HAMILTONIAN INCLUSIONS WITH CONVEX DISSIPATION WITH A VIEW TOWARDS APPLICATIONS[*]

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Abstract

We propose a generalization of Hamiltonian mechanics, as a Hamiltonian inclusion with convex dissipation function. We obtain a dynamical version of the approach of Mielke to quasistatic rate-independent processes. Then we show that a class of models of dynamical brittle damage can be formulated in this setting.

1 Introduction

We are interested in the modification of the Hamiltonian formalism by adding the subdifferential of a convex dissipation function. In the Lagrangian formalism this can be traced back to Rayleigh and Kelvin (cf. Thomson and Tait [28] or Chetayev [11]). For the case of autonomous Hamiltonian systems with a Rayleigh dissipation function added see the paper of Bloch,

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Krishnaprasad, Marsden and Ratiu [7]. Perturbation analysis of Hamiltonian systems is an old and much explored subject, which is beyond our scope in this paper.

Closer to our interests is Mielke theory of quasistatic rate-independent processes [20]. In fact one of our purposes is to reformulate Mielke theory in a dynamical context. From this point of view a dissipation perturbed Hamiltonian approach seems the most economical.

From the viewpoint of multivalued analysis, many generalizations of Hamiltonian and Lagrangian mechanics have been considered, like for example Rockafellar [25], Aubin, Cellina and Nohel [6] or Clarke [12]. The problem of solving a subdifferential inclusion of the type (14) for a 1-homogeneous dissipation function seems to be new. As a general problem this subdifferential inclusion seem to fall in the class of problems studied in the viabilty theory, [6] or the more recent [5], but the mathematical results in these papers do not apply here mainly because the dissipation is 1-homogeneous.

More specifically, concerning the particular form – (42) coupled with purely Hamiltonian equations (37), (39)– of this subdifferential inclusion, which is relevant for damage models in continuum mechanics, it seems that there are no mathematical results which could be applied to this problem as a perturbed Hamiltonian problem. We thank to one of the anonymous referees for pointing us to the paper [26]. From our viewpoint the results of this paper can be seen as leading to a solution of our problem, studied from the Lagrangian side, that is after reformulating it as a generalized Euler-Lagrange equation. Nevertheless we think that the Hamiltonian structure of this problem may lead to interesting discretization algorithms, maybe based on symplectic integrators, which are known to handle correctly the energy balance even in the discretized form.

Outline of the paper. In section 2 we propose and begin the study of a generalized Hamiltonian formalism, in the form of a subdifferential inclusion using a convex dissipation function. In section 3 we show that Mielke's theory of quasistatic evolutionary processes is the quasistatic approximation of the formalism presented here. As an application, in section 4 we use the formalism for a energy of the Ambrosio-Tortorelli type and a 1-homogeneous dissipation function and we obtain a dynamical model of brittle damage which may be of interest in continuum media mechanics.

2 Generalized Hamiltonian equations with convex dissipation

In the Lagrangian formalism we study the evolution of a system described by a variable q, which satisfies the Euler-Lagrange equation associated to a function $L = L(t, q, \dot{q})$:

$$D_q L(t, q, \dot{q}) - \frac{d}{dt} D_{\dot{q}} L(t, q, \dot{q}) = 0 \qquad . \tag{1}$$

The function L is called a Lagrangian and in many situation it has the form

$$L(t, q, \dot{q}) = \hat{T}(\dot{q}) - \mathcal{E}(t, q) \tag{2}$$

where: $\hat{T}$ represents the kinetic energy, is a smooth strictly convex smooth function (for example quadratic, positive definite), and $\mathcal{E}$ is a potential energy or stored energy.

In the formalism of Hamiltonian mechanics we double the variables: the system is described by a pair (q, p) where p has the meaning of a momentum associated with q. Instead of the Euler-Lagrange equation, the following system of equations is used:

$$\begin{cases} -\dot{p} & \in & D_q H(t, q, p) \\ \dot{q} & = & D_p H(t, q, p) \end{cases} \tag{3}$$

The function $H = H(t, q, p)$ is called a Hamiltonian.

Consider for simplicity that $q, p \in \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space with scalar product $(\cdot, \cdot)$. The equations of Hamiltonian mechanics can be written in a compact form if we use the notations $z = (q, p) \in \mathcal{H} \times \mathcal{H}$, $J(z) = J(q, p) = (-p, q)$:

$$J \dot{z} - D_z H(t, z) = 0 \tag{4}$$

In particular the Hamiltonian may take the form

$$H(t, q, p) = T(p) + \mathcal{E}(t, q) \tag{5}$$

where T represents again the kinetic energy, this time expressed as a function of p.

In this case the two formalisms are equivalent if we take T to be the Legendre-Fenchel conjugate of $\hat{T}$:

$$T(p) = \sup \left\{ (p, q) - \hat{T}(q) : q \in \mathcal{H} \right\}$$

2.1 Introducing dissipation

Consider a "dissipation function" $\mathcal{R}(q, \dot{q})$, convex in the second argument, and a Lagrangian function which is a sum of kinetic and potential energies. In the particular case of Rayleigh dissipation the function $\mathcal{R}$ has the form

$$\mathcal{R}(q, \dot{q}) = \frac{1}{2} \|\dot{q}\|^2$$

where $\|\cdot\|$ is a norm function. Then the Euler-Lagrange equations perturbed with the dissipation function $\mathcal{R}$ are, by definition:

$$D_q L(t, q, \dot{q}) - \frac{d}{dt} D_{\dot{q}} L(t, q, \dot{q}) \in \partial_{\dot{q}} \mathcal{R}(q, \dot{q}) \qquad . \tag{6}$$

where the ∂ symbol denotes the subdifferential from convex analysis.

