstyle\sum_{i=1}^{n_g} \eta^{q_i}_i \left(g^{(q_i-1)}_{i,y}(\bar y_t) f_y(\bar u_t, \bar y_t) + g^{(q_i-1)}_{i,yy}(\bar y_t) f(\bar u_t, \bar y_t)\right) \end{aligned} \tag{66}$$

with

$$\Delta_i = g^{(j-1)}_{i,y}(\bar y_t) f_y(\bar u_t, \bar y_t) - g^{(j)}_{i,y}(\bar y_t) + g^{(j-1)}_{i,yy}(\bar y_t) f(\bar u_t, \bar y_t). \tag{67}$$

But for $j < q_i$, $g^j(\bar y) = g^{(j-1)}_{i,y}(\bar y_t) f(\bar u_t, \bar y_t)$ and so

$$g^{(j)}_{i,y}(\bar y_t) = g^{(j-1)}_{i,yy}(\bar y_t) f(\bar u_t, \bar y_t) + g^{(j-1)}_{i,y}(\bar y_t) f_y(\bar u_t, \bar y_t) \tag{68}$$

proving that $\Delta_i = 0$. We conclude by noticing that

$$g_{i,y}^{(q_i)}(\bar{y}_t) = g_{i,y}^{(q_i-1)}(\bar{y}_t) f_y(\bar{u}_t, \bar{y}_t) + g_{i,yy}^{(q_i-1)}(\bar{y}_t) f(\bar{u}_t, \bar{y}_t), \tag{69}$$

so that (63) and (66) coincide.
We next prove (63). Eliminating p^q in (59), we obtain

$$\begin{aligned} \Delta(u) &:= H^q[\bar{\alpha}, p_t^q, \eta_t^q](u, \bar{y}_t) \\ &= H[\bar{\alpha}, \bar{p}_t, \bar{\lambda}_t](u, \bar{y}_t) + \eta^q G^q(u, \bar{y}_t) \\ &\quad - \textstyle\sum_{i=1}^{n_g} \sum_{j=1}^{q_i-1} \eta_{i,t}^j g_{i,y}^{(j-1)}(\bar{y}_t) f(u, \bar{y}_t). \end{aligned} \tag{70}$$

Since $g_{i,y}^{(j-1)} f(u, \bar{y}_t) = 0$ does not depend on u when $j < q_i$, and $G_i^q(u, \bar{y}_t) = g_{i,y}^{(q_i-1)}(\bar{y}_t) f(u, \bar{y}_t)$, the r.h.s. reduces to $H[\bar{\alpha}, \bar{p}_t, \bar{\lambda}](u, \bar{y}_t)$ plus a term not depending on u, as was to be proved. ■

In view of the previous relation, we see that stationarity or minimality w.r.t. u of $H^q[\bar{\alpha}, p_t^q, \eta_t^q](\cdot, \bar{y}_t)$ is, equivalent to the corresponding property for $H[\bar{\alpha}, \bar{p}_t, \bar{\lambda}_t](\cdot, \bar{y}_t)$.

Proof of theorem 3.5. Let $t \in [0, T]$. We partition the alternative multiplier at time t into $\eta_t^q = (\hat{\eta}_t, \tilde{\eta}_t)$, where $\hat{\eta}$ stands for the components in $I_{\{0,1\}}(t)$, and $\tilde{\eta}$ stands for the remaining components. We identify $\tilde{\eta}$ with its extension by zero for the components of η^q in $I_{\{0,1\}}(t)$. Consider the problem

$$\operatorname*{Min}_{u \in \mathbb{R}^m} \; H^q[\bar{\alpha}, p_t^q, \tilde{\eta}_t](u, \bar{y}_t) \; \text{ subject to } \; g_i^{(q)}(u, \bar{y}_t) = 0, \; i \in I_{\{0,1\}}(t). \tag{71}$$

Note that non active mixed constraints are not involved in this problem since they have zero associated Lagrange multipliers. Denote the set of state constraints with order greater than one by

$$I_{\{2-q\}} := \{1, \ldots, n_g\} \setminus I_1. \tag{72}$$

Due to a cancellation in the expression of the cost, the data of problem (71), apart from $I_{\{0,1\}}(t)$, are on the one hand, function $\bar{y}_t$, p_t^q, $\{\eta_i^q; i \in I_{\{2-q\}}\}$, which by construction, are Lipschitz function of time, and on the other hand, $\{\eta_i^1;\ i \in \bar{I}_1(t)\}$, where we denote by $\bar{I}_1(t) := I_1 \setminus I_1(t)$ the set of non active first-order constraints.

