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WEAK SOLVABILITY FOR A CLASS OF CONTACT PROBLEMS*

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Abstract

A unilateral frictionless contact model, under the small deformations hypothesis, for static processes is considered. We model the behavior of the material by a constitutive law stated in a subdifferential form. The contact is described with Signorini's condition. Our study focuses on the weak solvability of the model, based on a weak formulation with dual Lagrange multipliers.

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1 Introduction

The purpose of this paper is to investigate the weak solvability of a unilateral frictionless contact problem using a technique with dual Lagrange multipliers. The weak formulations with *dual Lagrange multipliers* allow to write

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efficient algorithms in order to approximate the weak solutions; see for example [6, 7, 8, 9] where contact problems involving linearly elastic materials were considered. An investigation of the weak solvability via dual Lagrange multipliers for a class of elasto-piezoelectric or viscoplastic contact problems, can be found in [6, 12, 13, 17].

In the present work, the behavior of the materials is described by using the subdifferential of a proper, convex, lower semicontinuous functional and the contact is modelled with Signorini's condition with zero gap. The results extend and improve the results obtained in the recent paper [13], where a unilateral frictionless contact model for nonlinearly elastic materials is analyzed.

Our investigation requires a background of convex analysis, functional analysis, variational calculus, mechanics of solids and contact mechanics; the reader can consult [3, 4, 5, 11, 14, 15, 16, 18].

The rest of the paper is structured as follows. In Section 2 we indicate some notation and preliminaries. In Section 3 we state the mechanical model and its weak formulation via dual Lagrange multipliers. In Section 4 we deliver two abstract results. These abstract results will be applied in Section 5 in order to prove the weak solvability of the considered model.

2 Notation and preliminaries

Let us denote by S^3 the space of second order symmetric tensors on R^3. Every field in R^3 or S^3 will be typeset in boldface. By $\cdot$ and $|\cdot|$ we will denote the inner product and the Euclidean norm on R^3 and S^3, respectively. Thus,

$$\boldsymbol{u}\cdot\boldsymbol{v}=u_iv_i,\ |\boldsymbol{v}|=(\boldsymbol{v}\cdot\boldsymbol{v})^{1/2},\ \boldsymbol{u},\boldsymbol{v}\in R^3,$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau}=\sigma_{ij}\tau_{ij},\ |\boldsymbol{\tau}|=(\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{1/2},\ \boldsymbol{\sigma},\boldsymbol{\tau}\in S^3.$$

Here and below, the indices i and j run between 1 and 3 and the summation convention over repeated indices is adopted.

Let $\Omega\subset R^3$ be a bounded domain. We introduce the following functional spaces on Ω,

$$\begin{aligned} H &= \{\boldsymbol{u}=(u_i)\mid u_i\in L^2(\Omega)\}, & \mathcal{H} &= \{\boldsymbol{\sigma}=(\sigma_{ij})\mid \sigma_{ij}=\sigma_{ji}\in L^2(\Omega)\},\\ H_1 &= \{\boldsymbol{u}\in H\mid \boldsymbol{\varepsilon}(\boldsymbol{u})\in\mathcal{H}\}, & \mathcal{H}_1 &= \{\boldsymbol{\sigma}\in\mathcal{H}\mid Div\,\boldsymbol{\sigma}\in H\}\end{aligned}$$

where

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad Div\,\boldsymbol{\sigma} = (\sigma_{ij,j}).$$

The index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. The spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with the inner products,

$$(\boldsymbol{u}, \boldsymbol{v})_H = \int_\Omega u_i v_i \, dx, \qquad (\boldsymbol{u}, \boldsymbol{v})_{H_1} = (\boldsymbol{u}, \boldsymbol{v})_H + (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}},$$
$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij} \, dx, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (Div\,\boldsymbol{\sigma}, Div\,\boldsymbol{\tau})_H.$$

The associated norms on the spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are denoted by $\|\cdot\|_H$, $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{H_1}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively.

We assume that the boundary of Ω, denoted by Γ, is Lipschitz continuous. We denote by $\boldsymbol{\nu}$ the unit outward normal vector on the boundary, defined almost everywhere.

Let us denote by $\boldsymbol{\gamma}$ the Sobolev trace operator,

$$\boldsymbol{\gamma} : H_1 \to L^2(\Gamma)^3,$$

and by H_Γ, the image of H_1 by γ, i.e., $H_\Gamma = \boldsymbol{\gamma}(H_1)$. We recall that $\boldsymbol{\gamma}$ is a linear, continuous and compact operator. Moreover, it is known that the space H_Γ is a Hilbert space. In addition, we recall that there exists a linear and continuous operator Z,

$$Z : H_\Gamma \to H_1,$$

such that

$$\gamma(Z(\boldsymbol{\zeta})) = \boldsymbol{\zeta} \qquad \forall \boldsymbol{\zeta} \in H_\Gamma. \tag{1}$$

The operator Z is called *the inverse to the right* of the operator $\boldsymbol{\gamma}$. We consider the Hilbert space

$$V = \{\, \boldsymbol{v} \in H_1 \mid \boldsymbol{\gamma v} = \boldsymbol{0} \text{ a.e. on } \Gamma_1 \}. \tag{2}$$

We underline that

$$\boldsymbol{\gamma}(Z(\boldsymbol{\gamma v})) = \boldsymbol{\gamma v} \quad \forall \boldsymbol{v} \in V.$$

For a vectorial field $\boldsymbol{v}$, we denote by v_ν and $\boldsymbol{v}_\tau$ the *normal* and the *tangential* components on the boundary, defined as follows,

$$v_\nu = \boldsymbol{\gamma v} \cdot \boldsymbol{\nu}, \quad \boldsymbol{v}_\tau = \boldsymbol{\gamma v} - v_\nu \boldsymbol{\nu}.$$

