A NOTE ON METRIC SPACES WITH CONTINUOUS MIDPOINTS*

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Abstract

A metric space (X, d) is a continuous midpoint space if there is a continuous map $\mu : X \times X \to X$ such that, for all $(a, b) \in X \times X$, $d(a, \mu(a, b)) = (1/2)d(a, b) = d(b, \mu(a, b))$. A closed subset C of a complete continuous midpoint space is convex if $\forall (a, b) \in C \times C$, $\mu(a, b) \in C$. Under suitable, but natural, assumptions continuous midpoint spaces are absolute retracts; Browder, Michael or Cellina like continuous selection theorems hold; bounded closed convex sets have the fixed point property for nonexpansive maps. Hyperconvex metric spaces, Cartan-Hadamard manifolds and more generally Hadamard spaces or metric spaces with non positive curvature in the sense of Busemann are continuous midpoint spaces.

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1 Introduction

Given two points a and b of a metric space (X, d) a point m of X is a **midpoint for the pair $(\mathbf{a}, \mathbf{b})$** if $d(a, m) = (1/2)d(a, b) = d(b, m)$. For all pairs of points of a complete metric space (X, d) to have a midpoint it is

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necessary and sufficient that the metric be strictly intrinsic[1], Lemma 2.4.8 and Theorem 2.4.16 in [8]. The construction of a shortest path is done through dyadic approximations, an idea which goes back at least to Elie Cartan (Theorem III page 360 of [10][2]). The unit circle with its intrinsic metric shows that it might be impossible to choose for each pair of points (a, b) a shortest path $\gamma_{(a,b)}$ which depends continuously on its end points. As we will see this is a consequence of the fact that one cannot choose for each pair of points a midpoint in a continuous way.

A **continuous midpoint map** on a metric space (X, d) is a continuous map $\mu : X \times X \to X$ such that, for all $(a, b) \in X \times X$, $d(a, \mu(a, b)) = (1/2)d(a, b) = d(b, \mu(a, b))$. If μ is a continuous midpoint then $\check{\mu}(a, b) = \mu(b, a)$ is also a continuous midpoint map. The triple (X, d, μ) is a **continuous midpoint space**. Given a continuous midpoint space (X, d, μ) it is natural to say that **a closed subset C of X is convex** if, for all $(a, b) \in C \times C$, $\mu(a, b) \in C$. We will see that the classical results associated to the names of Browder, Ky Fan, Michael, Cellina and others hold in the context of convex sets of continuous midpoint spaces, with for some of them an additional, but natural condition, on the metric. The class of continuous midpoint spaces is general enough to contain such different objects as hyperconvex metric spaces and Cartan-Hadamard manifolds, or more generally Hadamard spaces and metric spaces of nonpositive curvature in the sense of Busemann. Some of the results alluded to above have been known for some time in the realm of hyperconvex metric spaces - [18] or [21] for example offer two different approaches - or have been established more recently for spaces of nonpositive curvature, [26] and [27].

Examples of continuous midpoint spaces are given in Section 2. Section 3 characterizes complete continuous midpoint spaces as those spaces for which every two points can be joined by a geodesic path and the geodesic path can be chosen in such a way as to depend continuously on its endpoints. Convex sets in a complete continuous midpoint space are introduced in Section 4. Convexity structures are used here as a mean to an end - mainly establishing fixed point and selection theorems. From some given properties of the metric - Busemann, quasi-Busemann, quasiconvexity - corresponding

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[1]The metric d of a metric space (X, d) is strictly intrinsic if for all pairs of points (a, b) there exists a continuous rectifiable path $\gamma : [0, 1] \to X$ from a to b whose length is $d(a, b)$.

[2]As E. Cartan says, *"cet ouvrage est la reproduction d'un cours professé pendant le premier semestre 1925-1926 à la Faculté des Sciences de l'Université de Paris."*

properties of the convexity are derived which in turn, mostly from known results, imply the fixed point and selection theorems in question. The context of section 5 is that of complete continuous midpoint spaces whithout any other particular assumptions; it contains Fan's Intersection Theorem, Fan's Inequality, the Fan-Browder Fixed Point Theorem and Browder's Selection Theorem. Section 6 is about absolute retracts in complete continuous midpoint spaces. Continuous selections for lower semicontinuous multivalued maps are in Section 7 and approximate selections for upper semicontinuous multivalued maps are in Section 8. The fixed point property for nonexpansive maps in continuous midpoint spaces is treated in section 9.

2　Examples of continuous midpoint spaces

(**0**) Closed convex subsets of normed vector spaces.

(**1**) Let $F_1 = \left\{ \big((x_1, y_1), (x_2, y_2)\big) \in \mathbb{R}^2 \times \mathbb{R}^2 : |x_1 - x_2| \le |y_1 - y_2| \right\}$ and $F_2 = \left\{ \big((x_1, y_1), (x_2, y_2)\big) \in \mathbb{R}^2 \times \mathbb{R}^2 : |y_1 - y_2| \le |x_1 - x_2| \right\}$.

If $(a, b) = \big((x_1, y_1), (x_2, y_2)\big) \in F_1$ let

$$
\mu(a, b) = \begin{cases}
\left(x_2, \frac{1}{2}(y_1 + y_2) - \frac{1}{2}|x_1 - x_2|\right) & \text{if} \quad y_1 \le y_2 \\[2mm]
\left(x_2, \frac{1}{2}(y_1 + y_2) + \frac{1}{2}|x_1 - x_2|\right) & \text{if} \quad y_2 \le y_1
\end{cases}
$$

If $(a, b) = \big((x_1, y_1), (x_2, y_2)\big) \in F_2$ let

$$
\mu(a, b) = \begin{cases}
\left(\frac{1}{2}(x_1 + x_2) + \frac{1}{2}|y_1 - y_2|, y_1\right) & \text{if} \quad x_1 \le x_2 \\[2mm]
\left(\frac{1}{2}(x_1 + x_2) - \frac{1}{2}|y_1 - y_2|, y_1\right) & \text{if} \quad x_2 \le x_1
\end{cases}
$$

One can easily see that μ is continuous on $\mathbb{R}^2 \times \mathbb{R}^2$ and that with the metric $d(a, b) = |x_1 - x_2| + |y_1 - y_2|$ one has $d\big(a, \mu(a, b)\big) = d\big(b, \mu(a, b)\big) = (1/2)d\big(a, b\big)$.

This example shows first that, for a given metric, even a norm on a vector space, there might be many continuous midpoint maps (we could have chosen $m(a, b) = \big((1/2)(x_1 + x_2), (1/2)(y_1 + y_2)\big)$) and that a continuous midpoint map does not have to be symmetric (if $a = (0, 0)$ and $b = (1, 1)$ then $\mu(a, b) = (1, 0)$ while $\mu(b, a) = (0, 1)$).

(**3**) A metric space (X, d) is **hyperconvex** if $\bigcap_{\lambda \in \Lambda} B[x_\lambda, r_\lambda] \neq \emptyset$ [3] for all collections of closed balls $\{B[x_\lambda, r_\lambda] : \lambda \in \Lambda\}$ such that for all $\lambda, \lambda' \in \Lambda$ $d(x_\lambda, x_{\lambda'}) \leq r_\lambda + r_{\lambda'}$. Hyperconvex metric spaces are complete. They are characterized by the fact that they are nonexpansive retracts of Banach spaces in which they are embedded. Assume that (X, d) is hyperconvex and let $(E, \|\cdot\|)$ be a Banach space in which (X, d) is isometrically embedded. Let $r : E \to X$ be retraction such that, for all u and v in E, $d(r(u), r(v)) \leq \|u - v\|$ and, for x and y in X let $\mu(x, y) = r\left(\frac{1}{2}(x + y)\right)$. From $d(x, \mu(x, y)) d(r(x), \mu(x, y)) \leq (1/2)\|x - y\| = (1/2)d(x, y)$ and $\mu(x, y) = \mu(y, x)$ we obtain $d(x, \mu(x, y)) = d(y, \mu(x, y)) = (1/2)d(x, y)$. Hyperconvex metric spaces were introduced in [2]. More information relating to fixed points, selections and general properties of hyperconvex spaces can be found in [3], [18], [22], [23] and [24].

The previous argument shows that a nonexpansive retract of a continuous midpoint space is a continuous midpoint space.

(**4**) Hilbert spaces have a unique continuous midpoint map. This is a consequence of the parallelogramm law.

A metric space (X, d) is a **unique continuous midpoint space** if, for all pair of points there exists a unique midpoint and the midpoint map is continuous. If $\mu(x, y)$ is a continuous midpoint map then $\check{\mu}(x, y) = \mu(y, x)$ is also a continuous midpoint. In a unique continuous midpoint space the midpoint map is symmetric.

(**5**) A metric space (X, d) has the **Bruhat-Tits Property** if,

$$\forall (a, b) \in X \times X \ \exists z \in X \text{ such that } \forall x \in X$$
$$d(a, b)^2 + 4d(x, z)^2 \leq 2d(x, a)^2 + 2d(x, b)^2.$$

From the paralellogram law one can see that convex subsets of Hilbert spaces have the Bruhat-Tits Property.

A Bruhat-Tits metric space is a unique continuous midpoint space. This is more or less well known. For the reader's convenience we give a proof.

Taking $x = a$ in the Bruhat-Tits Property we obtain $4d(a, z)^2 \leq d(a, b)^2$ and therefore $2d(a, z) \leq d(a, b)$; similarly, with $x = b$ we obtain $2d(b, z) \leq d(a, b)$. These inequalities can not be strict. We have shown that $2d(a, z) = d(a, b) = 2d(b, z)$.

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[3] $B[x, r]$ denotes the closed ball of radius r centered at x

Given $(a, b) \in X \times X$ let us see that there is a unique point $z \in X$ for which the Bruhat-Tits Property holds.

If $d(a, b)^2 + 4d(x, z_i)^2 \leq 2d(x, a)^2 + 2d(x, b)^2$ holds for all $x \in X$ with $i \in \{1, 2\}$ then, taking $x = z_1$, we have

$$d(a, b)^2 + 4d(z_1, z_2)^2 \leq 2d(z_1, a)^2 + 2d(z_1, b)^2$$

and, from $2d(a, z_1) = d(a, b) = 2d(b, z_1)$,

$$d(a, b)^2 + 4d(z_1, z_2)^2 \leq d(a, b)^2$$

and therefore $d(z_1, z_2) = 0$.

The continuity of the midpoint map is a consequence of the following inequality, which can be easily derived from the Bruhat-Tits Property and the definition of a midpoint,

$$d\big(a, b\big)^2 + 4d\big(\mu(a, b), \mu(a', b')\big)^2 \leq$$
$$2\Big[\frac{1}{2}d(a', b') + d(a, a')\Big]^2 + 2\Big[\frac{1}{2}d(a', b') + d(b, b')\Big]^2.$$

Cartan-Hadamard manifolds, that is simply connected Riemannian manifolds of nonpositive curvature, Chapter XI of [25], are Bruhat-Tits spaces. The space of symmetric positive definite real matrices of a given dimension with the trace metric is a Cartan-Hadamard manifold, Chapter XII of [25].

The class of complete Bruhat-Tits spaces is exactly the class of Hadamard spaces, or complete and simply connected metric spaces of nonpositive curvature also known as CAT(0) spaces, see [8] for definitions and the meaning of "cat" and of CAT(0).