The Hamiltonian side of (6) is then

$$\begin{cases} -\dot{p} & \in & D_q H(t, q, p) + \partial_{\dot{q}} \mathcal{R}(q, \dot{q}) \\ \dot{q} & = & D_p T(p) \end{cases} \tag{7}$$

This motivates us to propose the following generalization of the Hamiltonian equations (4) in the form of a subdifferential inclusion:

$$J \dot{z} - D_z H(t, z) \in \partial_{\dot{z}} \mathcal{R}(z, \dot{z}) \tag{8}$$

where $\partial_{\dot{z}} \mathcal{R}(z, \dot{z})$ is the subdifferential of $\mathcal{R}$ with respect to $\dot{z}$:

$$\partial_{\dot{z}} \mathcal{R}(z, \dot{z}) = \big\{ (\bar{q}, \bar{p}) \in \mathcal{H} \times \mathcal{H} : \forall z' = (q', p') \in \mathcal{H} \times \mathcal{H} \tag{9}$$

$$\mathcal{R}(z, \dot{z} + z') \geq \mathcal{R}(z, \dot{z}) + (\bar{q}, q') + (\bar{p}, p') \big\} \quad .$$

We shall then be interested in the following particular case: suppose that we have a decomposition of the state variable $q = (q_1, q_2)$ into a non-dissipative q_1 variable and a dissipative q_2 variable. Then the momentum variable p decomposes as $p = (p_1, p_2)$. The Hamiltonian function H is taken as follows

$$H(t, q_1, q_2, p_1, p_2) = \mathcal{K}(p_1) + \frac{1}{2} \langle A p_2, p_2 \rangle + \mathcal{E}(t, q_1, q_2) \tag{10}$$

where $\mathcal{K}$ is the kinetic energy associated to the variable p_1, A is a strictly positive definite symmetric operator and $\mathcal{E}$ is a stored energy function. The dissipation function takes the form

$$\mathcal{R}(q_1, q_2, \dot{q}_1, \dot{q}_2) = \rho(\dot{q}_2) \tag{11}$$

with ρ a convex function.

With these choices of functions H and $\mathcal{R}$ the system of equations (7) becomes:

$$\begin{cases} -\dot{p}_1 & = & D_{q_1}\mathcal{E}(t, q_1, q_2) \\ \dot{q}_1 & = & D_p\mathcal{K}(p_1) \\ -\dot{p}_2 & \in & D_{q_2}\mathcal{E}(t, q_1, q_2) + \partial\rho(\dot{q}_2) \\ \dot{q}_2 & = & A\,p_2 \end{cases} \qquad (12)$$

We can see the first two equations as a Hamiltonian evolution of the variables (q_1, p_1) which has (q_2, p_2) as control parameters, coupled with a pair of evolution equations (the last two equations in (12)) for the control parameters. These last two equations can be seen as a differential inclusion:

$$-\dot{p}_2 - D_{q_2}\mathcal{E}(t, q_1, q_2) \in \partial\rho(A^{-1}p_2) \qquad . \qquad (13)$$

Interesting particular cases of dissipation function ρ are:

(a) $\rho = 0$, no dissipation, this corresponds to classical Hamiltonian equations,

(b) $\rho(\dot{q}) = \dfrac{1}{2}(\dot{q}, \dot{q})$, (where $(\cdot, \cdot)$ is a scalar product), which can be traced back to the Rayleigh dissipation function,

(c) $\rho(\dot{q}) = \|\dot{q}\|$, where $\|\cdot\|$ is a Banach space norm, or a more general 1-homogeneous convex function which, as we shall explain, is related to the approach of Mielke and collaborators – Mielke, Theil [22], Mielke, Theil and Levitas [23], [20] – to quasistatic rate-independent evolutionary processes.

2.2   The formalism in topological vector spaces

We shall precisely formulate relation (8) for a pair of locally compact topological vector spaces in duality. In particular this will cover the cases of Banach or Hilbert spaces.

Let X and Y be topological, locally convex, real vector spaces of dual variables $x \in X$ and $y \in Y$, with the duality product $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{R}$. We shall suppose that X, Y have topologies compatible with the duality product, that is: any continuous linear functional on X (resp. Y) has the form $x \mapsto \langle x, y \rangle$, for some $y \in Y$ (resp. $y \mapsto \langle x, y \rangle$, for some $x \in X$).

In this frame we don't have scalar products, neither an equivalent of the linear transformation J, therefore we start by introducing natural notations which make sense in this generality.

We want to study generalized Hamiltonian evolutions in the space $X \times Y$. For a general element of $X \times Y$ we shall use the notation $z = (x, y)$, or similar.

In order to properly formulate Hamiltonian equations or inclusions we need: a symplectic form, a Poisson bracket and a notion of subdifferential adapted in this setting. These will be the most natural objects one may think about and they were used many times before.

We shall use notations familiar in symplectic geometry, namely: ω for the symplectic form, $\{\cdot, \cdot\}$ for the Poisson bracket, X_f for the symplectic gradient of the function $f : X \times Y \to \mathbb{R}$ (if the linear J is available then $X_f = -J\,Df$, where Df is the differential of f). Instead of the usual subdifferential of a convex function F we shall use a "symplectic subdifferential" $X\,F$. In the usual setting in Hilbert spaces we have $J\,X\,F = \partial F$, where ∂F is the conventional subdifferential from convex analysis. In this general setting the definition of $X\,F$ is obtained from the definition of ∂F by replacing scalar products with the symplectic form.

Remark however that in this general setting the symplectic form and Poisson bracket have to be understood in a weaker sense than usual, let's say on a finite dimensional symplectic manifold. Indeed, a symplectic form is a non-degenerated 2-form which is closed (we renounce to the condition of being closed); a Poisson bracket is a Lie bracket over a algebra of functions, with supplementary properties, while here the "Poisson bracket" we define sends a pair of differentiable functions from $Der(X, Y)$ to a function which is not differentiable a priori.

Let us proceed with the introduction of the necessary objects.

Definition 1. *The space $X \times Y$ is endowed with a symplectic form: for any $z' = (x', y')$, $z'' = (x'', y'')$ we define the bilinear and anti-symmetric form*

$$\omega(z', z'') = \langle x', y'' \rangle - \langle x'', y' \rangle \qquad .$$

$Der(X, Y)$ is the linear space of functions $f : X \times Y \to \mathbb{R}$ which are continuously differentiable in each argument in the following sense: there are continuous functions $D_x f : X \times Y \to Y$ and $D_y f : X \times Y \to X$ such that for any $(x, y) \in X \times Y$ and

(a) for all $y' \in Y$ we have

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[f(x, y + \varepsilon y') - f(x, y) \right] = \langle D_y f(x, y), y' \rangle$$

(b) for all $x' \in X$ we have

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[f(x + \varepsilon x', y) - f(x, y) \right] = \langle x', D_x f(x, y) \rangle$$

The symplectic gradient of $f \in Der(X, Y)$ is the function $X_f : X \times Y \to X \times Y$ defined by

$$X_f(x, y) = (D_y f(x, y), -D_x f(x, y)) \qquad .$$

The Poisson bracket is the bilinear, antisymmetric form

$$\{\cdot, \cdot\} : Der(X, Y) \times Der(X, Y) \to \mathbb{R}^{X \times Y}$$

defined by: $\{f, g\} = \omega(X_f, X_g)$.