Let Γ_1 be a measurable part of Γ such that $meas(\Gamma_1) > 0$. Let us remember Korn's inequality: there exists $c_K = c_K(\Omega, \Gamma_1) > 0$ such that

$$\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}} \geq c_K \|\boldsymbol{v}\|_{H_1}, \qquad \forall \boldsymbol{v} \in V.$$

Using Korn's inequality, it can be proved that the space V is a Hilbert space endowed with the following scalar product,

$$(\cdot,\cdot)_V : V \times V \to R; \quad (\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}.$$

Keeping in mind (1), it is straightforward to verify that

$$Z(\boldsymbol{\zeta}) \in V \qquad \forall \boldsymbol{\zeta} \in \boldsymbol{\gamma}(V).$$

Furthermore,

$$R: \ \boldsymbol{\gamma}(V) \to V, \quad R(\boldsymbol{\zeta}) = Z(\boldsymbol{\zeta}), \tag{3}$$

is a linear and continuous operator.

According to [13], the space $\boldsymbol{\gamma}(V)$ is a closed subspace of H_Γ. Thus, $\boldsymbol{\gamma}(V)$ is a Hilbert space endowed with the inner product

$$(\cdot,\cdot)_{\gamma(V)} : \boldsymbol{\gamma}(V) \times \boldsymbol{\gamma}(V) \to R, \quad (\boldsymbol{\zeta}, \boldsymbol{\phi})_{\gamma(V)} = (\boldsymbol{\zeta}, \boldsymbol{\phi})_{H_\Gamma} \ \forall \boldsymbol{\zeta},\, \boldsymbol{\phi} \in \boldsymbol{\gamma}(V).$$

For a regular (say C^1) stress field $\boldsymbol{\sigma}$, the application of its trace on the boundary to $\boldsymbol{\nu}$ is the Cauchy stress vector $\boldsymbol{\sigma\nu}$. Furthermore, we define the *normal* and *tangential* components of the Cauchy vector on the boundary by the formulas

$$\sigma_\nu = (\boldsymbol{\sigma\nu}) \cdot \boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu \boldsymbol{\nu}$$

and we note that the following identity takes place,

$$\boldsymbol{\sigma\nu} \cdot \boldsymbol{v} = \sigma_\nu \, v_\nu + \boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau.$$

Finally, we recall the useful Green formula,

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (Div\, \boldsymbol{\sigma}, \boldsymbol{v})_H = \int_\Gamma \boldsymbol{\sigma\nu} \cdot \boldsymbol{v} \, da \quad \forall \boldsymbol{v} \in H_1. \tag{4}$$

For a proof of the formula (4) and more details related to this section, we send the reader to [4]. In order to facilitate the reading, we recall some elements of convex analysis.

Theorem 1. *Let* $(X, (\cdot,\cdot)_X, \|\cdot\|_X)$, $(Y, (\cdot,\cdot)_Y, \|\cdot\|_Y)$ *be two Hilbert spaces and let* $A \subseteq X$, $B \subseteq Y$ *be non-empty, closed, convex subsets. Assume that a real functional* $\mathcal{L} : A \times B \to R$ *satisfies the following conditions*

$$\forall \mu \in B, \quad v \to \mathcal{L}(v,\, \mu) \quad \textit{is convex and lower semicontinuous};$$
$$\forall v \in A, \quad \mu \to \mathcal{L}(v,\, \mu) \quad \textit{is concave and upper semicontinuous.}$$

Moreover,

$$A \textit{ is bounded or } \lim_{\|v\|_X \to \infty, v \in A} \mathcal{L}(v, \mu_0) = \infty \textit{ for some } \mu_0 \in B$$
$$\textit{and}$$
$$B \textit{ is bounded or } \lim_{\|\mu\|_Y \to \infty, \mu \in B} \inf_{v \in A} \mathcal{L}(v,\, \mu) = -\infty.$$

Then, the functional $\mathcal{L}$ *has at least one saddle point.*

For the proof of this theorem see [3]; more details on the saddle point theory and its applications can be found in [1, 2, 3, 5]. We also recall the definition of Gâteaux differentiable functions.

Definition 1. *Let* $(X, (\cdot,\cdot)_X, \|\cdot\|_X)$ *be a Hilbert space. Let* $\phi : X \to R$ *and let* $u \in X$. *Then* ϕ *is Gâteaux differentiable at* u *if there exists an element* $\nabla\phi(u) \in X$ *such that*

$$\lim_{t \to 0} \frac{\phi(u + tv) - \phi(u)}{t} = (\nabla\phi(u), v)_X, \quad \forall v \in X.$$

The element $\nabla\phi(u)$ which satisfies the relation above is unique and is called *the gradient of* ϕ *at* u. The function $\phi : X \to R$ is said to be *Gâteaux differentiable* if it is Gâteaux differentiable at every point of X. In this case, the operator $\nabla\phi : X \to X$ that maps every element $u \in X$ into the element $\nabla\phi(u)$ is called the *gradient operator* of ϕ. The convexity of Gâteaux differentiable functions can be characterized as follows.

Theorem 2. *Let* $(X, (\cdot,\cdot)_X, \|\cdot\|_X)$ *be a Hilbert space and let* $\phi : X \to R$ *be a Gâteaux differentiable function. Then,* ϕ *is convex if and only if*

$$\phi(v) - \phi(u) \geq (\nabla\phi(u), v - u)_X, \quad \forall v \in X.$$

For the proof of this theorem we refer e.g. to [10, 16].