The open unit disc in the complex plane - or the open unit ball of a complex Hilbert space - endowed with the hyperbolic metric is a unique continuous midpoint space, and also a Cartan-Hadamard manifold. Goebel, Sekowski and Stachura studied such spaces in relation to the fixed point property for nonexpansive maps in [14]; that same topic is also treated in Goebel and Reich's book [13] where the midpoint map for these spaces is explicitly written down.

The Bruhat-Tits condition appears in [7], Lemma 3.2.1. in connection with the structure of buildings associated to Bruhat-Tits systems, [7] Definition 2.5.5. According to Theorem 2.5.12 a building is a complete metric

space while according to Lemma 2.5.15 there is a geodesic between each pair of points and the geodesic map depends continuously on the end points. In other words, buildings are continuous midpoint spaces. The construction of the Bruhat-Tits metric on a given building can be found in Chapter 11 of [1].

(**6**) An $\mathbb{R}$-**tree** is a metric space (X, d) such that for x and y in X there exists a unique arc between x and y and this arc is isometric to an interval of $\mathbb{R}$. A metric space is a complete $\mathbb{R}$-tree if and only if it is a hyperconvex metric space with unique metric segments; this is due to Kirk [23]. An $\mathbb{R}$-tree is therefore a unique continuous midpoint space. $\mathbb{R}$-trees are treated in [23] and [24] and the references given therein.

(**7**) A midpoint space (X, d, μ) has **non-positive curvature in the sense of Busemann** if μ is symmetric and if

$$\forall u \in X \ \text{ and } \ \forall (x, y) \in X^2 \quad d(\mu(u, x), \mu(u, y)) \leq \frac{1}{2} d(x, y). \tag{1}$$

Property 1 is due to Busemann, [9] Chapter V where it is defined as a local condition whereas we require the non-positive curvature property to hold globaly. Let us call such midpoint spaces **Busemann midpoint spaces** or simply Busemann spaces.[4]

Busemann midpoint spaces (X, d, μ) are continuous midpoint spaces since

$$d(\mu(a, b), \mu(a', b')) \leq d(\mu(a, b), \mu(a, b')) + d(\mu(a, b'), \mu(a', b'))$$

and therefore

$$d(\mu(a, b), \mu(a', b')) \leq \frac{1}{2} \big(d(a, a') + d(b, b') \big). \tag{2}$$

If in 2 one takes $a = a' = u$, $b = x$ and $b' = y$ then one gets 1. In other words, *a symmetric midpoint space is a Busemann midpoint space if and only if 2 holds.*

Hyperconvex metric spaces (X, d) with the midpoint map associated to a given nonexpansive retraction r from a Banach space E in which X is embedded

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[4]There already are such things as "Busemann spaces" hence our somewhat cumbersome terminology "Busemann midpoint spaces". The relationship between the two structures is clarified in Section 3.

are Busemann midpoint spaces as can easily be seen from the definition of the midpoint map, that is $\mu(x, y) = r\big((1/2)(x + y)\big)$. More generally, a nonexpansive retract of a Busemann midpoint space is a Busemann midpoint space.

Bruhat-Tits spaces are Busemann midpoint spaces. This is Corollary 1.

Hyperbolic metric spaces in the sense of Reich and Shafrir [30], the definition of which includes property 1, *are Busemann midpoint spaces.*

(**8**) A midpoint space (X, d, μ) is a **quasi-Busemann** space if

$$\forall (x_1, x_2), (y_1, y_2) \in X^2 \quad d\big(\mu(y_1, y_2), \mu(x_1, x_2)\big) \leq \max\{d(y_1, x_1), d(y_2, x_2)\}. \tag{3}$$

(**9**) We will say that a midpoint space (X, d, μ) has a **quasiconvex metric** if

$$\forall u \in X \ \text{ and } \ \forall (x_1, x_2) \in X^2 \quad d(u, \mu(x_1, x_2)) \leq \max\{d(u, x_1), d(u, x_2)\}. \tag{4}$$

A quasi-Busemann space is a continuous midpoint space with a quasiconvex metric and it follows from 2 that a Busemann midpoint space is a quasi-Busemann space. Notice also that 2 (respectively 3) implies uniform continuity of the midpoint map $\mu : X \times X \to X$ therefore the midpoint map as a unique continuous extension $\mu^\star$ to $X^\star \times X^\star$, where $X^\star$ is the completion of X. Clearly $\mu^\star$ is a midpoint map for which 1, respectively 3 , holds. The completion of a Busemann midpoint space (respectively a quasi-Busemann space) is a Busemann midpoint space, respectively a quasi- Busemann space. Since a Bruhat-Tits metric space is also a Busemann midpoint space the completion of a Bruhat-Tits metric space is a Busemann midpoint space for which the Bruhat-Tits Property holds and is therefore a Bruhat-Tits metric space.

If (X_1, d_1, μ_1) and (X_2, d_2, μ_2) are continuous midpoint spaces then $\mu_\times$ defined by

$$\mu_\times\big((x_1, x_2), (y_1, y_2)\big) = \big(\mu_1(x_1, y_1), \mu_2(x_2, y_2)\big)$$

is a continuous midpoint map on the product $X_1 \times X_2$ for the metrics

$$D_p\big((x_1, x_2), (y_1, y_2)\big) = \big[d_1(x_1, y_1)^p + d_2(x_2, y_2)^p\big]^{1/p}, \quad 1 \leq p < \infty$$

and

$$D_\infty\big((x_1, x_2), (y_1, y_2)\big) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}.$$

If (X_1, d_1, μ_1) and (X_2, d_2, μ_2) are Busemann midpoint spaces then so is $(X_1 \times X_2, D_p, \mu_\times)$, $1 \leq p \leq \infty$.

3   The Main Lemma

All of the results of this paper are consequences Lemma 1 below and of suitable, but natural, assumptions on the metric depanding on the conclusion one wants to reach. For the reader's convenience we give a full proof of Lemma 1.

Lemma 1. *If (X, d, μ) is a complete continuous midpoint space then there exists a continuous map $\varphi : X \times X \times [0,1] \to X$ such that $\forall (a, b) \in X \times X$ and $\forall (t, t') \in [0,1] \times [0,1]$:*

(1) $d(\, a, \varphi(a, b, t)\,) = td(a, b)$ *and* $d(\, b, \varphi(a, b, t)\,) = (1 - t)d(a, b)$;

(2) $d(\, \varphi(a, b, t), \varphi(a, b, t')\,) = |t - t'|d(a, b)$

Futhermore, if (X, d) is also a unique midpoint space then there is a unique map $\varphi : X \times X \times [0,1] \to X$ for which (1) and (2) hold. It also has the following property

(3) $\varphi(a, b, t) = \varphi(b, a, 1 - t)$;

Proof. Let $\mathbb{D}_m = \{k2^{-m} : 0 \leq k \leq 2^m\}$; the set $\mathbb{D} = \bigcup_{m \in \mathbb{N}} \mathbb{D}_m$ of dyadic numbers is dense in $[0,1]$. If $t \in \mathbb{D}_{m+1} \setminus \mathbb{D}_m$ then $t = k/2^{m+1}$ where k is odd; let $t_s = (k-1)/2^{m+1}$ and $t_d = (k+1)/2^{m+1}$; t_s and t_d are both in $\mathbb{D}_m$.
(A) For a fixed pair $(a, b) \in X \times X$ the construction below defines by induction a sequence of maps $\varphi_m(a, b, -) : \mathbb{D}_m \to X$ such that the restriction of $\varphi_{m+1}(a, b, -)$ to $\mathbb{D}_m$ is $\varphi_m(a, b, -)$ and therefore a map $\varphi_\omega(a, b, -) : \mathbb{D} \to X$ whose restriction to $\mathbb{D}_m$ is $\varphi_m(a, b, -)$.

First, for all $m \in \mathbb{N}$, $\varphi_m(a, b, 0) = a$ and $\varphi_m(a, b, 1) = b$; this defines $\varphi_0(a, b, 0)$. Then

$\varphi_{m+1}(a, b, t) = \varphi_m(a, b, t)$ if $t \in \mathbb{D}_m$
 and
$\varphi_{m+1}(a, b, t) = \mu\big(\varphi_m(a, b, t_s), \varphi_m(a, b, t_d)\big)$ if $t \in \mathbb{D}_{m+1} \setminus \mathbb{D}_m$.

(B) If t_s and t_d are two consecutive elements of $\mathbb{D}_m$ then

$$d\big(\varphi_m(a, b, t_s), \varphi_m(a, b, t_d)\big) = \frac{1}{2^m}d(a, b).$$

For $m = 0$ this is a consequence of the definition of φ_0 and for $m = 1$ it follows from the definition of μ. From $t_d - t_s = 2^{-m}$ we have either $t_s \in \mathbb{D}_{m-1}$ or

$t_d \in \mathbb{D}_{m-1}$; assume that $t_s \in \mathbb{D}_{m-1}$, and therefore $t_s + 1/2^{m-1} \in \mathbb{D}_{m-1}$ and $t_d = 1/2\big[t_s + (t_s + 1/2^{m-1})\big] \in \mathbb{D}_m \setminus \mathbb{D}_{m-1}$.

$\varphi_m(a, b, t_s) = \varphi_{m-1}(a, b, t_s)$ and

$$\varphi_m(a, b, t_d) = \mu\big(\varphi_{m-1}(a, b, t_s), \varphi_{m-1}(a, b, t_s + 1/2^{m-1})\big)$$

We obtain

$$d\big(\varphi_m(a, b, t_s), \varphi_m(a, b, t_d)\big) d\big(\varphi_{m-1}(a, b, t_s), \mu\big(\varphi_{m-1}(a, b, t_s),$$

$$\varphi_{m-1}(a, b, t_s + 1/2^{m-1})\big)\big) = \frac{1}{2} d\big(\varphi_{m-1}(a, b, t_s),$$

$$\varphi_{m-1}(a, b, t_s + 1/2^{m-1})\big) \frac{1}{2}\Big(\frac{1}{2^{m-1}} d(a, b)\Big) = \frac{1}{2^m} d(a, b).$$

One proceeds similarly if $t_d \in \mathbb{D}_{m-1}$.

(C) By induction on m on shows that, for all $t \in \mathbb{D}$,
$$d\big(a, \varphi_\omega(a, b, t)\big) = td(a, b) \text{ and } d\big(b, \varphi_\omega(a, b, t)\big) = (1 - t)d(a, b).$$

If $t \in \mathbb{D}_0 = \{0, 1\}$ this is obvious; if $t \in \mathbb{D}_1$ it follows from the definition of μ. Let us assume that $t \in \mathbb{D}_{m+1} \setminus \mathbb{D}_m$.

$$d\big(a, \varphi_{m+1}(a, b, t)\big) = d\big(a, \mu\big(\varphi_m(a, b, t_s), \varphi_m(a, b, t_d)\big)\big) \leq$$
$$\leq d\big(a, \varphi_m(a, b, t_s)\big) + d\big(\varphi_m(a, b, t_s), \mu\big(\varphi_m(a, b, t_s), \varphi_m(a, b, t_d)\big)\big) \leq$$
$$\leq t_s d(a, b) + \frac{1}{2} d\big(\varphi_m(a, b, t_s), \varphi_m(a, b, t_d)\big) \leq \big(t_s + \frac{1}{2^{m+1}}\big) d(a, b) td(a, b).$$

This shows that for all $t \in \mathbb{D}$
$$d\big(a, \varphi_\omega(a, b, t)\big) \leq td(a, b).$$

Similarly, from $t_d = t + \dfrac{1}{2^{m+1}}$, one can show that, for all $t \in \mathbb{D}$,

$$d\big(\varphi_\omega(a, b, t), b\big) \leq (1 - t)d(a, b).$$

The triangle inequality implies that none of these inequalities can be strict, therefore, $t \in \mathbb{D}$,

$d\big(a, \varphi_\omega(a, b, t)\big) = td(a, b)$ and $d\big(\varphi_\omega(a, b, t), b\big) = (1 - t)d(a, b)$.