Definition 2. *Let $F : X \times Y \to \mathbb{R}$ be a convex lsc function. The symplectic subdifferential of F is the multivalued function which sends $z = (x, y) \in X \times Y$ to the set*

$$X F(z) = \left\{ z' \in X \times Y : \forall z'' \in X \times Y \quad F(z + z'') \geq F(z) + \omega(z', z'') \right\}$$

Remark that if $F \in Der(X, Y)$ and convex then we have $X F = \{X_F\}$. Indeed, if we use $z' = X_F = (D_y F(x, y), -D_x F(x, y))$ in the definition 2 of the symplectic differential we get

$$F(z + z'') \geq F(z) + \langle D_y F(x, y), y'' \rangle + \langle x'', D_x F(x, y) \rangle$$

which is true due to the convexity of F. Therefore $X_F(x, y) \in X F(x, y)$. The converse implication, that is $z' \in X F(x, y)$ implies $z' = X_F(x, y)$, is true by standard arguments of convex analysis.

We propose the following generalization of Hamiltonian evolution.

Definition 3. *Let $H : [0, T] \times X \times Y \to \mathbb{R}$ such that for all $t \in [0, T]$ we have $H(t, \cdot) \in Der(X, Y)$, and $\mathcal{R} : (X \times Y)^2 \to \mathbb{R} \cup \{+\infty\}$ be a "dissipation function" with the properties:*

(a) for any $z', z'' \in X \times Y$ we have $\mathcal{R}(z', z'') \geq 0$ and $\mathcal{R}(z', 0) = 0$,

(b) for any $z \in X \times Y$ the function $\mathcal{R}(z, \cdot)$ is convex, lsc.

Then a curve $z : [0, T] \to X \times Y$ is a solution of the evolution problem with Hamiltonian H and dissipation $\mathcal{R}$ if it is differentiable for all $t \in [0, T]$ (with derivative denoted by $\dot{z}$) and it satisfies the subdifferential inclusion:

$$\dot{z}(t) - X_{H(t,\cdot)}(z(t)) \in X\,(\mathcal{R}(z(t), \cdot))\,(\dot{z}(t)). \qquad (14)$$

We can give an equivalent characterization for a solution, which later will lead to a notion of weak solution. For any $f \in Der(X, Y)$ and any differentiable curve $z : [0, T] \to X \times Y$ we denote by $f \circ z : [0, T] \to \mathbb{R}$ the function composition of f and z, and by $\dfrac{d}{dt}\,[f \circ z]\,(t)$ the differential of this composition.

Proposition 1. *With the notations from definition 3, z is a solution of the evolution problem if and only if for any $f \in Der(X, Y)$ and for any $t \in [0, T]$ we have:*

$$\mathcal{R}(z(t), \dot{z}(t) - X_f(z(t))) \geq \mathcal{R}(z(t), \dot{z}(t)) + \frac{d}{dt}\,[f \circ z]\,(t) - \qquad (15)$$

$$- \{f, H(t, \cdot)\}\,(z(t)) \qquad .$$

Proof. For any $f \in Der(X, Y)$ and any differentiable curve $z : [0, T] \to X \times Y$ we have, by direct computation:

$$\frac{d}{dt}\,[f \circ z]\,(t) - \{f, H(t, \cdot)\}\,(z(t)) = \qquad (16)$$

$$= -\omega\left(\dot{z}(t) - X_{H(t,\cdot)}(z(t)), X_f(z(t))\right) \qquad .$$

Let z be a solution of the evolution problem. We choose then in (14) $z'' = -X_f(z(t))$ and use (16) to get (15).

Conversely, suppose that the curve z satisfies (15). For any $z'' \in X \times Y$ let us define $f \in Der(X, Y)$ by $f(z) = \omega(z, z'')$. It is easy to see then that $X_f = -z''$, that $\dfrac{d}{dt}\,[f \circ z]\,(t) = \omega(\dot{z}(t), z'')$ and that $\{f, H(t, \cdot)\}\,(z(t)) = \omega\left(X_{H(t,\cdot)}, z''\right)$. In conclusion the relation (15) for this choice of the function f becomes the relation (14) for z''. $\square$

It is visible that the functions $f \in Der(X, Y)$ play the role of test functions in (15). Let us consider curves $f : [0, T] \to Der(X, Y)$, which are

smooth in the sense that for any $t \in [0, T]$ there exists $\dfrac{\partial}{\partial t} f(t, z)$. We suppose that the Hamiltonian $H : [0, T] \to Der(X, Y)$ is such a curve. For an arbitrary $t \in [0, T]$, at each $\tau \in [0, t]$ we put $f(\tau, \cdot)$ in the relation (15) and then integrate with respect to $\tau \in [0, t]$. We obtain the following relation:

$$\int_0^t \mathcal{R}(z(\tau), \dot{z}(\tau) - X_{f(\tau, \cdot)}(z(\tau))) \, \mathrm{d}\tau \geq \int_0^t \mathcal{R}(z(\tau), \dot{z}(\tau)) \, \mathrm{d}\tau +$$

$$+ f(t, z(t)) - f(0, z(0)) - \qquad (17)$$

$$- \int_0^t \left[\frac{\partial}{\partial t} f(\tau, z(\tau)) + \{f(\tau, \cdot), H(\tau, \cdot)\} (z(\tau)) \right] \, \mathrm{d}\tau \quad .$$

The relation (17) makes sense if z is differentiable almost everywhere and

$$\int_0^T \mathcal{R}(z(\tau), \dot{z}(\tau)) \, \mathrm{d}\tau < +\infty \qquad (18)$$

$$\int_0^t \left[\frac{\partial}{\partial t} f(\tau, z(\tau)) + \{f(\tau, \cdot), H(\tau, \cdot)\} (z(\tau)) \right] \, \mathrm{d}\tau < +\infty \quad . \qquad (19)$$

This is leading us to the following definition of weak solution.

Definition 4. *Let $\mathcal{A}$ be a given vector space of smooth curves $f : [0, T] \to Der(X, Y)$ such that the Hamiltonian $H : [0, T] \to Der(X, Y)$ belongs to $\mathcal{A}$. Then let $\mathcal{S}(D, \mathcal{A})$ be the space of all curves $z : [0, T] \to X \times Y$ which are almost everywhere differentiable, such that $Diss(z, [0, T]) < +\infty$ and such that (19) is true for any $f \in \mathcal{A}$.*

A curve $z \in \mathcal{S}(D, \mathcal{A})$ is a weak solution of the evolution problem if for almost any $t \in [0, T]$ the inclusion (14) is true.