We claim that $\bar{u}_t$ is a local solution of this problem. Indeed, let $g_i(\bar{y}_t)$ be a first-order state constraint. Its first time derivative is continuous since

$\bar{u}$ is so, and is equal at zero whenever it is active since $g_i(\bar{y}_t)$ reaches a local maximum. It follows that $\bar{u}_t$ is feasible for problem (71).

By the qualification hypothesis (55), there exists a unique Lagrange multiplier. In view of the alternative optimality system, the latter is nothing but $\hat{\eta}_t$. The first-order optimality conditions are

$$H_u[\bar{\alpha}, p_t^q, \eta_t^q](u, \bar{y}_t) = 0; \quad g_i^{(q)}(u, \bar{y}_t) = 0, \ i \in I_{\{0,1\}}(t). \tag{73}$$

The Jacobian of these optimality conditions w.r.t. the unknowns $(u, \hat{\eta})$ is

$$\mathrm{Jac}_{I(t)} := \begin{pmatrix} H_{uu}[p_t^q, \eta_t^q](\bar{u}_t, \bar{y}_t) & g^{(q)}_{I_{\{0,1\}}(t)u}(\bar{u}_t, \bar{y}_t)^\top \\ g^{(q)}_{I_{\{0,1\}}(t)u}(\bar{u}_t, \bar{y}_t) & 0 \end{pmatrix} \tag{74}$$

In view of hypotheses (55)-(56), the latter being a well-known sufficient for local optimality for nonlinear programming problems, this Jacobian is invertible at $(\bar{u}_t, \hat{\eta}_t)$, and $\bar{u}_t$ is a local solution of (71) as claimed.

Let (a, b) be a compatible pair, for the set $\hat{I}(t) := I_{\{0,1\}}(t)$. Then $\bar{I}_1(a) = \bar{I}_1(b)$. It follows that the data of problem (71) satisfy a Lipschitz condition, with a constant not depending on the particular (a, b).

By the implicit function theorem, applied to (73), for each $t \in [0, T]$, there is a neighbourhood $\mathcal{V}_t$ of t such that, if a and b belong to $\mathcal{V}_t$, then since $\bar{u}$ is continuous and the data of problem (71) are Lipschitz, we have that for some $c_t > 0$

$$|\bar{u}_b - \bar{u}_a| + |\eta_b^q - \eta_a^q| \leq c_t(b - a). \tag{75}$$

Covering the compact set $[0, T]$ by a finite number of such neighbourhoods, and setting c as the maximum of constants c_t over these neighbourhoods, for all possible choices of I, we deduce that

$$|\bar{u}_b - \bar{u}_a| + |\eta_b^q - \eta_a^q| \leq c(b - a), \quad \text{for all compatible pairs } (a, b). \tag{76}$$

Using lemma 3.4 we deduce that $(\bar{u}, \eta^q)$ is Lipschitz. The conclusion follows. □

A Appendix

In the next lemma we establish in a general setting the L^∞ regularity of multipliers in the dual of an L^∞ space. The proof is an adaptation of the one in [5, Thm 3.1].

Lemma A.1. *Set* $X := L^\infty(0,T,I\!R^m)$*,* $Y := L^\infty(0,T,I\!R^s)$*, and* $K := Y_-$*. Let* $c \in K$ *and* $\lambda \in N_K(c)$*. Let* $A \in L(X,Y)$ *be defined by* $(Av)_t = M_t v_t$*, where* M_t *is an* $s \times m$ *matrix, measurable function of* t*, and essentially bounded. Assume that there exists* $\bar v \in X$ *and* $\beta > 0$ *such that*

$$\text{(i)}\ \ c + A\bar v \leq -\beta \mathbf{1}; \quad \text{(ii)}\ \ A^\top \lambda \in L^\infty(0,T,I\!R^m). \tag{77}$$

Then $\lambda \in L^\infty(0,T,I\!R^s)$*, and*

$$\|\lambda\|_\infty \leq \gamma, \quad \textit{where } \gamma := \beta^{-1}\|\bar v\|_\infty \|A^\top\lambda\|_\infty. \tag{78}$$

Proof. It suffices to check that $|\langle \lambda, a\rangle| \leq \gamma \|a\|_{L^1(0,T,\mathbb{R}^q)}$, for every $a \in Y$. Indeed, if this holds, since Y is a dense subset of $L^1(0,T,\mathbb{R}^s)$, λ has then a unique extension $\tilde\lambda$ in the dual space of $L^1(0,T,\mathbb{R}^q)$, i.e., $L^\infty(0,T,\mathbb{R}^{q*})$, that satisfies (78).