(D) Let us see that $t \mapsto \varphi_\omega(a, b, t)$ is uniformly continuous on $\mathbb{D}$.
For $t, t' \in \mathbb{D}$ we choose m such that $t, t' \in \mathbb{D}_m$, let us say $t = k/2^m$ and
$t = (k + j)/2^m$. We use (B) with the sequence of consecutive points $k/2^m$,
$(k + 1)/2^m, \cdots, (k + j)/2^m$ to obtain

$d\big(\varphi_\omega(a, b, t), \varphi_\omega(a, b, t')\big) \le (j/2^m)d(a, b)|t - t'|d(a, b)$.

(E) From (B), (C) and (D), and since $\mathbb{D}$ is dense in $[0, 1]$ and X is complete,
there is a unique uniformly continuous map $\varphi(a, b, -) : [0, 1] \to X$ such that

(1) $\varphi(a, b, 0) = a$ and $\varphi(a, b, 1) = b$;

(2) for all $t \in [0, 1]$, $d\big(a, \varphi(a, b, t)\big) = td(a, b)$ and $d\big(\varphi(a, b, t), b\big) = (1-t)d(a, b)$;

(3) for all $t, t' \in [0, 1]$, $d\big(\varphi(a, b, t), \varphi(a, b, t')\big) \le |t - t'|d(a, b)$.

The triangular inequality in conjonction with (2) yields

$d(a, b) \le td(a, b) + (1 - t')d(a, b) + d\big(\varphi(a, b, t), \varphi(a, b, t')\big)$ which shows that,
assuming $t < t'$, the inequality in (3) cannot be strict, and therefore,

(4) for all $t, t' \in [0, 1]$, $d\big(\varphi(a, b, t), \varphi(a, b, t')\big) = |t - t'|d(a, b)$.

(F) We show that φ is continuous on $X \times X \times [0, 1]$.

(a) First, if $t \in \mathbb{D}_m$ then $(a, b) \mapsto \varphi(a, b, t) = \varphi_m(a, b, t)$ is continuous as
one can see from the contunuity of the midpoint map μ, the definition of
$\varphi_{m+1}(a, b, t)$ and an induction on m.

(b) Let us see that for all $t \in [0, 1]$ $(a, b) \mapsto \varphi(a, b, t)$ is continuous. Let
$\Delta = d\big(\varphi(a, b, t), \varphi(a', b', t)\big)$; from (4) we have, for arbitrary $t' \in [0, 1]$,

$$\Delta \le |t - t'|\big(d(a, b) + d(a', b')\big) + d\big(\varphi(a, b, t'), \varphi(a', b', t')\big).$$

If $d(a, a') + d(b, b') < 1$ then $d(a', b') < d(a, b) + 1$ and therefore

$$\Delta \le |t - t'|\big(2d(a, b) + 1\big) + d\big(\varphi(a, b, t'), \varphi(a', b', t')\big).$$

We can choose $t' \in \mathbb{D}$ such that $|t - t'| < \varepsilon\big(2d(a, b) + 1\big)^{-1}$ and conclude from
the continuity of the map $\varphi(-, -, t')$ at (a, b).

(c) To see that φ is continuous at (a, b, t) let $\Delta = d\big(\varphi(a, b, t), \varphi(a', b', t')\big)$
and notice that

$$\Delta \le d\big(\varphi(a,b,t), \varphi(a',b',t)\big) + |t - t'| d(a',b').$$

(G) Let us assume now that (X,d) is a unique continuous midpoint space. We show that φ is the unique continuous map from $X \times X \times [0,1]$ to X for which (1) and (2) hold. Indeed, if ψ is such a map then $\psi(a,b,0) = \varphi(a,b,0)$, $\psi(a,b,1) = \varphi(a,b,1)$ and $\psi(a,b,1/2) = \mu(a,b)$.

To complete the proof let us see that

$$\psi\left(a,b, \frac{t_1 + t_2}{2}\right) = \mu\big(\psi(a,b,t_1), \psi(a,b,t_2)\big).$$

We can assume that $t_1 < t_2$ and we let

$$M = \psi\left(a,b, \frac{t_1 + t_2}{2}\right).$$

$$d\big(M, \psi(a,b,t_1)\big) = \left(\frac{t_1 + t_2}{2} - t_1\right) d(a,b) = \left(\frac{t_2 - t_1}{2}\right) d(a,b)$$

and similarly

$$d\big(M, \psi(a,b,t_2)\big) = \left(t_2 - \frac{t_1 + t_2}{2}\right) d(a,b) = \left(\frac{t_2 - t_1}{2}\right) d(a,b).$$

we also have

$$d\big(\psi(a,b,t_1), \psi(a,b,t_2)\big) = (t_2 - t_1) d(a,b).$$

This shows that M is the midpoint of $\psi(a,b,t_1)$ and $\psi(a,b,t_2)$.

If $t \in \mathbb{D} \setminus \{0, 1/2, 1\}$ let m be the smallest integer for which $t \in \mathbb{D}_m$, from $t = (1/2)(t_d + t_s)$ we have

$$\psi(a,b,t) = \mu\big(\psi(a,b,t_s), \psi(a,b,t_s)\big).$$

and an obvious induction on m shows that $\psi = \varphi$ on $X \times X \times \mathbb{D}$ and therefore $\psi = \varphi$.

Finally, notice that (1) and (2) hold for $\psi(a,b,t) = \varphi(b,a,1-t)$; this proves (3). $\qquad\square$

Given a continuous midpoint space (X,d,μ) the map φ constructed in Lemma 1 is the **equiconnecting map associated to** μ. At times we will also write $\varphi_{(a,b)}(t)$ for $\varphi(a,b,t)$. Since $d(a, \varphi(a,b,1/2)) = d(a,b)/2$ and $d(b, \varphi(a,b,1/2)) = d(a,b)/2$ we have the following characterization of complete spaces which carry a continuous midpoint map.

Proposition 1. *Let (X, d) be a complete metric space. Then there exists a continous midpoint map $\mu : X \times X \to X$ if and only if there exists a continuous map $\varphi : X \times X \times [0, 1] \to X$ such that $\forall (a, b) \in X \times X$ and $\forall (t, t') \in [0, 1] \times [0, 1]$:*

(1) $d(a, \varphi(a, b, t)) = td(a, b)$ and $d(b, \varphi(a, b, t)) = (1 - t)d(a, b)$;

(2) $d(\varphi(a, b, t), \varphi(a, b, t')) = |t - t'|d(a, b)$

For all pair (a, b) the map $\varphi_{(a,b)}$ clearly defines a rectifiable path whose length is $d(a, b)$. We call the set $\varphi_{(a,b)}([0, 1])$ the **geodesic segment** with extremities a and b and we denote by $\sigma_{(a,b)}^{\mu}$ or simply $\sigma_{(a,b)}$. Example 1 shows that we can possibly have $\sigma_{(a,b)} \neq \sigma_{(b,a)}$.

Going back to example (1) of section 2 we can compute $\varphi(a, b, t)$ using the fact that it is a constant speed geodesic with speed $d(a, b) = |x_1 - x_2| + |y_1 - y_2|$ where $a = (x_1, y_1)$ and $b = (x_2, y_2)$. One finds

$$\varphi(a, b, t) = \begin{cases} (x_1, y_1) + td(a, b)\left(\dfrac{x_2 - x_1}{|x_2 - x_1|}, 0\right) & \text{if } t \in \left[0, \dfrac{|x_2 - x_1|}{d(a, b)}\right] \\[3mm] (x_2, y_1) + \left(t - \dfrac{|x_2 - x_1|}{d(a, b)}\right)d(a, b)\left(0, \dfrac{y_2 - y_1}{|y_2 - y_1|}\right) & \text{if } t \in \left[\dfrac{|x_2 - x_1|}{d(a, b)}, 1\right] \end{cases}$$

If $x_1 = x_2$ or $y_1 = y_2$ then $\varphi(a, b, t) = (1 - t)a + bt$.

Busemann spaces can be characterized by the behaviour of the metric on geodesics, more precisely,

Proposition 2. *A symmetric and continuous midpoint space (X, d, μ) is a Busemann space if and only if, $\forall (a, b, a', b') \in X^4$ and $\forall (t, t') \in [0, 1] \times [0, 1]$, one has, with $t'' = (1/2)(t + t')$,*

$$d(\varphi_{a,b}(t''), \varphi_{a',b'}(t'')) \leq \frac{1}{2}d(\varphi_{a,b}(t), \varphi_{a',b'}(t)) + \frac{1}{2}d(\varphi_{a,b}(t'), \varphi_{a',b'}(t')) \quad (5)$$

Proof. If in 5 one takes $t = 0$ and $t' = 1$ then one obtains 2 which characterizes Busemann spaces. On the other hand, 2 is exactly 5 for $(t, t') \in \mathbb{D}_0 \times \mathbb{D}_0$. On shows by induction that 5 holds on $\mathbb{D} \times \mathbb{D}$ and therefore on $[0, 1] \times [0, 1]$. □

Proposition 3. *A symmetric and continuous midpoint space (X, d, μ) is a Busemann space if and only if, $\forall (u, x, y) \in X^3$ the map $t \mapsto d(\varphi(u, x, t), \varphi(u, y, t))$ is convex on $[0, 1]$.*

Proof. Let $\delta_{(w,a,b)}(t) = d\big(\varphi(w,a,t), \varphi(w,b,t)\big)$. If (X,d,μ) is a Busemann space then one can take in 5 $a = a' = u$, $b = x$ and $b' = y$ to see that $\delta_{(u,x,y)}$ is convex on $[0,1]$.

On the other hand, if $\delta_{(u,x,y)}$ is convex on $[0,1]$ then $\delta_{(u,x,y)}(1/2) \leq (1/2)\delta_{(u,x,y)}(0) + (1/2)\delta_{(u,x,y)}(1)$ which is exactly 1 since, for all $(u,z) \in X \times X$, $\varphi(u,z,0) = u$, $\varphi(u,z,1) = z$ and $\varphi(u,z,1/2) = \mu(u,z)$. $\qquad\square$.

For all $(a,b) \in X \times X$ of an arbitrary continuous midpoint space the map $\varphi_{a,b}$ is a constant speed geodesic - its speed is $d(a,b)$ - from which, assuming that we have a Bruhat-Tits space, the map $\delta_{(u,x,y)}$ is convex; this is Proposition 9.2.13 in [8][5].

Corollary 1. *Bruhat-Tits spaces are Busemann midpoint spaces.*

The standard definition of Busemann spaces, as opposed to our definition of Busemann midpoint spaces, is given in [28], Chapter 8: a metric space (X,d) is a Busemann space if it is a geodesic space (two arbitrary points can always be joinded by a geodesic path) and, given two geodesic paths $\gamma : [a,b] \to X$ and $\gamma' : [a',b'] \to X$ the map

$$(t,t') \mapsto d\big(\gamma(t), \gamma'(t')\big) \tag{6}$$

is convex on $[a,b] \times [a',b']$.