Let $z \in \mathcal{S}(D, \mathcal{A})$ be a weak solution. The dissipation along this solution is by definition the function:

$$\eta(t) = \int_0^t \omega\left(\dot{z}(\tau), X_{H(\tau, \cdot)}(z(\tau))\right) \, \mathrm{d}\tau \qquad . \qquad (20)$$

Proposition 2. *Let $z \in \mathcal{S}(D, \mathcal{A})$ be a weak solution and η the associated dissipation. Then for any $t \in [0, T]$ we have*

$$\eta(t) \geq \int_0^t \mathcal{R}(z(\tau), \dot{z}(\tau)) \, \mathrm{d}\tau \quad .$$

Proof. We shall use the inclusion (14), which means that for any $z' \in X \times Y$ and for almost any $t \in [0, T]$ we have

$$\mathcal{R}(z(t), \dot{z}(t) + z') \geq \mathcal{R}(z(t), \dot{z}(t)) + \omega(\dot{z}(t) - X_{H(t, \cdot)}(z(t)), z') \quad .$$

If we take for almost any $\tau \in [0, t]$ $z' = -\dot{z}(\tau)$ and use $\mathcal{R}(z, 0) = 0$ then we get

$$\omega\left(\dot{z}(\tau), X_{H(\tau, \cdot)}(z(\tau))\right) \geq \mathcal{R}(z(\tau), \dot{z}(\tau)) \geq 0 \quad .$$

The desired relation is obtained by integration. $\square$

2.3 The 1-homogeneous case

Suppose that X is a Banach space and $Y = X^*$. Then $X \times Y$ is a Banach space and the natural norm on $X \times Y$ induces a distance $d(z', z") = \|z' - z"\|$.

Suppose moreover that for any $z \in X \times Y$ the dissipation function $\mathcal{R}$ has the property that $\mathcal{R}(z, \cdot)$ is positively one-homogeneous. Then the dissipation function can be seen as a dissipation metric in the sense that it induces:

(a) a "dissipation length" defined for any curve $z : [0, T] \to X \times Y$ which is almost everywhere differentiable by:

$$L(z) = \int_0^T \mathcal{R}(z(t), \dot{z}(t)) \, \mathrm{d}t \quad .$$

The space of curves with finite dissipation length is denoted with $W_\mathcal{R}^{1,1}(X \times Y)$.

(b) a "dissipation distance" $D : (X \times Y)^2 \to \mathbb{R} \cup \{+\infty\}$, where $D(z', z")$ is defined as the infimum of the dissipation lengths of all curves joining z' and $z"$.

(c) a "dissipation variation" defined for any curve $z : [0, T] \to X \times Y$ as:

$$Diss(z, [0, T]) = \sup \left\{ \sum_1^N D(z(s_{j-1}), z(s_j)) \mid \text{all partitions of} [0, t] \right\} .$$

$BV_\mathcal{R}(X \times Y)$ denotes the space of curves with bounded dissipation variation.

The dissipation distance D is not really a distance, because it is not symmetric and it may take the value $+\infty$. It satisfies nevertheless the triangle inequality.

The dissipation length and dissipation variation are defined in principle for different classes of curves, but in particular cases they are the same. All in all this is a generalization of well-known facts in the analysis in metric spaces, see for the relevant results Gromov chapter 3 [19], or Ambrosio, Gigli, Savaré chapter 1[3], which has been developed by Mielke and collaborators in the theory of rate-independent evolution systems (see section 3 for further details and references). Enough is to mention that if z is a curve which is differentiable almost everywhere and of finite dissipation length then its dissipation length equals the dissipation variation.

In particular then any weak solution satisfies (17) with the term

$$\int_0^t \mathcal{R}(z(\tau), \dot{z}(\tau))\ \mathrm{d}\tau$$

replaced by $Diss(z, [0, t])$. If the class $\mathcal{A}$ is sufficiently rich then satisfaction of (17) will imply that z is a weak solution.

Theorem 1. *If $\mathcal{R}(z', z") = \mathcal{R}(z', x")$ for any $z', z" \in X \times Y$ then for any weak solution $z : [0, T] \to X \times Y$ in accord with definition 20 and for any $t \in [0, T]$ we have:*

$$H(0, z(0)) + \int_0^t \frac{\partial}{\partial t} H(\tau, z(\tau))\ \mathrm{d}\tau = H(t, z(t)) + Diss(z, [0, t]) \qquad (21)$$

Proof. In relation (17) let us take $f = \lambda H$ for an arbitrary $\lambda \in (-\infty, 1)$:

$$\int_0^t \mathcal{R}(z(\tau), \dot{z}(\tau) - \lambda X_{H(\tau, \cdot)}(z(\tau)))\ \mathrm{d}\tau \geq Diss(z, [0, t]) +$$

$$+ \lambda H(t, z(t)) - \lambda H(0, z(0)) - \lambda \int_0^t \left[\frac{\partial}{\partial t} H(\tau, z(\tau)) \right]\ \mathrm{d}\tau \quad . \qquad (22)$$

In the hypothesis of the theorem if z is a weak solution then it satisfies the following: for almost any $t \in [0, T]$ and for any $z" = (x", y") \in X \times Y$

$$\mathcal{R}(z(t), \dot{x}(t) + x") \geq \mathcal{R}(z(t), \dot{x}(t)) + \langle \dot{x}(t) - D_y H(t, \cdot)(x(t), y(t)), y" \rangle -$$

$$- \langle x", \dot{y}(t) + D_x H)t, \cdot)(x(t), y(t)) \rangle$$

It follows that for almost any $t \in [0, T]$ we have $\dot{x}(t) = D_y H(t, \cdot)(x(t), y(t))$, therefore for almost any $\tau \in [0, t]$ we have:

$$\mathcal{R}(z(\tau), \dot{z}(\tau) - \lambda X_{H(\tau, \cdot)}(z(\tau))) = \mathcal{R}(z(\tau), \dot{x}(\tau) - \lambda D_y H(\tau, \cdot)(x(\tau), y(\tau))) =$$

$$= \mathcal{R}\left(z(\tau), (1 - \lambda)(\dot{x}(\tau))\right) = (1 - \lambda)\mathcal{R}(z(\tau), \dot{x}(\tau))$$

We return to (22) and we use the information that we gained, as well as the equality between dissipation variation and dissipation distance. If we denote

$$\mathcal{A} = Diss(z, [0, t]) + H(t, z(t)) - H(0, z(0)) - \int_0^t \left[\frac{\partial}{\partial t} H(\tau, z(\tau)) \right] \, \mathrm{d}\tau$$

then for any $\lambda \in (-\infty, 1)$ we have $0 \geq \lambda \mathcal{A}$. The arbitrary λ can have any sign, therefore we deduce the desired equality (21) from the previous inequality. $\square$