Since the norm of $a \in L^1(0,T,\mathbb{R}^s)$ is the sum of the norms of its positive and negative parts, it suffices to check this inequality when $a \geq 0$, i.e., since $\lambda \geq 0$, to prove that $\langle \lambda, a\rangle \leq \gamma \|a\|_{L^1(0,T,\mathbb{R}^q)}$. We can write $a_t = \alpha_t \bar a_t$, with $\alpha_t = |a_t|$ and $|\bar a_t| = 1$. Set $h := -(c + A\bar v)$. Since $\beta \mathbf{1} \leq h_t$ and $a_{i,t} \leq \alpha_t$, $i = 1, \dots, q$, for a.a. t, we have that $\beta a_t \leq \alpha_t h_t$, and so, since $\lambda \geq 0$:

$$\beta\langle \lambda, a\rangle = \langle \lambda, \beta a\rangle \leq \langle \lambda, \alpha h\rangle. \tag{79}$$

Since $\lambda \geq 0$, $a \geq 0$ and $c \leq 0$, and the maximal ratio between the L^∞ and L^2 norms of $I\!R^{n_c}$ is $\sqrt{n_c}$, we have that:

$$0 \geq \langle \lambda, \alpha c\rangle \geq \sqrt{n_c}\langle \lambda, \|a\|_\infty c\rangle = \sqrt{n_c}\|a\|_\infty \langle \lambda, c\rangle = 0, \tag{80}$$

the last equality being the complementarity condition between elements of a convex cone and elements of the corresponding normal cone. It follows that $\langle \lambda, \alpha c\rangle = 0$. Combining with (79) (and using in the first equality the specific form of A) we obtain

$$\begin{aligned} \beta\langle \lambda, a\rangle \ &\leq -\langle \lambda, \alpha A\bar v\rangle = -\langle \lambda, A\alpha\bar v\rangle \\ &\leq \|A^\top\lambda\|_\infty \|\alpha\|_1 \|\bar v\|_\infty = \|A^\top\lambda\|_\infty \|\bar v\|_\infty \|a\|_1. \end{aligned} \tag{81}$$

The conclusion follows. ■

If K is a subset of a Banach space X, we define $\overline{\text{aff}}(K)$ as the smallest closed affine subspace of X containing K.

Lemma A.2. *Let X be a Banach space and $\mathcal{K}$ be a convex subset of $X \times E$, where E is an Euclidean space. Assume that there exists a convex cone C of X with nonempty interior, such that $\mathcal{K} = \mathcal{K} + C \times \{0\}$. Then* $\mathrm{ri}(\mathcal{K}) \neq \emptyset$.

Proof. Denote by K_1, K_2 the projection of $\mathcal{K}$ into X and $\mathbb{R}^n$, resp. Redefining E as $\overline{\mathrm{aff}}(K_2)$, we reduce the analysis to the case when $\mathrm{int}(K_2) \neq \emptyset$, and we will prove that $\mathcal{K}$ has a non empty interior. Let $k \in K$, $k = (k_1, k_2)$, be such that $k_2 \in \mathrm{int}(K_2)$. Let $c \in \mathrm{int}(C)$, and set $k' = k + (c, 0) = (k_1 + c, k_2)$.

Denote by n the dimension of E. Since $k_2 \in \mathrm{int}(K_2))$, there exists $n+1$ elements $f_1, \dots, f_{n+1}$ in K_2 such that $k_2 + \varepsilon_2 B_E \in \mathrm{conv}(f_1, \dots, f_{n+1})$. By definition of K_2 there exist $e_1, \dots, e_{n+1}$ in K_1 such that $(e_i, f_i) \in \mathcal{K}$, for all $i = 1$ to $n+1$. Since $\mathcal{K}$ is convex, $k' := (k'_1, k_2) \in \mathcal{K}$. In addition, let $\varepsilon_1 > 0$ be such that $B(c, \varepsilon_1) \in C$. Then $(e_i + B(c, \varepsilon_1), f_i) \in \mathcal{K}$ for all $i = 1$ to $n+1$. Since $\mathcal{K}$ is convex it contains the convex combinations say Δ of these sets, and we have

$$\Delta = \mathrm{conv}((e_1, f_1), \dots, (e_{n+1}, f_{n+1})) + B(c, \varepsilon_1) \times \{0\}. \tag{82}$$

Since the above convex hull has a second projection with a non empty interior, the conclusion follows. ■

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