Condition 6 implies that a Busemann space is a unique geodesic space (given a pair of points (x,y) there exists one and only one geodesic path parametrized by $t \in [0,1]$ joining x to y), this is Proposition 8.1.4 of [28]. Furthermore, from (vii) of Proposition 8.1.2, given two geodesics $\gamma : [0,1] \to X$ and $\gamma' : [0,1] \to X$ with $\gamma(0) = \gamma'(0)$ one has

$$d\big(\gamma(1/2), \gamma'(1/2)\big) \leq (1/2)d\big(\gamma(1), \gamma'(1)\big) \tag{7}$$

If $\gamma_{a,b} : [0,1] \to X$ is the unique geodesic of the Busemann space X such that $\gamma_{a,b}(0) = a$ and $\gamma_{a,b}(1) = b$ then $\gamma_{a,b}(1/2)$ is a midpoint for the pair (a,b) and 7 implies

$$d\big(\gamma_{u,x}(1/2), \gamma_{u,y}(1/2)\big) \leq (1/2)d(x,y)$$

which is exactly 1.

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[5]This is proved for the hyperbolic metric on the open unit disc in [13] Lemma 6.8

Furthermore, the path $\tilde{\gamma}_{a,b}(t) = \gamma_{a,b}(1-t)$ is a geodesic path from b to a; from the uniqueness of the geodesic we have $\gamma_{a,b}(1-t) = \gamma_{b,a}(t)$ and therefore $\gamma_{a,b}(1/2) = \gamma_{b,a}(1/2)$.

If $\mu : X \times X \to X$ is a continuous midpoint map on the Busemann space X then $\varphi(a,b,t) = \gamma_{a,b}(t)$, since $\varphi_{a,b}$ is a geodesic from a to b, in particular, $\mu(a,b) = \gamma_{a,b}(1/2)$. We can conclude this discussion with the following lemma:

Lemma 2. *A Busemann space is a unique continuous midpoint space and a Busemann midpoint space.*

A normed vector space $(E, \| \ \|)$ with the midpoint map $\mu(x,y) = (1/2)(x+y)$ is a Busemann midpoint space but, according to Proposition 7.2.1 of [28], a normed space is a unique geodesic space if and only if it is strictly convex. Consequently, a Busemann midpoint space does not have to be a unique geodesic space.

4   Convex structures

In this section (X, d, μ) is a complete continuous midpoint space and φ is the associated equiconnecting map. **A closed subset C of X is convex**, or more pricisely μ-convex, if, for all $(x,y) \in C \times C$, $\mu(x,y) \in C$. An obvious induction on dyadic numbers shows that if C is a closed μ-convex set then, for all $(x,y,t) \in C \times C \times [0,1]$, $\varphi(x,y,t) \in C$, in other words, a closed set C is convex if and only if,

$$\forall (a,b) \in C \times C \quad \sigma_{(a,b)} \subset C. \tag{8}$$

Such a set will be called **geodesically convex**. In a complete continous midpoint space the **geodesic hull** of an arbitrary set A is the smallest geodesically convex subset containing A, let us call it $\Delta_{geo}(A)$. One has

$$\Delta_{geo}(A) = \bigcup_{n \in \mathbb{N}} \Delta_{geo}^{(n)}(A) \tag{9}$$

where $\Delta_{geo}^{(0)}(A) = A$ and $\Delta_{geo}^{(n+1)}(A) = \bigcup \{\sigma_{(x,y)} : (x,y) \in \Delta_{geo}^{(n)}(A) \times \Delta_{geo}^{(n)}(A)\}$. It is clear that a subset C of a complete continuous midpoint space is geodesically convex if and only if $\Delta_{geo}(C) = C$ if and only if, for all

finite subsets A of C, $\Delta_{geo}(A) \subset C$. The geodesic hull of an arbitary finite set might not be closed, even for Bruhat-Tits spaces.

In a real topological vector space one can characterize a convex subset as a set that is geodesically convex or as a set that contains all the polytopes spanned by its finite subsets; in a real topological vector space polytopes are compact. In a complete continous midpoint space these two definitions do not have to be equivalent and polytopes - which will be defined below - do not have to be compact. If A is an arbitrary finite subset of a complete continuous midpoint space (X, d, μ) let

$$\Delta_{\mu}(A) = \bigcap \{C : A \subset C \text{ s.t. } C \text{ is closed and convex }\}. \qquad (10)$$

A set $P \subset X$ is a **polytope** if it is of the form $\Delta_{\mu}(A)$ for some finite subset A of X. We say that a subset C of a complete continuous midpoint space (X, d, μ) is **convex subset** of X if it contains all the polytopes spanned by its finite subsets, that is: if for all finite subsets A of C one has $\Delta_{\mu}(A) \subset C$. Let us see that for closed sets this definition is identical to the one given above.

Lemma 3. *A closed C subset of a complete continuous midpoint space (X, d, μ) is convex is and only if, for all finite subsets A of C one has $\Delta_{\mu}(A) \subset C$.*

Proof If C is a closed convex set and if $A \subset C$ then, from the definition of $\Delta_{\mu}(A)$, $\Delta_{\mu}(A) \subset C$. On the other hand, if $\Delta_{\mu}(A) \subset C$ for all finite subsets A of C then, for all x and y in C, $\Delta_{\mu}(\{x, y\}) \subset C$. By construction $\Delta_{\mu}(\{x, y\})$ is a closed convex set containing x and y and therefore $\mu(x, y) \in \Delta_{\mu}(\{x, y\})$. This shows that C is convex. $\square$

It is not even clear that a geodesic segment is geodesically convex. At least if the midpoint map is not unique.

Lemma 4. *In a unique continuous midpoint space geodesic segments are convex.*

Proof. Let u and v be two points of $\sigma_{(x,y)}$. We can write

$$u = \varphi(x, y, t_u) \text{ and } v = \varphi(x, y, t_v)$$

with t_u and t_v in $[0, 1]$. From 2 of Lemma 1 we have

$$d(u, v) = |t_u - t_v| d(x, y)$$

and also, with $t = \dfrac{t_u + t_v}{2}$,

$$d\big(u, \varphi(x, y, t)\big) = d\big(v, \varphi(x, y, t)\big) = \frac{|t_u - t_v|}{2} d(x, y).$$

From the uniqueness of the midpoint we can conclude that $\mu(u, v) = \varphi\big(x, y, \dfrac{t_u + t_v}{2}\big)$. $\qquad\square$

For finite subsets the almost obvious relationship between $\Delta_{geo}(A)$ and $\Delta_\mu(A)$ is given by the following lemma.

Lemma 5. *For all finite subsets A of a complete continuous midpoint space (X, d, μ) we have*

$$\Delta_\mu(A) = \overline{\Delta_{geo}(A)}. \tag{11}$$

Proof. Since $\Delta_\mu(A)$ is closed and geodesically convex we have $\Delta_\mu(A) \supset \overline{\Delta_{geo}(A)}$. For all $(x, y) \in \Delta_{geo}(A)$ we have $\mu(x, y) \in \Delta_{geo}(A)$. The continuity of μ implies that $\overline{\Delta_{geo}(A)}$ is convex, and therefore that $\Delta_\mu(A) \subset \overline{\Delta_{geo}(A)}$. $\qquad\square$

Lemma 5 implies that a convex set is geodesically convex, that a closed set is convex if and only if it is geodesically convex and that the closure of a geodesically convex set is convex. Also, the following observation, while evident, is crucial for all that follows.

Lemma 6. *Non empty geodesically convex subsets of a complete continuous midpoint space (X, d, μ) are contractible.*

Given a convex (resp. geodesically convex) subset C of a complete continuous midpoint space (X, d, μ) let $\mathcal{C}_\mu(C)$ (resp. $\mathcal{C}_{geo}(C)$) be the collection of those convex (resp. geodesically convex) subsets of X which are contained in C. Let $\mathcal{C}$ be either $\mathcal{C}_\mu(C)$ or $\mathcal{C}_{geo}(C)$. The following properties are easily established:

Conv1: $\emptyset \in \mathcal{C}$, $C \in \mathcal{C}$ and, for all $x \in C$, $\{x\} \in \mathcal{C}$;

Conv2: For all subfamilies $\mathcal{A}$ of $\mathcal{C}$, $\bigcap \mathcal{A} \in \mathcal{C}$;

Conv3: if $\mathcal{A}$ is a subfamily of $\mathcal{C}$ such that, for all $A_1, A_2 \in \mathcal{A}$ there exists $A_3 \in \mathcal{A}$ such that $A_1 \cup A_2 \subset A_3$ then $\bigcup \mathcal{A} \in \mathcal{C}$.

From (Conv 2) the smallest convex subset of X containing a given subset S is the intersection of all the elements of $\mathcal{C}_\mu(X)$, closed or not, containing S. Since $\Delta_\mu(A) \in \mathcal{C}_\mu(X)$ for all finite subsets A of X we have

$$\Delta_\mu(A) = \bigcap \{ C \in \mathcal{C}_\mu(X) : A \subset C \} \tag{12}$$

For an arbitrary subset S of X we define the convex hull $\Delta_\mu(S)$ of S as the intersection of all the convex subsets of X containing S.

Given a nonempty set S let $\langle S \rangle$ be the family of finite and nonempty subsets of S. Let Δ be either Δ_μ or Δ_{geo}. One can easily verify that property (Conv 3) implies that for all subsets S of a complete continuous midpoint space (X, d, μ) the following holds

$$\Delta(S) = \bigcup_{A \in \langle S \rangle} \Delta(A). \tag{13}$$

Finally, notice that a complete continuous midpoint space (X, d, μ) is always locally contractible - if $d(x_0, x) < \varepsilon$ then $d(x_0, \varphi(x_0, x, t)) = t d(x_0, x) < t\varepsilon$ and therefore $\varphi(x_0, x, t) \in B(x_0, \varepsilon)$ which implies that open balls are contractible - and, as has been shown, contractible.

Convex structures will be used as a mean to derive certain properties of complete continuous midpoint spaces whose metric satisfies some explicit conditions.

5   Around the KKM Lemma in complete continuous midpoint spaces

In this section (X, d, μ) is a complete continuous midpoint space without any other particular property. We will see that some fundamental results from nonlinear analysis hold in this very general framework. Only the basic statements - relating mainly to variationnal inequalities, fixed points and selections - will be given, often without proof since they can be directly derived from propositions and theorems already available.

A **c-structure on a topological space** X is a map Δ from the family of non empty finite subsets of X to the family of non empty homotopically trivial subsets of X such, $\forall A, B \in \langle X \rangle$, $\Delta(A)$ is homotopically trivial and if $A \subset B$ then $\Delta(A) \subset \Delta(B)$. The pair (X, Δ) is called a **c-space**, [17], [18], [16].

Given a c-space (X, Δ), a subset C of X is a **Δ-convex set** if, $\forall A \in \langle C \rangle$, $\Delta(A) \subset C$.