This theorem shows a great advantage of Hamiltonian formulations upon Lagrangian formulations: a weak Hamiltonian formulation naturally conserves quantities of interest, like the energy. In Lagrangian formulations this can be achieved in two ways: by a convexity hypothesis on the energy function and a process of integration by parts coupled with stronger regularity of solutions or, like in the case of energetic formulations where the solution may have jumps, this has to be imposed by hand. The case of 1-homogeneous dissipation functions is very particular because is introducing more structure in the general symplectic formulation, namely the dissipation distance. Accordingly, the difference between weak and energetic solutions in the Lagrangian formulations is technically related to the use of the space $BV_\mathcal{R}(X \times Y)$ rather than $W_\mathcal{R}^{1,1}(X \times Y)$. In definition 20 and in theorem 1 we have used $W_\mathcal{R}^{1,1}(X \times Y)$. The result of theorem 1 can be interpreted as: weak solutions of the Hamiltonian formulation are energetic solutions in the Lagrangian formulation. We believe that this result is still true if we modify the definition 20, in the particular case of 1-homogeneous dissipation, in order that weak solutions belong to $BV_\mathcal{R}(X \times Y)$. This is not done however in this paper and it is left for further research.

3   Connection with Mielke's theory of quasistatic evolutionary processes

Consider a physical system with the state space $\mathcal{Q}$. This space may have a manifold structure, or it may be a space of functions $q : \Omega \to \mathcal{M}$, with given regularity, where $\mathcal{M}$ is a manifold. In this case the bounded Lipschitz domain Ω represents the reference configuration of a continuous body. We shall denote a generic point of $\mathcal{Q}$ by the letter q and $\dot{q}$ denotes a vector in the tangent space to $\mathcal{Q}$ at $q \in Q$.

For the first time in the proceedings paper Mielke, Theil [22], then in Mielke, Theil and Levitas [23], the notion of an energetic solution of a quasistatic evolutionary process was introduced, based on a energy function

$$\mathcal{E} : [0, T] \times \mathcal{Q} \to \mathbb{R} \cup \{+\infty\} \quad , \quad \mathcal{E} = \mathcal{E}(t, q)$$

and a "dissipation metric"

$$\mathcal{R} : T\mathcal{Q} \to [0, +\infty] \quad , \quad \mathcal{R} = \mathcal{R}(q, \dot{q})$$

Here $T\mathcal{Q} = \{(q, \dot{q}) \mid \dot{q} \in T_q \mathcal{Q}\}$ is the tangent space space to $\mathcal{Q}$ at $q \in Q$, in a generalized sense.

The dissipation metric is convex and lower semicontinuous with respect to the second variable. For the case of rate-independent processes the dissipation metric is 1-homogeneous (i.e. it can really be interpreted as a metric). The **force balance** equation is:

$$0 \in \partial_{\dot{q}} \mathcal{R}(q, \dot{q}) + D_q \mathcal{E}(t, q) \tag{23}$$

To the dissipation metric $\mathcal{R}$ is associated a non symmetric dissipation distance $D : \mathcal{Q} \times \mathcal{Q} \to [0, +\infty]$, defined by:

$$D(q_1, q_2) = \inf \int_0^1 \mathcal{R}(q(s), \dot{q}(s)) \mathrm{d}s$$

over all $q \in W^{1,1}([0, 1], \mathcal{Q})$, with $q(0) = q_1$ and $q(1) = q_2$.

Definition 5. *An evolution $q : [0, T] \to \mathcal{Q}$ is an energetic solution associated with $\mathcal{E}$ and D if*

   (a) *the function $t \in [0, T] \mapsto \partial_t \mathcal{E}(t, q(t))$ belongs to $L^1((0, T))$, and for every $t \in [0, T]$ we have $\mathcal{E}(t, q(t)) < +\infty$,*

(b) the stability condition holds: for any $\hat{q} \in \mathcal{Q}$

$$\mathcal{E}(t, q(t)) \leq \mathcal{E}(t, \hat{q}) + D(q(t), \hat{q})$$

(c) the energy balance holds:

$$\mathcal{E}(t, q(t)) + Diss(q, [0, t]) = \mathcal{E}(0, q(0)) + \int_0^T \partial_t \mathcal{E}(s, q(s)) \, ds$$

where

$$Diss(q, [0, t]) = \sup \left\{ \sum_1^N D(q(s_{j-1}), q(s_j)) \mid \text{all partitions of } [0, t] \right\}$$

We can recover the force balance equation (23) from the generalized Hamiltonian formalism with dissipation proposed in section 2. Indeed, suppose that the state space of the physical system is $\mathcal{Q} = B$, a reflexive Banach space. Consider the phase space $X \times Y = B \times B^*$. A generic element of $z \in \mathcal{B}$ has the form $z = (q, p)$ with $q \in B$, $p \in B^*$.

We shall take Hamiltonian and dissipation functions almost as in (10), (11). The Hamiltonian function H has the form $H(t, q, p) = \mathcal{K}(p) + \mathcal{E}(t, q)$ where $\mathcal{K}$ is a smooth function (kinetic energy) and $\mathcal{E}$ is the energy function of Mielke (which can be characterized as a Gibbs-type stored energy). We take a dissipation function $\mathcal{R}(q, p, \dot{q}, \dot{p}) = \mathcal{R}(q, \dot{q})$ with $\mathcal{R}$ the dissipation metric.

With these choices of functions H and D the equation (14) takes the form:

$$\begin{cases} -\dot{p} & \in & D_q \mathcal{E}(t, q) + \partial_{\dot{q}} \mathcal{R}(q, \dot{q}) \\ \dot{q} & = & D_p \mathcal{K}(p) \end{cases} . \tag{24}$$

The quasistatic version of (24) is just the force balance equation of Mielke (23). We are also in the hypothesis of theorem 1. If we neglect the inertial terms in (21) we obtain the energy balance condition (c) from the definition of energetic solution 5.