3 The model and its weak formulation

We consider a body that occupies the bounded domain $\Omega \subset R^3$, with the boundary partitioned into three measurable parts, Γ_1, Γ_2 and Γ_3, such that $meas(\Gamma_1) > 0$. The unit outward normal to Γ is denoted by $\boldsymbol{\nu}$ and is defined almost everywhere. The body Ω is clamped on Γ_1, body forces of density $\boldsymbol{f}_0$ act on Ω and surface traction of density $\boldsymbol{f}_2$ act on Γ_2. On Γ_3 the body can be in contact with a rigid foundation. In order to describe the behavior of the materials, we use a nonlinear constitutive law expressed by the subdifferential of a proper, convex, lower semicontinuous functional and the contact will be modelled using Signorini's condition with zero gap. We denote by $\boldsymbol{u} = (u_i)$ the displacement field, by $\boldsymbol{\varepsilon} = (\varepsilon_{ij})$ the infinitesimal strain tensor and by $\boldsymbol{\sigma} = (\sigma_{ij})$ the Cauchy stress tensor. To resume, we are interested to study the following problem.

Problem 1. *Find $\boldsymbol{u} : \bar{\Omega} \to R^3$ and $\boldsymbol{\sigma} : \bar{\Omega} \to S^3$, such that*

$$\begin{array}{cl}
Div\, \boldsymbol{\sigma}(\boldsymbol{x}) + \boldsymbol{f}_0(\boldsymbol{x}) = \boldsymbol{0} & in\ \Omega, \\
\boldsymbol{\sigma}(\boldsymbol{x}) \in \partial\omega(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))) & in\ \Omega, \\
\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{0} & on\ \Gamma_1, \\
\boldsymbol{\sigma\nu}(\boldsymbol{x}) = \boldsymbol{f}_2(\boldsymbol{x}) & on\ \Gamma_2, \\
\boldsymbol{\sigma}_\tau = \boldsymbol{0},\ u_\nu(\boldsymbol{x}) \leq 0, \sigma_\nu(\boldsymbol{x}) \leq 0, \sigma_\nu(\boldsymbol{x}) u_\nu(\boldsymbol{x}) = 0 & on\ \Gamma_3.
\end{array}$$

Let us assume that the densities of the volume and surface forces verify

$$\boldsymbol{f}_0 \in H; \quad \boldsymbol{f}_2 \in L^2(\Gamma_2)^3. \tag{5}$$

Concerning the constitutive function ω we assume:

$$\left.\begin{array}{l}
\omega : S^3 \to [0, \infty) \text{ is a convex, lower semicontinuous functional,} \\
\text{there exists } \alpha_1,\, \alpha_2 > 0 :\ \alpha_1 |\boldsymbol{\varepsilon}|^2 \geq \omega(\boldsymbol{\varepsilon}) \geq \alpha_2 |\boldsymbol{\varepsilon}|^2 \quad \forall \boldsymbol{\varepsilon} \in S^3.
\end{array}\right\} \tag{6}$$

To give an example of such a function, we can consider

$$\omega : S^3 \to [0, \infty), \quad \omega(\boldsymbol{\varepsilon}) = \frac{1}{2} \mathcal{A} \boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} + \frac{\beta}{2} |\boldsymbol{\varepsilon} - P_K \boldsymbol{\varepsilon}|^2 \tag{7}$$

where $\mathcal{A}$ is a fourth order symmetric tensor satisfying the ellipticity condition, β is a strictly positive constant, $K \subset S^3$ denotes a closed, convex set containing the element 0_{S^3} and $P_K : S^3 \to K$ is the projection operator.

We are interested to write a weak formulation of Problem 1.

Let us define a functional as follows,

$$W : \mathcal{H} \to [0, \infty), \quad W(\boldsymbol{\tau}) = \int_\Omega \omega(\boldsymbol{\tau}(\boldsymbol{x}))dx.$$

Since $\boldsymbol{\sigma}(\boldsymbol{x}) \in \partial\omega(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))$, we have

$$\begin{aligned} \omega(\boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x}))) - \omega(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))) \ &\geq \ \boldsymbol{\sigma}(\boldsymbol{x}) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x})) - \boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))), \\ &\text{a.e. in } \ \Omega, \forall \boldsymbol{v} \in H_1. \end{aligned}$$

For regular enough functions involved in the writing of Problem 1, after integration on Ω and using Green's formula (4), we get

$$\begin{aligned} W(\boldsymbol{\varepsilon}(\boldsymbol{v})) - W(\boldsymbol{\varepsilon}(\boldsymbol{u})) \geq \int_\Gamma \boldsymbol{\sigma\nu} \cdot (\boldsymbol{\gamma v} - \boldsymbol{\gamma u})da \\ -(Div\boldsymbol{\sigma}, \boldsymbol{v} - \boldsymbol{u})_H, \qquad \forall \boldsymbol{v} \in H_1, \end{aligned}$$

and from this

$$\begin{aligned} W(\boldsymbol{\varepsilon}(\boldsymbol{v})) - W(\boldsymbol{\varepsilon}(\boldsymbol{u})) \geq \int_{\Gamma_2} \boldsymbol{f}_2 \cdot (\boldsymbol{\gamma v} - \boldsymbol{\gamma u})da \\ + \int_{\Gamma_3} \boldsymbol{\sigma\nu} \cdot (\boldsymbol{\gamma v} - \boldsymbol{\gamma u})da + (\boldsymbol{f}_0, \boldsymbol{v} - \boldsymbol{u})_H \quad \forall \boldsymbol{v} \in V, \end{aligned}$$

where V is the functional space defined by (2).