If (X, d, μ) is a complete continuous midpoint space then, for all geodesically convex (respectively, convex) subsets C of X, (C, Δ_{geo}) (respectively, (C, Δ_μ)) is a c-space whose convex subsets are exactly the geodesically convex (respectively, convex) subsets of X contained in C. Hyperconvex metric spaces carry a natural c-structure which does not depend on the midpoint map associated to a given retraction; it is defined as follows : for all finite subsets A of X

$$\Delta_{hyperconv}(A) = \bigcap \{ B[x, r] : A \subset B[x, r] \}. \tag{14}$$

Of course $\Delta_{hyperconv}(A)$ makes sense for bounded sets, they are the admissible sets of [22] and [21].The proof that $(X, \Delta_{hyperconv})$ is indeed a c-space can be found in [18].

Let us call the convexity associated to $\Delta_{hyperconv}$ the **natural convexity of the hyperconvex metric space** (X, d).

Let (X, d) be an hyperconvex metric space and let $\mu(x, y) = r \left(\dfrac{x + y}{2} \right)$ be the continuous midpoint map associated to an arbitrary nonexpansive retraction $r : E \to X$ from a Banach space E onto X. We have seen that (X, d, μ) is quasi-Busemann and therefore closed balls are Δ_μ-convex. From (Conv 2) and 14 and we have, $\forall A \in \langle X \rangle$, $\Delta_\mu(A) \subset \Delta_{hyperconv}(A)$.

The relevance of this remark is due to the following simple observation: if, in the context of complete continuous midpoint spaces, a statement - for example a fixed point theorem or a selection theorem - is true under general topological assumptions for sets, or families of sets, that are convex then, in the context of hyperconvex metric spaces, under those same general topological assumptions, it is true for sets, or families of sets, that are convex with respect to the natural convexity of (X, d). The passage from Proposition 9 to Corollary 4, which is a theorem of Khamsi [21], is a typical example.

The first and basic ingredient of this section is the following statement whose proof can be obtained from Theorem 2 in [20].

Lemma 7 (KKM)**.** *Let $\{ M_0, \cdots, M_l \}$ be a family of subsets of a geodesically convex subset C of X. Assume that either they are all closed or all open in C and that there exists a finite subset $\{ x_0, \cdots, x_l \}$ of C such that for all non empty set of indices $J \subset \{ 0, \cdots, l \}$, $\Delta_{geo} \{ x_j : j \in J \}) \subset \cup_{j \in J} M_j$. Then $\cap_{j=0}^l M_j \neq \emptyset$.*

It is well known that the classical Knaster-Kuratowski-Mazurkiewicz Lemma for convex subsets of $\mathbb{R}^n$ can be used to prove Brouwer's Fixed Point Theorem, and reciprocally [15]. In [26] Niculescu and Rovenţa have given a proof of the KKM Lemma for closed sets in complete Bruhat-Tits spaces assuming that, for all non empty finite subsets A, $\Delta_\mu(A)$ has the fixed point property for continuous maps.

   The first, and maybe most fundamental, application of Lemma 7 is the following version of Ky Fan's Intersection Theorem. A proof in the standard setting of topological vector spaces can be found in [15]. The proof of the slightly more general formulation can be found in [17], Theorem 1 page 350.

Proposition 4 (Fan's Intersection Theorem). *Let C be a geodesically convex subset of a complete continuous midpoint space and let $\Gamma : C \to C$ be a multivalued map with closed values, at least one of which is compact. Assume that Γ has a multivalued selection $\Theta : C \to C$ such that, $\forall x \in C$, $x \in \Theta x$ and $C \setminus \Theta^{-1}x$ is geodesically convex. Then $\cap_{x \in C} \Gamma x \neq \emptyset$.*

   The importance of this result, at least in the standard framework of topological vector spaces, is due to its use in the study of variational inequalities. For an illustration, the following inequality is a direct and easy application of Proposition 4.

Corollary 2. *Let C be a geodesically convex subset of a complete continuous midpoint space and $f, g : C \times C \to \mathbb{R}$ be two functions such that: (1) $f \leq g$, (2) $\forall y \in C$ $f(-, y)$ is lower semicontinuous, (3) $\forall x_1, x_2, y \in C$ and $\forall t \in [0, 1]$, $\min\{g(x_1, y), g(x_2, y)\} \leq g(\varphi(x_1, x_2, t), y)$, (4) $\forall x \in C$ $g(x, x) \leq 0$ and (5) $\exists x_0 \in C$ such that $\{y \in C : f(x_0, y) \leq 0\}$ is compact. Then, $\exists y_0 \in C$ such that, $\forall x \in C$, $f(x, y_0) \leq 0$.*

Taking $f = g$ in Corollary 2 we obtain the classical inequality of Ky Fan a proof of which in the context of complete Bruhat-Tits spaces, under the assumption that C is compact and polytopes have the fixed point property for continuous maps, can be obtained from the more general results of [26].

An elementary manipulation of Proposition 4 yields the following midpoint spaces version of the Browder-Fan Fixed Point Theorem.

Proposition 5. *Let $\Phi : C \to C$ be a multivalued map from a geodesically convex subset of a complete continuous midpoint space to itself. Assume that (1) Φ has geodesically convex values, (2) Φ has a multivalued selection*

$\Psi : C \to C$ *with open fibers and non empty values,* (3) $\exists x_0 \in C$ *such that* $C \setminus \Psi^{-1} x_0$ *is compact. Then* $\exists \hat{x} \in C$ *such that* $\hat{x} \in \Phi \hat{x}$.

[6] A proof of Browder's Selection Theorem in c-spaces can be found in [16] or in [17], Theorem 2 page 348. For midpoint spaces Browder's Selection Theorem takes the following form.

Proposition 6. *A Browder map* $\Phi : Z \to C$ *from a paracompact topological space* Z *to a geodesically convex subset* C *of a complete continuous midpoint space has a continuous selection.*

Most of the classical results whose proof depends on the KKM Lemma, for example those presented in [15] on page 142 and subsequent pages, can be formulted in the framework of midpoint spaces. For more results of this type in the context of c-spaces one could look at [17], at [20] for Klee type theorems or, for even more general formulations, at [29] and the references given therein.

6 Absolute retracts and fixed points

The generalized Schauder's Theorem, Theorem 7.4 Page 291 in [15], states that a compact continuous map $f : C \to C$ [7] from an absolute retract to itself has a fixed point. A convex subset of a normed space is an absolute retract; for closed convex sets this is a consequence of Michael's Selection Theorem. More generally, metrizable convex subsets of locally convex topological vector space are absolute retract; this is a consequence of Dugundji's Extension Theorem. What can be said with regard to convex subsets of complete continuous midpoint spaces? This is the question we partially answer in this section.

Since a complete continuous midpoint space is locally contractible and contractible the following statement is a direct consequence of a Theorem of

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[6]The terminology is still somewhat unstable. In [16], and in the context of general c-spaces, maps Φ for which the assumptions of Proposition 5 hold are called Browder maps. In [15], definition 1.1 page 142, in the context of linear topological spaces, a map with nonempty convex values and open fibers is an $\mathbb{F}$ map; still in [15], maps of type Φ, which are defined on page 176, are what we just called Browder maps while the Browder maps of [15], which are the maps of type $\mathbb{B}$ on page 176, are still something else. We will keep the terminology of [16].

[7]$f : C \to C$ is compact if there exists a compact subset K of C such that $f(C) \subset K$.

Dugundji, Theorem 3.1 in [11], or Borsuk, Corollary 10.5 in [4], for compact spaces.

Proposition 7. *A finite dimensional continuous midpoint space is an absolute retract.*

Proposition 7 for finite dimensional compact spaces is essentially due to Borsuk, [4] page 219 where it is stated for compact metric spaces with a unique midpoint map (in which case the continuity of the midpoint map is a consequence of the compactness of the space). Borsuk asked if a compact metric space with a unique midpoint map is always an absolute retract, Problem 10.1 on page 219 of [4]. We will see that this is indeed the case, without assuming compactness, if the metric is quasiconvex.

Lemma 8. *In a complete continuous midpoint space (X, d, μ) the following statements are equivalent:* (1) *The metric is quasiconvex;* (2) *closed balls are convex;* (3) *open balls are convex;* (4) *open ball are geodesically convex.*

Proof. The equivalence between (1) and (2) is obvious from 4. The equivalence of (2) and (3) follows from (conv2) and (conv3). To see that (4) implies (2) notice that if open balls are geodesically convex then, by (conv3), closed balls are geodesically convex and therefore convex. Obviously (3) implies (4)   □

Proposition 8. *A geodesically convex subset of a complete continuous midpoint space whose metric is quasiconvex is an absolute retract.*

Proof. Let C be a geodesically convex subset of a complete continuous midpoint space (X, d, μ) whose metric is quasiconvex. In X open balls are geodesically convex, Lemma 8. The family of geodesically convex sets $\{C \cap B(x, r) : x \in C, r > 0\}$ is an open basis for the metric space C. By a theorem of Dugundji, Theorem 3.4 in [11], C is an ANR and since it is contractible it is an absolute retract.   □

Corollary 3 (Schauder's Theorem)**.** *A geodesically convex subset of a complete continuous midpoint space which is either finite dimensional or whose metric is quasiconvex has the fixed point property for continuous maps.*

In [27] Niculescu and Rovenţa have given a direct proof of Schauder's Theorem in complete Bruhat-Tits spaces using a Schauder like approximation method. Notice that Corollary 3 implies that in complete Bruhat-Tits

spaces, and more generally in quasi-Busemann spaces, a set of the form $\Delta_\mu(A)$, that is the closure of geodesic hulls of finite sets A, has the fixed point property for continuous maps.

In a complete continuous midpoint space whose metric is quasiconvex inequality 4 can be written

$$\forall u \in X, \ \forall (x_1, x_2, t) \in X^2 \times [0,1], \ d(u, \varphi(x_1, x_2, t)) \le \max\{d(u, x_1), d(u, x_2)\}. \tag{15}$$

From this remark we have a fixed point theorem originally proved by Ky Fan in normed spaces, Theorem 1.9 on page 146 of [15] and as a Corollary Khamsi's extension of that theorem to hyperconvex metric spaces, Theorem 6 in [21].

Proposition 9. *Let C be a compact convex subset of a complete continuous midpoint space (X, d, μ) whose metric is quasiconvex. Let $f : C \to X$ be a continuous map. Then, $\exists y_0 \in C$ such that, $\forall x \in C$*

$$d(y_0, f(y_0)) \le d(x, f(y_0)). \tag{16}$$

Furthermore, if for all $x \in C$ such that $f(x) \ne x$ one of the following conditions holds then f has a fixed point:

(1) $\exists t \in (0,1)$ such that $C \cap B[f(x), td(x, f(x)] \ne \emptyset$.

(2)(Fan's condition) The geodesic segment $\sigma_{(f(x),x)}$ contains at least two points of C.