Let us see what is the expression of the dissipation along a solution of (24), as defined by (20). We have

$$\dot{\eta}(t) = -\langle D_p \mathcal{K}(p(t)), \dot{p}(t) \rangle - \langle \dot{q}(t), D_q \mathcal{E}(t, q(t)) \rangle$$

As in the proof of proposition 2, we arrive to the inequality

$$0 \geq \mathcal{R}(q(t), \dot{q}(t)) + \langle D_p \mathcal{K}(p(t)), \dot{p}(t) \rangle + \langle \dot{q}(t), D_q \mathcal{E}(t, q(t)) \rangle$$

therefore we get $\mathcal{R}(q(t), \dot{q}(t)) \leq \dot{\eta}(t)$. We integrate this inequality and we obtain:

$$\eta(t) \geq \int_0^t \mathcal{R}(q(s), \dot{q}(s)) \mathrm{d}s$$

We finally obtain that $\eta(t) \geq D(q(0), q(t)) \geq 0$, which means that the dissipation η along a solution of (24) is always greater or equal to the dissipation distance (in fact greater than the dissipation length).

4   Application: a dynamical model of brittle damage using the Ambrosio-Tortorelli functional

Mielke and Roubíček [21] proposed a rate-independent brittle damage model based on the theory of rate-independent evolutionary processes [20]. The model of Mielke and Roubíček is a quasistatic particular case of the more general dynamical model of Stumpf and Hackl [27].

By using the generalized Hamiltonian formalism we are able to obtain a dynamical model of brittle damage, which is also a particular case of the general dynamical model of Stumpf and Hackl.

The model is based on an energy of Ambrosio-Tortorelli type and a dissipation function as in the model of Mielke and Roubíček.

4.1   The Ambrosio-Tortorelli functional

Let us consider $n \in \mathbb{N}^*$ and $\Omega \subset \mathbb{R}^n$ a bounded, open set, with piecewise smooth boundary. The Mumford-Shah functional [24] is

$$E(u, S) \;=\; \int_\Omega \frac{1}{2} K \mid \nabla u \mid^2 \; + \gamma \, \mathcal{H}^{n-1}(S) \tag{25}$$

defined over all pairs (u, S) such that $u \in \mathcal{C}^1(\Omega \setminus S, \mathbb{R})$. The set S is a $n-1$-dimensional surface in $\mathbb{R}^n$, or a countable union of such surfaces. In the case $n = 2$ this functional can be seen as the energy of a brittle body suffering an antiplane displacement u and presenting a crack S.

For $n = 3$ the state of a brittle body is described by a pair displacement-crack. $(\mathbf{u}, S)$ is such a pair if S is a crack (a 2D surface) which appears in the body and $\mathbf{u} \in \mathcal{C}^1(\Omega \setminus S, \mathbb{R}^3)$ is a displacement of the broken body, that is $\mathbf{u}$ is smooth in the exterior of the surface S, but it may have jumps over

S. The total energy of a brittle body is a Mumford-Shah functional of the form:

$$E(\mathbf{u}, S) = \int_\Omega w(\nabla \mathbf{u}) \, \mathrm{d}x + G \mathcal{H}^2(S) \qquad . \qquad (26)$$

The first term of the functional E represents the elastic energy of the body with the displacement $\mathbf{u}$. The second term represents the energy consumed to produce the crack S in the body. Here his energy is taken to be proportional with the area of the crack S (technically this is the 2 dimensional Hausdorff measure of S), with the proportionality factor G, which is the Griffith constant.

Starting with the foundational papers of Mumford, Shah [24], De Giorgi, Ambrosio [14], Ambrosio [1], [2], the development of models of quasistatic brittle fracture based on Mumford-Shah functionals continues with Francfort, Marigo [16], [17], Mielke [20], Dal Maso, Francfort, Toader, [13], Buliga [8], [9], [10].

All these models are based on a technique of time discretization followed by a sequence of incremental minimization problems. These models are either seen as applications of De Giorgi method of energy minimizing movements, or in the frame of the theory of Mielke of rate-independent evolutionary processes [20].

The functional

$$E_c(u, d) = \int_\Omega \left\{ \phi(d) \frac{1}{2} K \mid \nabla u \mid^2 + \frac{1}{2} \gamma c \mid \nabla d \mid^2 + \frac{\gamma}{2c} d^2 \right\} \qquad (27)$$

was introduced by Ambrosio and Tortorelli [4], as a variational approximation of the Mumford-Shah functional (25). Here d is a field which approximates the characteristic function of a crack, that is $d : \Omega \to [0, 1]$ and the set

$$S_c = \left\{ x \in \bar\Omega : 1 \geq d_c(x) \geq 1 - \mathcal{O}(c) \right\}$$

approximates the crack. More precisely, if (u_c, d_c) is a minimizer of the Ambrosio-Tortorelli functional (27) then as $c \to 0$ the displacement u_c converges (in some norm) to a displacement u, the set S_c shrinks to a surface S and (u, S) is a minimizer of the Mumford-Shah functional (25).

The variable d plays the role of a brittle damage variable, because it takes values in $[0, 1]$ and also because it is coupled with the antiplane displacement u through the term

$$\int_\Omega \left\{ \phi(d) \frac{1}{2} K \mid \nabla u \mid^2 \right\}$$

which represents the elastic energy of the body with elasticity coefficient $\phi(d)K$. The function ϕ is taken as a decreasing function from $[0, 1]$ to $[0, 1]$, such that $\phi(1) = 0$, $\phi(0) = 1$.

Focardi [15] proved that there is an Ambrosio-Tortorelli functional suitable for approximating the 3D Mumford-Shah functional (26), namely:

$$E_c(u, d) \; = \; \int_\Omega \left\{ \phi(d) \, w(\nabla \mathbf{u}) \, + \, \frac{1}{2} \gamma \, c \mid \nabla d \mid^2 \, + \, \frac{\gamma}{2c} \, d^2 \right\} \qquad (28)$$

under certain growth conditions on the elastic energy function w.

4.2 Quasistatic model, using Mielke's theory

In this subsection we obtain an interpretation of a mathematical result of Giacomini [18], which shows that models of damage based on the Ambrosio-Tortorelli functional have the important property of being compatible with brittle damage from the energetic point of view. This is a desirable feature of a model of brittle damage, as there are many "classical" models of brittle damage which allow the creation of a brittle crack (seen a concentrated total damaged region) with zero consumed energy.

We shall look at the equations coming from the force balance equation of Mielke (23) and the Ambrosio-Tortorelli functional taken as the potential energy. The state of the system is described by a pair $(\mathbf{u}, d)$, where $\mathbf{u}$ is the displacement and d a scalar damage variable taking values in $[0, 1]$.