Next, we define the functional

$$J : V \to [0, \infty), \quad J(\boldsymbol{v}) = W(\boldsymbol{\varepsilon}(\boldsymbol{v})). \tag{8}$$

Using Riesz's representation theorem, we define $\boldsymbol{f} \in V$ as follows,

$$(\boldsymbol{f}, \boldsymbol{v})_V = \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{\gamma v} da + \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v} dx, \quad \forall v \in V.$$

Let us denote by D the dual of the space $\boldsymbol{\gamma}(V)$. We define the following subset of D,

$$\Lambda = \{\boldsymbol{\mu} \in D \ : \ \langle \boldsymbol{\mu}, \boldsymbol{\gamma v} \rangle \leq 0, \forall \boldsymbol{\gamma v} \in \mathcal{K}\}, \tag{9}$$

where

$$\mathcal{K} = \{\boldsymbol{\gamma v} \in \boldsymbol{\gamma}(V) \ : \ v_\nu \leq 0 \text{ a.e. on } \Gamma_3\}.$$

Here and everywhere below, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between D and $\boldsymbol{\gamma}(V)$.

In addition, we define the bilinear form

$$b : V \times D \to R, \quad b(\boldsymbol{v}, \boldsymbol{\mu}) = \langle \boldsymbol{\mu}, \boldsymbol{\gamma v} \rangle \qquad \forall \boldsymbol{v} \in V, \mu \in D \tag{10}$$

and the Lagrange multiplier $\lambda \in D$,

$$\langle \lambda, \boldsymbol{\gamma v} \rangle = -\int_{\Gamma_3} \sigma_\nu v_\nu da, \qquad \forall \boldsymbol{\gamma v} \in \boldsymbol{\gamma}(V).$$

Thus, we are led to the following weak formulation.

Problem 2. *Find $\boldsymbol{u} \in V$ and $\boldsymbol{\lambda} \in \Lambda$ such that*

$$\begin{aligned} J(\boldsymbol{v}) - J(\boldsymbol{u}) + b(\boldsymbol{v} - \boldsymbol{u}, \boldsymbol{\lambda}) &\geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V, \quad && \forall \boldsymbol{v} \in V \\ b(\boldsymbol{u}, \boldsymbol{\mu} - \boldsymbol{\lambda}) &\leq 0, && \forall \boldsymbol{\mu} \in \Lambda. \end{aligned}$$

The solvability of this weak formulation of Problem 2 will be analyzed in Section 5, based on the abstract results in Section 4 .

4 Abstract results

Let $(X, (\cdot,\cdot)_X, ||\cdot||_X)$ and $(Y, (\cdot,\cdot)_Y, ||\cdot||_Y)$ be Hilbert spaces. We consider a functional J as follows,

$$\left.\begin{array}{l} J : X \to [0, \infty) \text{ convex, lower semicontinuous,} \\ \text{there exists } m_1,\, m_2 > 0 : m_1\|v\|_X^2 \geq J(v) \geq m_2\|v\|_X^2 \quad \forall v \in X. \end{array}\right\} \tag{11}$$

In addition, we consider

$$\left.\begin{array}{l} b : X \times Y \to R \;\text{ a bilinear form such that} \\ \text{i) there exists } M_b > 0 : |b(v, \mu)| \leq M_b\|v\|_X\|\mu\|_Y \forall v \in X,\, \mu \in Y, \\ \text{ii) there exists } \alpha > 0 : \displaystyle\inf_{\mu \in Y, \mu \neq 0_Y} \sup_{v \in X, v \neq 0_X} \frac{b(v, \mu)}{\|v\|_X\|\mu\|_Y} \geq \alpha. \end{array}\right\} \tag{12}$$

Finally,

$$\Lambda \text{ is a closed, convex, unbounded subset of } Y \text{ that contains } 0_Y. \tag{13}$$

We are interested to prove the existence of the solutions of the following variational problem.

Problem 3. *Find $u \in X$ and $\lambda \in \Lambda$ such that*

$$\begin{aligned} J(v) - J(u) + b(v-u,\lambda) &\geq (f, v-u)_X \quad && \forall v \in X \\ b(u, \mu - \lambda) &\leq 0 && \forall \mu \in \Lambda. \end{aligned}$$

Theorem 3. *Assume* (11), (12) *and* (13)*. Then, Problem* 3 *has at least one solution.*

The proof of this theorem will be carried out in several steps.

We underline that Λ is an unbounded subset of the space Y. Let us define

$$\Lambda_n = \{\mu \in \Lambda, \|\mu\|_Y \leq n\},$$

a bounded subset of Λ, for every $n \in N^*$.

We state the following auxiliary problem.

Problem 4. *Find $u_n \in X$ and $\lambda_n \in \Lambda_n$ such that*

$$J(v) - J(u_n) + b(v-u_n,\lambda_n) \geq (f, v-u_n)_X \quad \forall v \in X, \tag{14}$$

$$b(u_n, \mu - \lambda_n) \leq 0 \quad \forall \mu \in \Lambda_n. \tag{15}$$

Lemma 1. *Problem* 4 *has at least one solution.*

Proof. Let us define

$$\mathcal{L} : X \times \Lambda_n \to R \ , \quad \mathcal{L}(v,\mu) := J(v) - (f,v)_X + b(v,\mu).$$

A pair (u_n, λ_n) is a solution of Problem 4 if and only if it is a saddle point of the functional $\mathcal{L}$, i. e.

$$\mathcal{L}(u_n,\mu) \leq \mathcal{L}(u_n,\lambda_n) \leq \mathcal{L}(v,\lambda_n), \qquad \forall v \in X, \forall \mu \in \Lambda_n. \tag{16}$$

Indeed, the first of the inequalities above is equivalent to the inequality

$$b(u_n, \mu - \lambda_n) \leq 0, \qquad \forall \mu \in \Lambda_n.$$

After replacing $\mathcal{L}$ from its definition, the second of the two inequalities (16) becomes

$$J(v) - J(u_n) + b(v-u_n,\lambda_n) \geq (f, v-u_n)_X, \qquad \forall v \in X,$$

so the equality between the set of solutions of the Problem 4 and the set of saddle points of the functional $\mathcal{L}$ is proved.