    Proof. For all $x \in C$ let $\Gamma x = \{y \in C : d(y, f(y)) \le d(x, f(y))\}$. Notice that $x \in \Gamma x$ and that Γx is closed, and therefore compact.
Also, $C \setminus \Gamma^{-1} y = \{x \in C : d(x, f(y)) < d(y, f(y))\}$ is, by 15, geodesically convex. Applying Proposition 4 with $\Theta = \Gamma$ we find a point y_0 in $\cap_{x \in C} \Gamma x$. This proves the first part. If (1) holds and if $0 < d(y_0, f(y_0))$ then there exist $y_1 \in C$ and $t \in (0,1)$ such that $d(y_1, f(y_0)) \le td(y_0, f(y_0))$. Taking $x = y_1$ in 16 yields a contradiction. If (2) holds and if $y_0 \ne f(y_0)$ then there exists $y_1 \ne y_0$ which belongs to $C \cap \sigma_{(f(y_0), y_0)}$. We have $y_1 = \varphi(f(y_0), y_0, t)$ with $t < 1$ and therefore $d(y_1, f(y_0)) = d(\varphi(f(y_0), y_0, t), f(y_0)) = td(f(y_0), y_0)$ which shows that (1) holds. $\qquad\square$

Corollary 4. *Let C be a compact subset of a hyperconvex metric space (X, d) which is convex with respect to the natural convexity of (X, d), that is*

$\Delta_{hyperconv}(C) = C$, *and let* $f : C \to X$ *be a continous map such that,* $\forall x \in C$, *either* $f(x) = x$ *or* $\exists t \in (0,1)$ *such that*

$$C \cap B[f(x), td(x, f(x)] \neq \emptyset. \tag{17}$$

Then f *has a fixed point.*

Proof. Let μ be the midpoint point map associated to an arbitrary nonexpansive retraction $r : E \to X$ where E is a Banach space in which X is isometrically embedded. We have seen that (X, d, μ) is a Busemann midpoint space and that the metric is therefore quasiconvex.

Since C is $\Delta_{hyperconv}$-convex it is also Δ_μ-convex. By Proposition 9 there exists $y_0 \in C$ such that, for all $x \in C$, $d(y_0, f(y_0)) \leq d(x, f(y_0))$.

If $y_0 \neq f(y_0)$ then, by 17, we have a contradiction. $\qquad\square$

Instead of condition 17 above Khamsi in [21] writes $C \cap B[x, \alpha d(x, f(x)] \cap B[f(x), (1 - \alpha)d(x, f(x)] \neq \emptyset$ with $\alpha \in (0,1)$. Notice that, for $x \in C$, $C \cap B[x, \alpha d(x, f(x)] \neq \emptyset$ holds trivially and, for $\alpha \in (0,1)$, $B[x, \alpha d(x, f(x)] \cap B[f(x), (1 - \alpha)d(x, f(x)] \neq \emptyset$ holds by hyperconvexity. If we also have 17 then, by hyperconvexity, we also have Khamsi's condition which is therefore equivalent to 17.

7   Selections for lower semicontinuous maps

A metric space (X, d) endowed with a c-structure for which balls and arbitrary ε-neighborhoods of c-convex sets are c-convex is called an lc-space. Michael's selection theorem holds in lc-spaces. A proof can be found in [18] and a more recent proof using "small selections" can be found in [16], Theorem 6.1, or, in a slightly different context in [19], Theorem 3.4.

Since we are considering here only c-structures which are associated to a complete continuous midpoint space (X, d, μ) let us say that **the metric and the midpoint map are strongly compatible** if arbitrary ε-neighborhoods of convex sets are convex that is, if (X, Δ_μ) is an lc space.

For Bruhat-Tits spaces compatibility of the metric and the midpoint map is also a consequence of the following condition (**1-convexity** of the square of the distance to a point)[8] :

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[8]A function $f : X \to \mathbb{R}$ is λ-convex if $f(\mu(x, y)) \leq (f(x) + f(y))/2 - \lambda d(x, y)^2/4$ or, equivalently, if, for all x and y, the function $t \mapsto f(\varphi(x, y, t)) - \lambda t^2$ is convex.

$$\forall w \in X \ \text{ and } \ \forall(x,y) \in X^2 \quad d\big(w, \mu(x,y)\big)^2 \leq$$

$$\leq \frac{d(w,x)^2 + d(w,y)^2}{2} - \frac{d(x,y)^2}{4}. \tag{18}$$

Condition 7 is just another way to write the Bruhat-Tits Property.

We will now see that the metric and the midpoint map are strongly compatible exactly when, for all convex sets C, the distance to the convex set C, that is the function d_C, is quasiconvex, or, equivalently that, for all polytopes $\Delta_\mu(A)$, the distance function $d_{\Delta_\mu(A)}$ is quasiconvex. For unique midpoint spaces this condition reduces to the quasiconvexity of the distance functions to arbitrary geodesic segment .

The ε-neighborhood of a subset S of a metric space (X, d), denoted by $\mathcal{N}(S, \varepsilon)$, is the set of points of X whose distance to S is strictly smaller than ε, that is $\mathcal{N}(S, \varepsilon) = \{x \in X : d_S(x) < \varepsilon\}$ where $d_S(x) = \inf\limits_{y \in S} d(x, y)$. We write $\mathcal{N}[S, \varepsilon]$ for the set $\{x \in X : d_S(x) \leq \varepsilon\}$

Lemma 9. *The following statements are equivalent:*

(1) *The metric and the midpoint map of a complete continuous midpoint space (X, d, μ) are strongly compatible.*

(2) $\forall \varepsilon > 0$ *and for all convex subset C of X $\mathcal{N}[C, \varepsilon]$ is convex.*

(3) $\forall \varepsilon > 0$ *and $\forall A \in \langle X \rangle$ $\mathcal{N}(\Delta_\mu(A), \varepsilon)$ is convex.*

(4) $\forall \varepsilon > 0$ *and $\forall A \in \langle X \rangle$ $\mathcal{N}[\Delta_\mu(A), \varepsilon]$ is convex.*

(5) $\forall A \in \langle X \rangle$ *and $\forall(x_1, x_2) \in X \times X$*

$$d_{\Delta_\mu(A)}(\mu(x_1, x_2)) \leq \max\{d_{\Delta_\mu(A)}(x_1), d_{\Delta_\mu(A)}(x_2)\}.$$

(6) $\forall(y_1, y_2), (x_1, x_2) \in X^2$

$$d_{\Delta_\mu(\{y_1, y_2\})}(\mu(x_1, x_2)) \leq \max\{d_{\Delta_\mu(\{y_1, y_2\})}(x_1), d_{\Delta_\mu(\{y_1, y_2\})}(x_2)\}.$$

(7) *For all closed convex subset C of X and $\forall(x_1, x_2) \in X^2$*

$$d_C(\mu(x_1, x_2)) \leq \max\{d_C(x_1), d_C(x_2)\}.$$

Proof. The equivalence between (1) and (2) as well as that between (3) and (4) is a consequence of (Conv2), (Conv3), $\mathcal{N}(S, \varepsilon) = \bigcup_{\varepsilon' < \varepsilon} \mathcal{N}[S, \varepsilon']$ and $\mathcal{N}[S, \varepsilon] = \bigcap_{\varepsilon' > \varepsilon} \mathcal{N}(S, \varepsilon')$. The equivalence of (1) and (3) follows from 13 and (Conv3) applied to the family $\{\mathcal{N}(\Delta_\mu(A), \varepsilon) : A \in \langle C \rangle\}$. If (5) holds then, for all x_1 and x_2 in $\mathcal{N}[\Delta_\mu(A), \varepsilon]$, $\mu(x_1, x_2) \in \mathcal{N}[\Delta_\mu(A), \varepsilon]$. Since $\mathcal{N}[\Delta_\mu(A), \varepsilon]$ is closed it is convex. Assume that (4) holds. If x_1 and x_2 both belong to $\Delta_\mu(A)$, which is closed and convex, then $\mu(x_1, x_2) \in \Delta_\mu(A)$ and (5) trivially holds. If either x_1 or x_2 does not belong to $\Delta_\mu(A)$ let $\varepsilon = \max\{d_{\Delta_\mu(A)}(x_1), d_{\Delta_\mu(A)}(x_2)\}$; both x_1 and x_2 are in $\mathcal{N}[\Delta_\mu(A), \varepsilon]$, which is closed and convex, and therefore $\mu(x_1, x_2) \in \mathcal{N}[\Delta_\mu(A), \varepsilon]$ from which (5) follows. If (5) holds then so does (6).

Assume that (6) holds and let A be an arbitrary finite and nonempty subset of X. For all $\eta > \max\{d_{\Delta_\mu(A)}(x_1), d_{\Delta_\mu(A)}(x_2)\}$ we can find $y_{1,\eta}$ and $y_{2,\eta}$ in $\Delta_\mu(A)$ such that $d(y_{i,\eta}, x_i) \leq \eta$.

With $A_\eta = \{y_{1,\eta}, y_{2,\eta}\}$ we have $\max\{d_{\Delta_\mu(A_\eta)}(x_1), d_{\Delta_\mu(A_\eta)}(x_2)\} \leq \eta$. From (6) we obtain $d_{\Delta_\mu(A_\eta)}(\mu(x_1, x_2)) \leq \max\{d_{\Delta_\mu(A_\eta)}(x_1), d_{\Delta_\mu(A_\eta)}(x_2)\}$.

Since $\Delta_\mu(A_\eta) \subset \Delta_\mu(A)$ we also have $d_{\Delta_\mu(A)}(\mu(x_1, x_2)) \leq d_{\Delta_\mu(A_\eta)}(\mu(x_1, x_2))$ and therefore $d_{\Delta_\mu(A)}(\mu(x_1, x_2)) \leq \eta$.

One proceeds similarly to establish the equivalence of (5) and (7)   □

Lemma 10. *In a complete and continuous unique midpoint space (X, d, μ) the following statements are equivalent:*

(1) *The metric and the midpoint map are strongly compatible.*

(2) $\forall (x_1, x_2) \in X^2$ *and* $\forall \varepsilon > 0$, *the ε-neigborhood of the geodesic segment $\sigma_{(x_1, x_2)}$ is convex.*

(3) $\forall (y_1, y_2), (x_1, x_2) \in X^2$

$$d_{\sigma_{(y_1, y_2)}}(\mu(x_1, x_2)) \leq \max\{d_{\sigma_{(y_1, y_2)}}(x_1), d_{\sigma_{(y_1, y_2)}}(x_2)\}.$$

Proof. From Lemmas 4 and 9.   □

Recall that a continuous midpoint space (X, d, μ) is quasi-Busemann if, for all (u, v) and (x_1, x_2) in $X \times X$, $d(\mu(u, v), \mu(x_1, x_2)) \leq \max\{d(u, x_1), d(v, x_2)\}$.