We shall take a dissipation metric which is almost the same as in Mielke and Roubíček model [21], relation (2.5) (see also the discussion at the end of the section 2.2 from [21]), which gives the dissipation functional

$$\mathcal{R}(\mathbf{u}, d, \dot{\mathbf{u}}, \dot{d}) \; = \; \int_\Omega \left\{ \beta \dot{d} \, + \, \chi_1(d) \, + \, \chi_2(\dot{d}) \right\}$$

The functions χ_1, χ_2 are indicator functions of convex sets:

$$\chi_1(d) = \left\{ \begin{array}{ccc} 0 & , & \text{if } d \in [0, 1] \\ +\infty & , & \text{else} \end{array} \right. , \chi_1(\dot{d}) = \left\{ \begin{array}{ccc} 0 & , & \text{if } \dot{d} \in [0, +\infty) \\ +\infty & , & \text{else} \end{array} \right.$$

Alternatively, we may take χ_1 as the indicator function of the set $(-\infty, 0]$. Formally integrating by parts the force balance equation of Mielke (23), we arrive to the evolution equations:

$$\left\{ \begin{array}{ccl} 0 & = & \frac{\partial}{\partial x_i} \left(\phi(d) \, K \, \frac{\partial u}{\partial x_i} \right) \\ 0 & \in & \beta - \gamma \, c \, \Delta d + \phi'(d) \frac{1}{2} K \mid \nabla u \mid^2 + \frac{\gamma}{2c} \, d + \partial \chi_2(\dot{d}) \end{array} \right. . \qquad (29)$$

In order to have $d \in [0,1]$ at any moment, it is sufficient to impose $d_{|t=0} \in [0,1]$. The term

$$-\phi'(d)\,\frac{1}{2}K\mid\nabla u\mid^2$$

is greater or equal than 0, due to the fact that ϕ is decreasing, thus $\phi' \leq 0$. This term represents the variation of the elastic energy density due to damage.

The paper [18] can be seen as an investigation of the limit to the fracture model of the bulk damage model of Mielke and Roubíček, that is in the limit when the damage variable equals 0 almost everywhere (therefore the value of the parameter β is not important in the sense that $a > 0$ makes the same effect as $\beta = 0$). This result can be described as follows: for any parameter c let $q_c = (\mathbf{u}_c, d_c)$ denote an energetic solution associated with the Ambrosio-Tortorelli energy E_c and dissipation distance D coming from the dissipation metric $\mathcal{R}$. Then as c converges to 0, the evolution q_c converges to an evolution $(\mathbf{u}, S)$ of the energetic formulation of brittle fracture of Francfort, Marigo [16] or Buliga [9].

From the point of view of mechanics fracture is a manifestation of concentrated damage. Therefore a good (bulk) damage model should have the property that it is not possible to produce arbitrarily concentrated damage with arbitrarily small expense of energy. Such models are said to be compatible with brittle fracture from the viewpoint of energy balance. There are many models of brittle damage in use, not all of them compatible with brittle damage. The mathematical result of Giacomini means that the Ambrosio-Tortorelli functional leads to brittle damage models which are compatible with brittle fracture from the point of view of energy balance.

4.3   Hamiltonian brittle damage

We shall apply the generalized Hamiltonian approach to a functional of the Ambrosio-Tortorelli type.

We take as state $q = (\mathbf{u}, d)$ the pair formed by the displacement $\mathbf{u}$ and the scalar damage variable $d \in [0,1]$. The space of this pairs corresponds to the space X from the general model.

The dual variable, in the sense of Hamiltonian mechanics, is $p = (\mathbf{p}, y) \in Y$, where $\mathbf{p}$ is the momentum and y is a scalar variable dual to d (which will turn out to be linearly dependent on $\dot{d}$).

The space of all pairs (q, p) is a product of two symplectic vector spaces $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$. The space of non-dissipative variables $B_1 \times B_1^*$ is a space of pairs of (weak) functions $(\mathbf{u}, \mathbf{p})$ defined over $\Omega \subset \mathbb{R}^3$. Therefore $\mathbf{u} \in B_1$ and $\mathbf{p} \in B_1^*$, where B_1 is a Banach space (for example a suitably chosen Sobolev space of functions over Ω) and B_1^* is its dual. The duality product is

$$\langle \mathbf{p}, \mathbf{u} \rangle_1 = \int_\Omega \mathbf{p} \cdot \mathbf{u}$$

Similarly, the space of dissipative variables (d, y) is $\mathcal{B}_2 = B_2 \times B_2^*$, a space of pairs of (weak) functions (d, y) defined over $\Omega \subset \mathbb{R}^3$, with B_2 another Banach space of functions over Ω, B_2^* is its dual. The duality product is

$$\langle y, d \rangle_2 = \int_\Omega y \, d$$

Let us define the Hamiltonian as:

$$H(t, \mathbf{u}, \mathbf{p}, d, y) = \mathcal{E}(\mathbf{u}, d) + T(\mathbf{p}, y) - \langle l(t), \mathbf{u} \rangle \tag{30}$$

where $\mathcal{E}$ is the stored energy, T the kinetic energy and $l(t)$ the external forces, seen as:

$$\langle l(t), \mathbf{u} \rangle = \int_\Omega \mathbf{f}(t) \cdot \mathbf{u} + \int_\Gamma \bar{\mathbf{f}}(t) \cdot \mathbf{u}$$

Here $\Gamma \subset \partial\Omega$ is that part of the boundary where surface forces $\bar{\mathbf{f}}(t)$ are imposed at the moment t. Displacements may be imposed on another part Γ' of the boundary $\partial\Omega$. This is done by imposing that at every moment $t \in [0, T]$ the displacement $\mathbf{u}(t)$ belongs to a subspace $B_1(t) \subset B_1$ of **kinematically admissible** displacements.

The stored energy $\mathcal{E}$ is therefore:

$$\mathcal{E}(t, \mathbf{u}, d) = \Psi(\mathbf{u}, d) - \langle l(t), \mathbf{u} \rangle \quad .$$

The expression of the free energy Ψ is the following:

$$\Psi(\mathbf{u}, d) = \int_\Omega \left[\phi(d) w(\nabla \mathbf{u}) + \frac{1}{2} K \|\nabla d\|^2 + \frac{1}{2} L \mid d \mid^2 \right] \tag{31}$$

Here ϕ is a smooth, decreasing function with values in the interval $[0, 1]$. The term $\frac{1}{2} L \mid d \mid^2$ is not present in Stumpf and Hackl [27] formula (3.34) for the

free energy, but it is motivated by the expression of the Ambrosio-Tortorelli type functional (28). From there we may deduce a physical meaning of the constant L, as γ/c, where c is a characteristic length and γ is a Griffith-type constant.