Now we prove that the functional $\mathcal{L}$ has at least one saddle point.

Keeping in mind the definition of the functional $\mathcal{L}$, as J is convex and lower semicontinuous and the functional b is bilinear and continuous, it is straightforward to deduce that, for all $\mu \in \Lambda$, $v \to \mathcal{L}(v, \mu)$ is convex and lower semicontinuous, and, for all $v \in X$, $\mu \to \mathcal{L}(v, \mu)$ is concave and upper semicontinuous. In addition, we note that

$$\mathcal{L}(v, 0_Y) = J(v) + b(v, 0_Y) - (f, v)_X \geq m_2\|v\|_X^2 - \|f\|_X\|v\|_X,$$

which allows us to say that

$$\lim_{\|v\|_X \to \infty} \mathcal{L}(v, 0_Y) = \infty.$$

Then, also taking into account that Λ_n is a bounded subset of the space Y, we apply Theorem 1 to deduce that the functional $\mathcal{L}$ has at least one solution. Since the set of the solutions of Problem 4 is the same with the set of the saddle points of the functional $\mathcal{L}$, we conclude that Problem 4 has at least one solution.

Lemma 2. *There is some $n_0 > 0$ such that $\|\lambda_{n_0}\|_Y < n_0$.*

Proof. Let us assume that $\|\lambda_n\|_Y = n, \forall n \geq 1$. By using the inf-sup property of the form b, see (12), we get

$$\alpha\|\lambda_n\|_Y \quad \leq \quad \sup_{w \in X,\, w \neq 0_X} \frac{b(w, \lambda_n)}{\|w\|_X}.$$

Let us take $w \in X$ $w \neq 0_X$, arbitrarily chosen. Putting $v = u_n - tw$ with $t > 0$ in (14), we have

$$b(tw, \lambda_n) \leq (f, tw)_X + J(u_n - tw) - J(u_n).$$

Therefore, taking into account (11) we reach to

$$tb(w, \lambda_n) \leq t(f, w)_X + m_1\|u_n - tw\|_X^2.$$

Consequently, we get

$$\begin{aligned}\frac{b(w, \lambda_n)}{\|w\|_X} &\leq \|f\|_X + \frac{m_1\|u_n - tw\|_X^2}{t\|w\|_X} \\ &\leq \|f\|_X + \frac{2m_1(\|u_n\|_X^2 + t^2\|w\|_X^2)}{t\|w\|_X}.\end{aligned}$$

As we have considered $t > 0$, now we put $t = \frac{1}{\|w\|_X}$ and so

$$\frac{b(w, \lambda_n)}{\|w\|_X} \leq \|f\|_X + 2m_1(\|u_n\|_X^2 + 1). \tag{17}$$

If $v = 0_X$ in (14), then

$$J(u_n) \leq -b(u_n, \lambda_n) + (f, u_n)_X.$$

Then, if we put $\mu = 0_Y$ in (15),

$$-b(u_n, \lambda_n) \leq 0.$$

So,

$$J(u_n) \leq (f, u_n)_X \leq \|f\|_X \|u_n\|_X.$$

Since

$$J(u_n) \geq m_2 \|u_n\|_X^2$$

we infer that

$$\|u_n\|_X \leq \frac{\|f\|_X}{m_2}, \quad \forall n \geq 1.$$

So, by (17) , we deduce

$$\frac{b(w, \lambda_n)}{\|w\|_X} \leq \|f\|_X + 2m_1\Big(\frac{\|f\|_X^2}{m_2^2} + 1\Big),$$

and then

$$\sup_{w \in X,\, w \neq 0_X} \frac{b(w, \lambda_n)}{\|w\|_X} \leq \|f\|_X + 2m_1\Big(\frac{\|f\|_X^2}{m_2^2} + 1\Big).$$

Thus, by our assumption, for all $n \geq 1$,

$$\alpha\, n \leq \|f\|_X + 2m_1\Big(\frac{\|f\|_X^2}{m_2^2} + 1\Big),$$

which is impossible.

[Proof of Theorem 3] Let $n_0 > 0$ be a positive integer such that $\|\lambda_{n_0}\|_Y < n_0$ and let $\mu \in \Lambda$ be arbitrarily fixed. We define

$$\sigma = \lambda_{n_0} + \varepsilon(\mu - \lambda_{n_0}).$$

This element σ is an element of Λ_{n_0}. Indeed, if $\mu = \lambda_{n_0}$, we can take $\varepsilon = 1$; if $\mu \neq \lambda_{n_0}$, then, we take ε such that $|\varepsilon| < \frac{n_0 - \|\lambda_{n_0}\|_Y}{\|\mu - \lambda_{n_0}\|_Y}$. Since

$$b(u_{n_0}, \sigma - \lambda_{n_0}) \leq 0,$$

we have

$$\varepsilon b(u_{n_0}, \mu - \lambda_{n_0}) \leq 0.$$

As μ was arbitrarily fixed, we deduce

$$b(u_{n_0}, \mu - \lambda_{n_0}) \leq 0, \qquad \forall \mu \in \Lambda.$$

It follows that (u_{n_0}, λ_{n_0}) is a solution of Problem 3.