Lemma 11. *In a quasi-Busemann continuous midpoint space (X, d, μ) the metric and the midpoint map are strongly compatible.*

Proof. Let $\varepsilon > 0$ and choose $u_{i,\varepsilon} \in \Delta_\mu(\{y_1, y_2\})$ such that
$d(u_{i,\varepsilon}, x_i) \leq d_{\Delta_\mu(\{y_1,y_2\})}(x_i) + \varepsilon$.
From $d\big(\mu(u_{1,\varepsilon}, u_{2,\varepsilon}), \mu(x_1, x_2)\big) \leq \max\{d(u_{1,\varepsilon}, x_1), d(u_{2,\varepsilon}, x_2)\}$ we have
$d\big(\mu(u_{1,\varepsilon}, u_{2,\varepsilon}), \mu(x_1, x_2)\big) \leq \max\{d_{\Delta_\mu(\{y_1,y_2\})}(x_1), d_{\Delta_\mu(\{y_1,y_2\})}(x_2)\} + \varepsilon$ and
therefore $d_{\Delta_\mu(\{y_1,y_2\})}(\mu(x_1, x_2)) \leq \max\{d_{\Delta_\mu(\{y_1,y_2\})}(x_1), d_{\Delta_\mu(\{y_1,y_2\})}(x_2)\} + \varepsilon$.
We have shown that (6) of Lemma 9 holds. $\qquad\square$

We can now state the main result of this section, that is Michael's Selection Theorem for midpoint spaces :

Proposition 10. *Let Y be a paracompact topological space, A a nonempty closed subspace of Y and (X, d, μ) a quasi-Busemann complete space. If $\Omega : Y \to X$ is a lower semicontinuous map with nonempty convex closed values then any continous selection $g : A \to X$ of Ω restricted to A can be extended to a continuous selection $f : Y \to X$ of Ω.*

8 Approximate selections and fixed points for upper semicontinuous maps

Approximate selections for upper semicontinuous maps with convex values in convex subsets of continuous midpoint spaces can be obtained from Theorem 3.5 of [19] which, adapted to complete continuous midpoint spaces, gives the following result:

Proposition 11. *Let (X, d, μ) be a complete quasi-Busemann space and Y be a paracompact topological space. Let $C \in \mathcal{C}_\mu(X)$ be a convex set and $\Omega : Y \to C$ an upper semicontinuous map with non empty convex values. Then, $\forall \varepsilon > 0$ there exists a continuous map $f : Y \to C$ such that, $\forall y \in Y$, the distance from $f(y)$ to Ωy is less that ε.*

Furthermore, if the values are convex and compact then, for all neighbourhood $\Theta \subset Y \times X$ of the graph of Ω there exists a continuous map $f : Y \to X$ whose graph is contained in Θ.

Corollary 5 (Kakutani's Fixed Point Theorem). *Let C be a convex and compact subset of a complete quasi-Busemann space. If $\Omega : C \to C$ is an upper semicontinuous map with non empty closed convex values then $\exists x^\star \in C$ such that $x^\star \in \Omega x^\star$.*

Proof. If, for all $x \in C$, $x \notin \Omega x$ then $(C \times C) \setminus \Delta_C$, where $\Delta_C = \{(x, x) : x \in C\}$, is a neighborhood of the graph of Ω. By Proposition 11 there is a

continuous map $f : C \to C$ whose graph is contained in $(C \times C) \setminus \Delta_C$ and this contradicts Theorem 1.   □

To obtain the Generalized Schauder Theorem for upper semicontinuous maps, that is the multivalued version of Corollary 3, we will have to ask a little more of the convexity associated to a midpoint map. We will need the following fact which is a consequence of Theorem 2.8 in [31].

Lemma 12. *In a complete quasi-Busemann space whose polytopes are all compact the convex hull of a compact set is compact.*

Proposition 12. *If C is a convex subset of a complete quasi-Busemann space whose polytopes are all compact and if $\Omega : C \to C$ is a compact[9] upper semicontinuous map with nonempty closed convex values then $\exists x^\star \in C$ such that $x^\star \in \Omega x^\star$.*

Proof.   There exists a compact subset K of C such that, for all $x \in C$, $\Omega x \subset K$. By Lemma 12 $\Delta_\mu(K)$ is compact; it is also a convex subset of C since C is itself convex. The restriction of Ω to $\Delta_\mu(K)$ is upper semicontinuous with compact and convex values. From Corollary 5 it has a fixed point.   □

In [19] the convex hull of a set S is defined as the intersection of all the convex sets containing it. So, for a subset S of C the convex hull of S in C, as defined in [19], is the intersection of all the convex subsets C' of C containing S, and since C is itself convex convex, this convex hull is also the intersection of all the convex subsets C' of X containing S. We have implicitly used in the proof the fact that for a on empty finite subset A of X the intersection of all the convex subsets C' of X containing A is exactly $\Delta_\mu(A)$; 12 says that this is indeed the case.

9   Fixed points for nonexpansive maps

A well known result of F. Browder states that bounded closed convex subsets of Hilbert spaces have the fixed point property for nonexpansive maps; that result was extended to uniformly convex normed spaces by both Browder and Goehde. Bounded hyperconvex metric spaces also have the fixed point property for nonexpansive maps. Closed bounded convex subsets of uniformly

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[9]$\Omega : C \to C$ is a compact if there exists a compact subset $K \subset C$ such that $\cup_{x \in C} \Omega x \subset K$

convex normed spaces and hyperconvex bounded spaces belong to a class of metric spaces, all of which share a kind of weak compactness property and a so called normality condition (the Chebyshev radius not greater than the diameter); Kirk showed that those spaces have the fixed point property for nonexpansive maps, Section 4 of [12], more particularly Theorem 4.5 and 4.6.

We follow another path, that of the purely metric proof given in [15]. The subclass of the class of continuous metric spaces for which that method works contains the class of complete Bruhat-Tits spaces.

Lemma 13. *In all complete Bruhat-Tits spaces* (X, d) *the following properties hold:*

(1) $\forall u \in X \quad \forall (a, b) \in X^2 \quad \forall R, r \geq 0 \;\; if \; R \geq \max\{d(u, a), d(u, b)\}$ *and* $r \leq d(u, \mu(a, b))$ *then*
$$\frac{d(a, b)}{2} \leq \sqrt{R^2 - r^2};$$

(2) $\forall (u, t) \in X \times]0, 1[$ *the map* $x \mapsto \varphi(u, x, t)$ *is contractive.*

Proof. Statement (1) is easily obtained from the Bruhat-Tits Property. Statement (2) is true of all midpoint Busemann spaces, and therefore of Bruhat-Tits spaces.
Indeed, by Proposition 3 the map $t \mapsto d\big(\varphi(u, x, t), \varphi(u, y, t)\big)$ is convex, we can therefore write

$$d\big(\varphi(u, x, t), \varphi(u, y, t)\big) \leq (1-t)d\big(\varphi(u, x, 0), \varphi(u, y, 0)\big) + td\big(\varphi(u, x, 1), \varphi(u, y, 1)\big).$$

Finally, from $\varphi(a, b, 0) = a$ and $\varphi(a, b, 1) = b$ we obtain

$$d\big(\varphi(u, x, t), \varphi(u, y, t)\big) \leq td(x, y).$$

$\square$

Using the first part of the preceding lemma and proceeding exactly as in [15] one can proves the following statement:

Lemma 14. *Let* C *be a convex and bounded subset of a complete continuous midpoint space* (X, d, μ) *for which* (1) *and* (2) *of Lemma 13 hold and let* $F : C \to C$ *be nonexpansive. Then,* $\forall x, y \in C$ *and* $\forall R$ *such that* $d(x, F(x)) \leq R$ *and* $d(y, F(y)) \leq R$

$$d\big(\mu(x, y), F(\mu(x, y))\big)^2 \leq 8R\,diam(C).$$

We now have all the necessary ingredients to prove Browder's Fixed Point Theorem for closed convex subsets of complete Bruhat-Tits spaces as in [15].

Theorem 1. *Closed bounded convex subsets of a complete continuous midpoint spaces (X, d, μ) for which (1) and (2) of Lemma 13 hold have the fixed point property for continuous nonexpansive maps.*

Proof. Let C be a closed bounded convex subset of X and let $F : C \to C$ be a nonexpansive map. We proceed as in the proof of Browder's Theorem in [15], with some very minor adaptations, to show that F has a fixed point.

Fix an arbitrary point $u \in C$ and, for all $n > 0$ let

$$F_n(x) = \varphi(u, F(x), 1 - 1/n).$$

By the proof of second part of Lemma 13, F_n is contractive with contraction constant equal to $1/n$; let x_n be its fixed point. From Lemma 1 we have

$$d(x_n, F(x_n)) = d(\varphi(u, F(x_n), 1 - 1/n), F(x_n)) = \frac{d(u, F(x_n))}{n}$$

from which we obtain

$$d(x_n, F(x_n)) \leq \frac{\mathrm{diam}(C)}{n}.$$

As in [15], for $n \geq 2$, let $Q_n = \{x \in C : d(x, F(x)) \leq (1/n)\mathrm{diam}(C)\}$; Q_n is closed and not empty. Let $d_n = \inf_{x \in Q_n} d(u, x)$ and notice that, for all $n \geq 2$, $d_n \leq d_{n+1} \leq \mathrm{diam}(C)$. Let $d = \lim_{n \to \infty} d_n$ and

$$A_n = Q_{8n^2} \cap B[u, d + 1/n].$$

Using Lemma 14 one shows that, for all x, y in Q_{8n^2}, $\mu(x, y) \in Q_n$. Therefore, if x and y are in A_n, we must have $d(u, x) \leq d + \dfrac{1}{n}$ and $d(u, y) \leq d + \dfrac{1}{n}$ as well as $d(u, \mu(x, y)) \geq d_n$. The first part of Lemma 13 yields the following estimation for the diameter of A_n:

$$\mathrm{diam} A_n \leq 2\sqrt{\frac{2d}{n} + \frac{1}{n^2} + d^2 - d_n^2}.$$

To complete the proof notice that $A_{n+1} \subset A_n$ and consequently that $\bigcap_{n \geq 2} A_n$ reduces to a single point which is clearly a fixed point of F. □

Of course, as has already been said, this is the proof of [15]. But having said so it might be interesting to notice that it entirely relies on the two conditions of Lemma 13, which as we have seen hold in all complete Bruhat-Tits spaces. The second condition says that "*the translation operators* $x \mapsto \varphi(u, x, t)$ *along the geodesics issuing from a given point* u *are contractive*". It holds whenever the map $t \mapsto d\big(\varphi(u, x, t), \varphi(u, y, t)\big)$ is convex that is, by Proposition 3 in Busemann midpoint spaces. The first condition of Lemma 13 says that closed balls are "*at least as uniformly convex as balls in a Hilbert space*". To make that statement precise define the **modulus of convexity** in a continuous midpoint space (X, d) as in [13], that is

$$\delta(R, \varepsilon) = \inf \left(1 - \frac{d(u, \mu(a, b))}{R} \right)$$

where $R > 0$, $\varepsilon \in [0, 2]$ and the infimum is taken over all triples $(u, a, b) \in X^3$ such that $\max\{d(u, a), d(u, b)\} \leq R$ and $d(a, b) \geq \varepsilon R$. For a Hilbert space the modulus of convexity does not depend on R and its value is $\delta_H(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}$.[10]

Proposition 13. *In all continuous midpoint spaces* (X, d, μ) *the following conditions are equivalent:*

(1) $\forall u \in X \quad \forall (a, b) \in X^2 \quad \forall R, r \geq 0 \;$ *if* $R \geq \max\{d(u, a), d(u, b)\}$ *and* $r \leq d(u, \mu(a, b))$ *then*
$$\frac{d(a, b)}{2} \leq \sqrt{R^2 - r^2};$$

(2) $\forall R > 0$ *and* $\forall \varepsilon \in [0, 2]$
$$\delta(R, \varepsilon) \geq \delta_H(\varepsilon);$$

(3) $\forall u \in X$ *and* $\forall (a, b) \in X^2$
$$4d\big(u, \mu(a, b)\big)^2 + d(a, b)^2 \leq 4 \max\{d(u, a), d(u, b)\}^2. \tag{19}$$

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[10]The modulus of convexity with respect to the hyperbolic metric is defined and computed in [13] where it is shown that $\delta(R, \varepsilon) \geq \delta_H(\varepsilon)$.