The kinetic energy has the form:

$$T(\mathbf{p}, y) = \int_{\Omega} \left[\frac{1}{2} b \mid y \mid^2 + \frac{1}{2\rho} \|\mathbf{p}\|^2 \right] \qquad . \tag{32}$$

The second term in the expression of the kinetic energy is just the usual kinetic energy expresses as a function of momentum $\mathbf{p}$, as it is usual in the Hamiltonian formalism. Similarly, y is a momentum variable corresponding to d and b is the scalar version of a microinertia tensor (we use the same name as Stumpf and Hackl [27] concerning the kinetic energy described in their formula (2.4)). We suppose that the constants K, L and b are positive.

The dissipation function is the same as in the previous section:

$$\mathcal{R}(d, y, \dot{d}, \dot{y}) = \int_{\Omega} \left[\chi_{[0,1]}(d) + \chi_{[0,+\infty)}(\dot{d}) + \beta \mid \dot{d} \mid \right] \qquad . \tag{33}$$

We shall find the equations satisfied by any curve of evolution $(\mathbf{u}, \mathbf{p}, d, y) : [0, T] \to \mathcal{B}$ which is a solution of the generalized Hamiltonian equations (14), for the Hamiltonian (30) and dissipation (33). By using the expressions of the free energy (31) and kinetic energy (32), we obtain:

$$\langle \dot{\mathbf{p}}, \hat{\mathbf{u}} \rangle_1 + \langle D_{\mathbf{u}} \Psi(\mathbf{u}, d), \hat{\mathbf{u}} \rangle_1 = \langle l(t), \hat{\mathbf{u}} \rangle_1 \qquad \forall \hat{\mathbf{u}} \in B_1(t) \quad , \tag{34}$$

$$\langle \hat{\mathbf{p}}, \dot{\mathbf{u}} \rangle_1 - \langle \hat{\mathbf{p}}, D_{\mathbf{p}} K(\mathbf{p}, y) \rangle_1 = 0 \qquad \forall \hat{\mathbf{p}} \in B_1^* \quad . \tag{35}$$

There are two more equations, for the evolution of d and y. Due to the non smooth dissipation, these are in fact expressed as subdifferential inequalities: for almost any $t \in [0, T]$ $d(t, x) \in [0, 1]$ and $y(t, x) \in [0, +\infty)$ for almost every $x \in \Omega$, at any $t \in [0, T]$ the displacement $\mathbf{u}(t)$ is kinematically admissible, i.e. $\mathbf{u}(t) \in B_1(t)$, and moreover for any $\hat{d} \in B_2$, such that $\hat{d}(x) + \dot{d} \geq 0$ for almost every $x \in \Omega$, and for any $\hat{y} \in B_2^*$ we have:

$$\beta \int_{\Omega} \left[\mid \dot{d} + \hat{d} \mid - \mid \dot{d} \mid \right] \geq \tag{36}$$

$$\geq \int_{\Omega} \left[\hat{y}(\dot{d} - by) - (Ld + \phi'(d)w(\nabla \mathbf{u}) + \dot{y})\hat{d} - K \nabla d \, \nabla \hat{d} \right] \qquad .$$

The equation (34) gives the usual momentum balance: for any $\hat{\mathbf{u}}$ kinematically admissible we have

$$\int_\Omega [-\dot{p} \cdot \hat{\mathbf{u}} - \phi(d)\,Dw(\nabla\mathbf{u}):\nabla\hat{\mathbf{u}}] \;=\; \int_\Omega \mathbf{f}(t) \cdot \hat{\mathbf{u}} \;+\; \int_\Gamma \bar{\mathbf{f}}(t) \cdot \hat{\mathbf{u}}$$

Denote by $\mathbf{S} = Dw(\nabla\mathbf{u})$ the stress variable given by the elastic energy w. Integration by parts leads us to a balance equation and boundary conditions:

$$div\,(\phi(d)\,\mathbf{S}) + \mathbf{f}(t) = \dot{\mathbf{p}} \quad \text{in } \Omega \tag{37}$$

$$\phi(d)\,\mathbf{Sn} = \bar{\mathbf{f}}(t) \text{ on } \Gamma\,,\; \phi(d)\,\mathbf{Sn} = 0 \text{ on } \partial\Omega \setminus (\Gamma \cup \Gamma')\,,\; \mathbf{u} = \mathbf{u}_0(t) \text{ on } \Gamma' \tag{38}$$

Equation (35) gives us the momentum $\mathbf{p}$ as function of $\dot{\mathbf{u}}$:

$$\mathbf{p} = \rho\,\dot{\mathbf{u}} \tag{39}$$

Equation (36) is equivalent to the following two relations:

$$\dot{d} = by \tag{40}$$

and for all $\hat{d} \in B_2$, such that $\hat{d}(x) + \dot{d} \geq 0$ for almost every $x \in \Omega$

$$\beta \int_\Omega \left[|\,\dot{d} + \hat{d}\,| - |\,\dot{d}\,| \right] \geq \tag{41}$$

$$\geq -\int_\Omega \left[(Ld + \phi'(d)w(\nabla\mathbf{u}) + \dot{y})\hat{d} + K\nabla d\,\nabla\hat{d} \right] $$

Let $S : [0,+\infty) \to 2^{\mathbb{R}}$ be the multivalued function defined by:

$$S(v) = \begin{cases} \beta & , \quad v > 0 \\ (-\infty, \beta] & , \quad v = 0 \end{cases} $$

The function S is the subdifferential of a convex function. By using the definition of S and relation (40) we obtain the following equivalent form of the inequality (41): for almost every $x \in \Omega$ we have:

$$-\left(\dot{y} + Ld + \phi'(d)w(\nabla\mathbf{u}) - K\Delta d\right) \in S(y) \tag{42}$$

We may add the boundary condition (which is not strictly speaking a consequence of the formalism): on $\partial\Omega$ we have:

$$y \geq 0\,, \quad -K\frac{d}{d\mathbf{n}}d \in S(y) \tag{43}$$

In the particular case of a functional of the Ambrosio-Tortorelli type (28) we may take:

$$K = \gamma\, c \quad , \quad L = \frac{\gamma}{c} \quad , \quad b = \gamma\, c$$

The function ϕ which enters in the expression of the free energy is chosen as in the Ambrosio-Tortorelli functional. With this choice of constants we obtain from (42) and (40) the differential inclusion:

$$- \left(\ddot{d} + \gamma^2 d + \gamma c\, \phi'(d) w(\nabla \mathbf{u}) - \gamma^2 c^2 \Delta d \right) \in \gamma c\, S(\dot{d}) \qquad .$$

This inclusion suggests that in this model there is a maximal speed of propagation of damage of order γ.

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