Under additional assumptions, we investigate the uniqueness and the stability of the solution of Problem 3. More precisely, in addition to (11), (12) and (13), we consider the following assumptions

$$J : X \to [0, \infty) \text{ is Gâteaux differentiable,} \tag{18}$$

$$\text{there exists } L > 0 : \|\nabla J(u) - \nabla J(v)\|_X \leq L\|u - v\|_X \quad \forall u, v \in X, \tag{19}$$

$$\text{there exists } m > 0 : (\nabla J(u) - \nabla J(v), u - v)_X \geq m\|u - v\|_X^2 \quad \forall u, v \in X. \tag{20}$$

Theorem 4. *Assume* (11), (12), (13), (18) *and* (20). *Then, Problem* 3 *has a unique solution. If we assume in addition that* (19) *holds, then the solution depends Lipschitz continuously on the data f.*

Proof. The set of the solutions of Problem 3 coincide with the set of the solutions of the following problem: find $u \in X$ and $\lambda \in \Lambda$ such that

$$(P) : \begin{cases} (\nabla J(u), v)_X + b(v, \lambda) = (f, v)_X & \forall v \in X, \\ b(u, \mu - \lambda) \leq 0 & \forall \mu \in \Lambda. \end{cases}$$

Indeed, if (u, λ) is a solution of the problem (P), then

$$(\nabla J(u), v-u)_X + b(v-u, \lambda) = (f, v-u)_X \quad \forall v \in X.$$

As J is Gâteaux differentiable and convex we know that

$$(\nabla J(u), v-u)_X \leq J(v) - J(u), \quad \forall v \in X.$$

Taking into account the last two relations, we conclude that

$$J(v) - J(u) + b(v-u, \lambda) \geq (f, v-u)_X \quad \forall v \in X,$$

and due to the fact that $b(u, \mu - \lambda) \leq 0 \quad \forall \mu \in \Lambda$, this means that (u, λ) is a solution of Problem 3.

Conversely, let (u, λ) be a solution of Problem 3. Consequently,

$$J(v) - J(u) + b(v-u, \lambda) \geq (f, v-u)_X \quad \forall v \in X. \tag{21}$$

Let $t > 0$ be arbitrarily chosen. Using the above inequality with $u + tv$ instead of v we get

$$\frac{J(u+tv) - J(u)}{t} \geq (f, v)_X - b(v, \lambda) \quad \forall v \in X.$$

By taking $t \to 0$ we have

$$(\nabla J(u), v)_X \geq (f, v)_X - b(v, \lambda) \quad \forall v \in X. \tag{22}$$

On the other hand, if we put $u - tv$ instead of v in (21) it follows

$$\frac{J(u+(-t)v) - J(u)}{-t} \leq (f, v)_X - b(v, \lambda) \quad \forall v \in X,$$

and taking again $t \to 0$ we have

$$(\nabla J(u), v)_X \leq (f, v)_X - b(v, \lambda) \quad \forall v \in X. \tag{23}$$

The relations (22) and (23) lead to the equality

$$(\nabla J(u), v)_X = (f, v)_X - b(v, \lambda) \quad \forall v \in X,$$

which allows us to conclude that (u, λ) is a solution of the problem (P). The problem (P) has at least one solution since we already know that Problem 3 has at least one solution.

We prove now that the solution of the problem (P) is unique. For this purpose we consider (u_1, λ_1) and (u_2, λ_2) solutions of the problem (P). Thus,

$$(\nabla J(u_1) - \nabla J(u_2), v)_X + b(v, \lambda_1 - \lambda_2) = 0 \quad \forall v \in X,$$

and for $v = u_2 - u_1$

$$(\nabla J(u_1) - \nabla J(u_2), u_2 - u_1)_X + b(u_2 - u_1, \lambda_1 - \lambda_2) = 0.$$

Since

$$b(u_2 - u_1, \lambda_1 - \lambda_2) \leq 0,$$

we deduce that

$$(\nabla J(u_1) - \nabla J(u_2), u_1 - u_2)_X \leq 0.$$

Using (20), we obtain

$$m\|u_1 - u_2\|_X^2 \leq (\nabla J(u_1) - \nabla J(u_2), u_1 - u_2)_X \leq 0,$$

and from this, we conclude $u_1 = u_2$. Moreover, as

$$(\nabla J(u_1) - \nabla J(u_2), v)_X + b(v, \lambda_1 - \lambda_2) = 0 \quad \forall v \in X,$$

we get

$$b(v, \lambda_1 - \lambda_2) = 0 \quad \forall v \in X.$$

Thus, by the inf-sup property of the form b we get

$$\alpha\|\lambda_1 - \lambda_2\|_Y \leq \sup_{v \in X, v \neq 0_X} \frac{b(v, \lambda_1 - \lambda_2)}{\|v\|_X} = 0,$$

and so it follows that $\lambda_1 = \lambda_2$. Finally, we conclude that the problem (P) has a unique solution. Consequently, Problem 3 has a unique solution.

Next we prove that the solution of the problem (P) depends Lipschitz continuously on the data f. For this purpose we consider (u_1, λ_1) and (u_2, λ_2) solutions of the problem (P) corresponding to the data f_1 and f_2, respectively. Therefore

$$\begin{aligned}(\nabla J(u_1), v)_X + b(v, \lambda_1) &= (f_1, v)_X \quad \forall v \in X,\\ (\nabla J(u_2), v)_X + b(v, \lambda_2) &= (f_2, v)_X \quad \forall v \in X,\end{aligned}$$

and then

$$b(v, \lambda_1 - \lambda_2) = (f_1 - f_2, v)_X + (\nabla J(u_2) - \nabla J(u_1), v)_X \quad \forall v \in X,$$

which, for $v = u_1 - u_2$, leads to

$$(\nabla J(u_1) - \nabla J(u_2), u_1 - u_2)_X = (f_1 - f_2, u_1 - u_2)_X + b(u_2 - u_1, \lambda_1 - \lambda_2). \quad (24)$$