Proof. Assume that (1) holds. If $R \geq \max\{d(u,a), d(u,b)\}$ then, taking $r = d(u, \mu(a,b))$, we have

$$d(u, \mu(a,b))^2 \leq R^2 - \frac{d(a,b)^2}{4}.$$

If we also have $d(a,b) \geq \varepsilon R$ then

$$d(u, \mu(a,b)) \leq R\sqrt{1 - \frac{\varepsilon^2}{4}} = R\left[1 - \left(1 - \sqrt{1 - \frac{\varepsilon^2}{4}}\right)\right]$$

and consequently $\delta(R, \varepsilon) \geq \delta_H(\varepsilon)$.

Assume that $\delta(R, \varepsilon) \geq \delta_H(\varepsilon)$ for all $R > 0$ and $\varepsilon \in [0, 2]$. If $R \geq \max\{d(u,a), d(u,b)\}$ then $d(a,b)/R \leq 2$. We take $\varepsilon = d(a,b)/R$ to obtain

$$d\big(u, \mu(a,b)\big)^2 \leq R^2 - \frac{d(a,b)^2}{4}$$

from which (1) follows. The equivalence between (1) and (3) is clear.   □

The left hand side of condition 19 is identical the left hand side of the Bruhat-Tits condition. The Bruhat-Tits condition clearly implies 19 which can be seen as a kind of "quasi-Bruhat-Tits" condition. Notice also that condition 19 implies that closed balls are convex, since $d\big(u, \mu(a,b)\big) \leq \max\{d(u,a), d(u,b)\}$.

Let us say that a continuous midpoint space (X, d, μ) is **strictly convex** if closed balls are convex and for all convergent sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of points of X and all points $u \in X$ such that

$$\lim_{n\to\infty} d(u, x_n) = \lim_{n\to\infty} d(u, y_n) = \lim_{n\to\infty} d(u, \mu(x_n, y_n))$$

one has

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

With $a = x_n$ and $b = y_n$ in 19 one sees that a continuous midpoint space for which one of the Properties of Proposition 13 holds is strictly convex. In particular, complete Bruhat-Tits spaces are strictly convex. The following proposition shows that "*closed and bounded convex subsets of strictly convex midpoint spaces are weakly compact*".

Proposition 14. *Let $\mathcal{F} = \{C_\lambda : \lambda \in \Lambda\}$ be a family of closed and bounded convex subsets of a continuous midpoint space which is complete and strictly convex. If $\mathcal{F}$ has the finite intersection property then $\bigcap_{\lambda \in \Lambda} C_\lambda \neq \emptyset$.*

Proof. (A) Let us first assume that the elements of $\mathcal{F}$ form a decreasing family $C_0 \supset C_1 \supset \cdots$ of nonempty closed convex susbets of X. Fix an arbitrary point u_0 of C_0 and let d_n be the distance from u_0 to C_n. From $d_0 \leq d_1 \cdots \leq d_n \leq d_{n+1} \leq \cdots \leq \operatorname{diam}C_0$ we have a convergent sequence. Let $d = \lim_{n \to \infty} d_n$. We verify that there is a unique point $\bar{u} \in \bigcap_{n \in \mathbb{N}} C_n$ such that $d(u_0, \bar{u}) = d$.

If such a point exists it has to belong to $\bigcap_{n \in \mathbb{N}} \left[B[u_0, d + 2^{-n}] \cap C_n \right]$. Taking $A_n = B[u_0, d + 2^{-n}] \cap C_n$ we obtain a decreasing sequence of nonempty closed convex sets. For all $n \in \mathbb{N}$ choose x_n and y_n in A_n such that

$$\operatorname{diam}A_n \leq d(x_n, y_n) + 2^{-n}.$$

From $\mu(x_n, y_n) \in A_n$ and $d_n \leq d(u_0, v_n)$ for all points v_n of C_n we have

$$\lim_{n \to \infty} d(u_0, x_n) = \lim_{n \to \infty} d(u_0, y_n) = \lim_{n \to \infty} d(u_0, \mu(x_n, y_n))$$

and therefore $\lim_{n \to \infty} d(x_n, y_n) = 0$. This shows that $\lim_{n \to \infty} \operatorname{diam}A_n = 0$ and consequently that there is a point $\bar{u}$ such that $\bigcap_{n \in \mathbb{N}} A_n = \{\bar{u}\}$.

(B) To complete the proof let C_0 be an arbitrary element of $\mathcal{F}$ and an element u_0 of C_0. Replacing each C of $\mathcal{F}$ by $C \cap C_0$ we we can assume that all the elements of $\mathcal{F}$ are contained in C_0. Let $d = \sup_{C \in \mathcal{F}} d(u_0, C)$ and fix a decreasing sequence $\{\hat{C}_n : n \in \mathbb{N}\}$ of elements of $\mathcal{F}$ such that $d - 2^{-n} \leq d(u_0, C_n) \leq d$. Let $\bar{u}$ be the unique point (A) such that $\bar{u} \in \bigcap_{n \in \mathbb{N}} C_n$ and $d(u_0, \bar{u}) = d$. To see that $\bar{u} \in \bigcap_{C \in \mathcal{F}} C$ take an arbitrary element C of $\mathcal{F}$ and consider the sequence given by $C'_n = C \cap C_n$. There is a unique point u' such that $u' \in \bigcap_{n \in \mathbb{N}} C'_n$ and $d(u_0, u') = \sup_{n \in \mathbb{N}} d(u_0, C'_n)$. From $d(u_0, C_n) \leq d(u_0, C'_n) \leq d$ we have $d(u_0, u') = d$ and therefore $u' = \bar{u}$. $\square$

The proof of Proposition 14 given above is essentially identical to the proof of Theorem 4.2 for closed bounded subsets of Hilbert spaces given in [15]. The first part of the proof of Proposition 14 has the following interesting byproduct:

Proposition 15. *Let (X, d, μ) be a complete continuous midpoint space which is strictlly convex. Then, for all closed nonempty convex subset C of X and for all point $u \in X$ there exists a unique point $\bar{u} \in C$ such that $d(u, \bar{u}) = d(u, C)$.*

Proof. Let $d = d(u, C)$ and apply (A) of Proposition 14 to the constant sequence $C_n = B[u, d + 1] \cap C$.   □

Under the conditions of Proposition 15, closed convex sets are Chebyshev. This is proved for the hyperbolic metric on the open unit disc in [13]. An important feature of the projection onto a closed convex subset of a Hilbert space is that it is nonexpansive. The same is true for the hyperbolic metric, Theorem 6.10 in [13]. Given a complete continuous strictly convex midpoint space (X, d, μ) and a closed nonempty convex subset C of X let $P_C : X \to C$ be the closest point map. For complete Bruhat-Tits spaces P_C is nonexpansive, for a proof see Theorem 2.4 on page 176 of [5] which we partially state below.

Theorem 2. *Let (X, d) be a complete Bruhat-Tits space and C a closed nonempty convex subset of X. Then*

(1) $P_C : X \to C$ is a nonexpansive retraction and

(2) the map $H : X \times [0, 1] \to X$ defined by $H(x, t) = \varphi(x, P_C(x), t)$ is a continuous homotopy from the identity map of X to C.

The second part of Theorem 2 implies that a closed convex subset of a complete Bruhat-Tits space is a deformation retract of X.

From Theorem 2 some well known fixed points theorems for nonexpansive maps in Hilbert spaces, Theorem 1.5 and Corollary 1.6 Page 54 of [15] can easily be extended to complete Bruhat-Tits spaces. First, we prove the following lemma.

Lemma 15. *Let $B[u_0, r]$ be the closed ball of radius r centered at u_0 of a complete Bruhat-Tits space (X, d). Then $\forall x \in X \setminus B[u_0, r]$, the geodesic segment $\sigma_{(u_0, x)}$ and the boundary of $B[u_0, r]$ intersect in a single point which is the projection of x on $B[u_0, r]$.*

Proof. Let $\bar{x}$ be the projection of x on $B[u_0, r]$. Since $d(u_0, \varphi(\bar{x}, x, 0)) \leq r$ and $d(u_0, \varphi(\bar{x}, x, 1)) > r$ there exists $t \in [0, 1]$ such that $d(u_0, \varphi(\bar{x}, x, t)) = r$.

If $t \neq 0$ then, from $d(\varphi(\bar{x}, x, t)), x) = (1-t)d(\bar{x}, x)$, we have $d(\varphi(\bar{x}, x, t)), x) < d(\bar{x}, x)$, which would contradict the minimality of $d(\bar{x}, x)$. Therefore $t = 0$ and $d(u_0, \bar{x}) = r$. □

Theorem 3. *Let $f : B[u_0, r] \to X$ be a continuous map defined on a closed ball of a complete Bruhat-Tits space. Assume that f is either compact or nonexpansive. Then either*

(1) *f has a fixed point or*

(2) *there exists $t \in]0, 1[$ such that the map $x \mapsto \varphi(u_0, t, f(x))$ has a fixed point on the boundary of $B[u_0, r]$ or, equivalently, there exists $\hat{x}$ such that $\{\hat{x}\} = \sigma_{(u_0, f(\hat{x}))} \cap \partial B[u_0, r]$.*

Proof. Let $C = B[u_0, r]$ and $g(x) = P_C(f(x))$. If f is a compact map then g is a compact map from $B[u_0, r]$ to itself. If f is nonexpansive then g is nonexpansive. In either case there exists $\hat{x} \in C$ such $g(\hat{x}) = \hat{x}$. If $f(\hat{x}) \in C$ then $g(\hat{x}) = f(\hat{x})$; if not let $\alpha(t) = \varphi(u_0, f(\hat{x}), t)$. It is an homeomorphism from $[0, 1]$ to the geodesic segment $\sigma_{(u_0, f(\hat{x}))}$. Since $d(u_0, \alpha(0)) = 0$ and $d(u_0, \alpha(1)) > r$ there is unique $\hat{t} \in]0, 1[$ such that $d(u_0, \alpha(\hat{t})) = r$. From Lemma 15 we have $\alpha(\hat{t}) = P_C(f(\hat{x}))$. In conclusion $d(u_0, \hat{x}) = r$ and $\hat{x} = \varphi(u_0, f(\hat{x})), \hat{t})$ with $0 < \hat{t} < 1$ that is $\{\hat{x}\} = \sigma_{(u_0, f(\hat{x}))} \cap \partial B[u_0, r]$. □

Corollary 6. *Let (X, d, μ) be a complete Bruhat-Tits space and $f : X \to X$ a continuous map that is either absolutely compact[11] or nonexpansive. Assume that $\exists u_0 \in X$ and $r > 0$ such that $\forall x \in \partial B[u_0, r]$ one of the following two conditions below holds:*

(1) *$d(u_0, f(x))^2 \leq r^2 + d(x, f(x))^2$;*

(2) *$d(u_0, f(x)) \leq d(x, f(x))$.*

Then f has a fixed point.

Proof. Assume that (1) holds. If f does not have a fixed point then there exists $t \in]0, 1[$ and $x \in \partial B[u_0, r]$ such $x = \varphi(u_0, t, f(x))$. From

$$d(x, f(x)) = d(\varphi(u_0, t, f(x)), f(x)) = (1 - t)d(u_0, f(x))$$

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[11] f is absolutely compact if for all closed balls B the restriction of f to B is a compact map.

and
$$r = d(u_0, x) = d(u_0, \varphi(u_0, t, f(x)) = td(u_0, f(x))$$

we have $d(u_0, f(x)) = r + d(x, f(x))$ with $d(x, f(x)) > 0$ which contradicts (1).

If (2) holds then so does (1). □

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