From the inf-sup property of the form b we get

$$\begin{aligned}\alpha\|\lambda_1 - \lambda_2\|_Y &\leq \sup_{v \in X, v \neq 0_X} \frac{b(v, \lambda_1 - \lambda_2)}{\|v\|_X}\\ &= \sup_{v \in X, v \neq 0_X} \frac{(f_1 - f_2, v)_X + (\nabla J(u_2) - \nabla J(u_1), v)_X}{\|v\|_X}\\ &\leq \|\nabla J(u_1) - \nabla J(u_2)\|_X + \|f_1 - f_2\|_X,\end{aligned}$$

and, using (19), it follows that

$$\alpha\|\lambda_1 - \lambda_2\|_Y \leq L\|u_1 - u_2\|_X + \|f_1 - f_2\|_X. \quad (25)$$

Now, from (24) and $b(u_2 - u_1, \lambda_1 - \lambda_2) \leq 0$, we can write

$$\begin{aligned}(\nabla J(u_1) - \nabla J(u_2), u_1 - u_2)_X &\leq (f_1 - f_2, u_1 - u_2)_X\\ &\leq \|f_1 - f_2\|_X \|u_1 - u_2\|_X.\end{aligned}$$

This inequality together with (20) leads us to

$$m\|u_1 - u_2\|_X \leq \|f_1 - f_2\|_X. \quad (26)$$

Thus, from (25) and (26), we conclude that there exists $A = A(L, \alpha, m) > 0$ such that

$$\|u_1 - u_2\|_X + \|\lambda_1 - \lambda_2\|_Y \leq A\|f_1 - f_2\|_X.$$

5 The solvability of Problem 2

The purpose of this section is to study the solvability of Problem 2 using the abstract results delivered in the previous section.

Theorem 5 (An existence result)**.** *Assume* (5) *and* (6). *Then, Problem* 2 *has at least one solution.*

Proof. Using (5) and (6) we deduce that the functional J, see (8), has the properties (11). As ω is a convex lower semicontinuous function, it follows that J is also a convex lower semicontinuous function. In addition,

$$J(\boldsymbol{v}) = \int_\Omega \omega(\boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x})))dx \geq \alpha_2 \int_\Omega |\boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x}))|^2 dx = \alpha_2(\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} = \alpha_2 \|\boldsymbol{v}\|_V^2.$$

Also,

$$J(\boldsymbol{v}) = \int_\Omega \omega(\boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x})))d\boldsymbol{x} \leq \alpha_1 \|\boldsymbol{v}\|_V^2.$$

The bilinear form b, see (10), and the subset Λ, see (9), verify the required properties (12) with $X = V$ and $Y = D$. We can write

$$|b(\boldsymbol{v}, \boldsymbol{\mu})| \leq \|\boldsymbol{\mu}\|_D \|\boldsymbol{v}\|_{\boldsymbol{\gamma}(V)},$$

and from this, due to the fact that $\boldsymbol{\gamma}$ is a linear and continuous operator, we deduce that there exists $M_b > 0$ such that (12)-i) is verified.

Using the operator R, see (3), it can be proved that there exists $\alpha > 0$ such that the form b defined in (10), verifies (12)-ii). Indeed, there exists $\bar{c} > 0$ such that

$$\begin{aligned} \|\boldsymbol{\mu}\|_D &= \sup_{\boldsymbol{w}\in\boldsymbol{\gamma}(V),\, \boldsymbol{w}\neq\mathbf{0}_{\boldsymbol{\gamma}(V)}} \frac{<\boldsymbol{\mu}, \boldsymbol{w}>}{\|\boldsymbol{w}\|_{\boldsymbol{\gamma}(V)}} \\ &\leq \bar{c} \sup_{\boldsymbol{w}\in\boldsymbol{\gamma}(V),\, \boldsymbol{w}\neq\mathbf{0}_{\gamma(V)}} \frac{b(R\boldsymbol{w}, \boldsymbol{\mu})}{\|R\boldsymbol{w}\|_V} \\ &\leq \bar{c} \sup_{v\in V,\, v\neq\mathbf{0}_V} \frac{b(\boldsymbol{v}, \boldsymbol{\mu})}{\|\boldsymbol{v}\|_V}, \end{aligned}$$

and we can take $\alpha = \dfrac{1}{\bar{c}}$.

We end the proof of this theorem by applying Theorem 3, thus getting the existence of the solution of Problem 2.

Let us assume in addition to the hypotheses (5) and (6) the following hypotheses.

$$\omega \text{ is Gâteaux differentiable,} \tag{27}$$

$$\text{there exists } L > 0 : |\nabla\omega(\boldsymbol{\varepsilon}) - \nabla\omega(\boldsymbol{\tau})| \leq L|\boldsymbol{\varepsilon} - \boldsymbol{\tau}| \quad \forall \boldsymbol{\varepsilon}, \boldsymbol{\tau} \in S^3, \tag{28}$$

$$\text{there exists } m > 0 : (\nabla\omega(\boldsymbol{\varepsilon}) - \nabla\omega(\boldsymbol{\tau})) \cdot (\boldsymbol{\varepsilon} - \boldsymbol{\tau}) \geq m|\boldsymbol{\varepsilon} - \boldsymbol{\tau}|^2 \quad \forall \boldsymbol{\varepsilon}, \boldsymbol{\tau} \in S^3. \tag{29}$$

Taking into account the properties of the projection operator, it can be proved that the function ω given by example (7), verifies also (27)-(29).

Let us prove the following theorem.

Theorem 6 (An existence, uniqueness and stability result)**.** *Let us assume* (5)*,* (6)*,* (27)-(29)*. Then, Problem* 2 *has a unique solution. Moreover, if* $(\boldsymbol{u}_1, \boldsymbol{\lambda}_1)$ *and* $(\boldsymbol{u}_2, \boldsymbol{\lambda}_2)$ *are two solutions of Problem* 2 *corresponding to the data* $\boldsymbol{f}_1, \boldsymbol{f}_2 \in V$*, then there exists* $C > 0$ *such that*

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V + \|\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2\|_D \leq C\|\boldsymbol{f}_1 - \boldsymbol{f}_2\|_V. \tag{30}$$

Proof. The set of the solutions of Problem 2 coincides with the set of the solutions of the following problem: find $\boldsymbol{u} \in V$ and $\boldsymbol{\lambda} \in \Lambda$ such that

$$\begin{aligned} (\nabla J(\boldsymbol{u}), \boldsymbol{v})_V + b(\boldsymbol{v}, \boldsymbol{\lambda}) &= (\boldsymbol{f}, \boldsymbol{v})_V \qquad && \forall \boldsymbol{v} \in V \\ b(\boldsymbol{u}, \boldsymbol{\mu} - \boldsymbol{\lambda}) &\leq 0 && \forall \boldsymbol{\mu} \in \Lambda. \end{aligned}$$

Using (27)-(29), we deduce that $J : V \to [0, \infty)$ verifies (18)-(20). Indeed,

$$\frac{J(\boldsymbol{u} + t\boldsymbol{v}) - J(\boldsymbol{u})}{t} = \int_\Omega \frac{\omega(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}) + t\boldsymbol{v}(\boldsymbol{x}))) - \omega(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})))}{t} d\boldsymbol{x},$$

and, as ω is Gâteaux differentiable, for any $\delta > 0$ there exists some $\eta > 0$ such that for $|t| < \eta$ we have

$$\frac{J(\boldsymbol{u} + t\boldsymbol{v}) - J(\boldsymbol{u})}{t} \geq \int_\Omega \nabla\omega(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x})) d\boldsymbol{x} - \delta meas(\Omega),$$

$$\frac{J(\boldsymbol{u} + t\boldsymbol{v}) - J(\boldsymbol{u})}{t} \leq \int_\Omega \nabla\omega(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x})) d\boldsymbol{x} + \delta meas(\Omega).$$

Consequently, J is Gâteaux differentiable and

$$(\nabla J(\boldsymbol{u}), \boldsymbol{v})_V = \int_\Omega \nabla\omega(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x}))d\boldsymbol{x}.$$

Then, in order to prove (19) , we consider

$$\begin{aligned}\|\nabla J(\boldsymbol{u}) - \nabla J(\boldsymbol{v})\|_V &= \sup_{\boldsymbol{w}\in V, \boldsymbol{w}\neq 0_V} \frac{(\nabla J(\boldsymbol{u}) - \nabla J(\boldsymbol{v}), \boldsymbol{w})_V}{\|\boldsymbol{w}\|_V} \\ &= \sup_{\boldsymbol{w}\in V, \boldsymbol{w}\neq 0_V} \frac{\int_\Omega (\nabla\omega(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})) - \nabla\omega(\boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x})))) \cdot \boldsymbol{\varepsilon}(\boldsymbol{w}(\boldsymbol{x}))d\boldsymbol{x}}{\|\boldsymbol{w}\|_V} \\ &= \sup_{\boldsymbol{w}\in V, \boldsymbol{w}\neq 0_V} \frac{(\nabla\omega(\boldsymbol{\varepsilon}(\boldsymbol{u})) - \nabla\omega(\boldsymbol{\varepsilon}(\boldsymbol{v})), \boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}}}{\|\boldsymbol{w}\|_V} \\ &\le \sup_{\boldsymbol{w}\in V, \boldsymbol{w}\neq 0_V} \frac{\|\nabla\omega(\boldsymbol{\varepsilon}(\boldsymbol{u})) - \nabla\omega(\boldsymbol{\varepsilon}(\boldsymbol{v}))\|_{\mathcal{H}} \|\boldsymbol{\varepsilon}(\boldsymbol{w})\|_{\mathcal{H}}}{\|\boldsymbol{w}\|_V},\end{aligned}$$

which, together with $\|\boldsymbol{\varepsilon}(\boldsymbol{w})\|_{\mathcal{H}} = \|\boldsymbol{w}\|_V$ and (28), implies that the next relations hold true:

$$\begin{aligned}\|\nabla J(\boldsymbol{u}) - \nabla J(\boldsymbol{v})\|_V &\le \|\nabla\omega(\boldsymbol{\varepsilon}(\boldsymbol{u})) - \nabla\omega(\boldsymbol{\varepsilon}(\boldsymbol{v}))\|_{\mathcal{H}} \\ &\le L\|\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}} \\ &= L\|\boldsymbol{u} - \boldsymbol{v}\|_V.\end{aligned}$$

Now, using (29), we prove that J verifies (20):

$$\begin{aligned}&(\nabla J(\boldsymbol{u}) - \nabla J(\boldsymbol{v}), \boldsymbol{u} - \boldsymbol{v})_V = \\ &\quad = \int_\Omega (\nabla\omega(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))) - \nabla\omega(\boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x})))) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})) - \boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x})))d\boldsymbol{x} \\ &\quad\quad \ge m \int_\Omega |\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})) - \boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x}))|^2 d\boldsymbol{x} = \\ &\quad\quad = m\|\boldsymbol{u} - \boldsymbol{v}\|_V^2.\end{aligned}$$

At this point, we recall that J is also convex and lower semicontinuous and the form b verifies (12) with $X = V$ and $Y = D$. Thus, we may apply Theorem 4, getting that Problem 2 has a unique solution and also, for $(\boldsymbol{u}_1, \boldsymbol{\lambda}_1)$ and $(\boldsymbol{u}_2, \boldsymbol{\lambda}_2)$ two solutions of Problem 2 corresponding to the data $\boldsymbol{f}_1$, $\boldsymbol{f}_2 \in V$, there exists $C > 0$ such that (30) holds.

Hence, the weak solvability of Problem 1 is substantiated.

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