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THE EXISTENCE OF A GLOBAL ATTRACTOR FOR A CLASS OF RATIONAL MAPS*

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Abstract

In this paper we prove the global asymptotic stability of a class of rational iterative processes. Our approach combines the presence of a group of symmetries with certain a priori estimates.

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keywords: Rational map, global attractor, global asymptotic stability

1 Introduction

During recent years a great deal of research has been done in an attempt to understand the dynamics of rational maps of several real variables. Valuable information can be found in the monographs of G. Ladas and his coworkers, devoted to this subject. See [2], [6], [7]. However, the theory of higher order rational difference equations is still in its infancy and new examples could be helpful for future progress.

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The aim of this paper is to discuss the asymptotic behavior of the following nonlinear iterative process

$$x_i = \frac{(1+w)x_{i-7}x_{i-6}x_{i-5}x_{i-4} + \sum_{i-3\leq p<q\leq i-1} x_p x_q}{\sum_{i-7\leq p<q<r\leq i-4} x_p x_q x_r + w x_{i-3}x_{i-2}x_{i-1}}, \quad i \geq 8, \qquad (1)$$

where the initial data $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ and the parameter w are all positive. This process describes the dynamics of the map $T_w : \mathcal{M} \to \mathcal{M}$ that acts on

$$\mathcal{M} = \underbrace{(0,\infty) \times \cdots \times (0,\infty)}_{7 \text{ times}}$$

by the formula

$$\begin{aligned} &T_w((x_1, x_2, x_3, x_4, x_5, x_6, x_7)) \\ &= (x_2, x_3, x_4, x_5, x_6, x_7, \frac{(1+w)x_1x_2x_3x_4 + x_5x_6 + x_5x_7 + x_6x_7}{x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 + wx_5x_6x_7}). \end{aligned}$$

It is easy to verify that this map admits a unique equilibrium point, $C = (1,1,1,1,1,1,1)$. In order to discuss the nature of this point we need to precise a topology on $\mathcal{M}$. The natural metric on $\mathcal{M}$ is the metric associated to the supremum norm,

$$d(X,Y) = \max\{|x_i - y_i| : i = 1, \ldots, 7\},$$

where x_i and y_i are the components respectively of X and Y. However this metric is not suitable to our purposes and we shall use instead the so called *multiplicative metric,*

$$\begin{aligned} d_\star(X,Y) &= -\log \min\left\{\frac{x_i}{y_i}, \frac{y_i}{x_i} : i = 1, \ldots, 7\right\} \\ &= \max\{|\log x_i - \log y_i| : i = 1, \ldots, 7\}. \end{aligned}$$

These two metrics are equivalent (since the convergent sequences relative to each of them are the same). Besides, the multiplicative metric is invariant under the action of any of the maps

$$i_k : \mathcal{M} \to \mathcal{M}, \; i_k((x_1, \ldots, x_k, \ldots, x_7)) = (x_1, \ldots, 1/x_k, \ldots, x_7),$$

where $k \in \{1, ..., 7\}$.

The usefulness of $d_\star$ rests on the following nice behavior of the iterates of T_w :

Lemma 1. *For $w \in [2,3]$, the map $T_w : \mathcal{M} \to \mathcal{M}$ verifies the condition*

$$d_\star(T_w^8(X), C) < d_\star(X, C) \quad \text{for all } X \neq C.$$

This lemma (which reminds of the Krasovskii-LaSalle invariance principle [12]) shows that the general criterion of global asymptotic stability due to N. Kruse and T. Nesermann [5] applies to T_w. As a consequence, the following result holds true:

Theorem 1. *For $w \in [2,3]$, the point C is globally asymptotically stable. In particular, C is the global attractor of the dynamical system associated to T_w.*

The dynamics of T_w outside this range of parameter w is briefly sketched in Section 3.

The proof of Lemma 1 makes the objective of the next section. Its main feature is the use of certain inequalities to estimate how the trajectories of T_w spray out.

In a sense, the result of Theorem 1 is very natural as (1.1) is a higher order analogue of the difference equation

$$x_i = \frac{x_{i-1}x_{i-2} + 1}{x_{i-1} + x_{i-2}}, \quad i \geq 3,\ x_1, x_2 > 0,$$

for which it is known that the positive equilibrium is globally asymptotically stable. See [8].

Our approach (which was inspired by a recent paper by X. Yang, M. Yang and H. Liu [14]) can be easily adapted to study some rational maps of a greater complexity, for example, all maps that can be obtained from T_w by interchanging two variables and/or replacing a variable by its inverse. In the same time we may consider rational maps that correspond to rational iterative processes of higher order such as

$$x_i = \frac{(1+w)x_{i-2N+1}x_{i-2N+2}\cdots x_{i-N} + \sum_{i-N+1 \leq p_1 < \cdots < p_{N-2} \leq i-1} x_{p_1}\cdots x_{p_{N-2}}}{\sum_{i-2N+1 \leq p_1 < \cdots < p_{N-1} \leq i-N} x_{p_1}\cdots x_{p_{N-1}} + w x_{i-N+1}\cdots x_{i-1}},$$

for $i \geq 2N$, where $w > 0$ and $N \in \mathbb{N}$, $N \geq 3$, are parameters. Their global asymptotic stability will occur at least for w in the interval $[N-2, N-1]$.

Last but not least, our approach has some interesting consequences to the problem of interlacing sequences. This is discussed in the final section of this paper.

2 Proof of Lemma 1

The proof of Lemma 1 is based on the following result that provides sharp estimates for the deviation of the iterates x_i from the initial data:

Lemma 2. *Let $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ be positive numbers not all equal to 1 and let $w \in [2, 3]$. Then each of the numbers*

$$x_i = \frac{(1+w)x_{i-7}x_{i-6}x_{i-5}x_{i-4} + \sum_{i-3\le p<q\le i-1} x_p x_q}{\sum_{i-7\le p<q<r\le i-4} x_p x_q x_r + w x_{i-3}x_{i-2}x_{i-1}}, \quad i \ge 8,$$

lies in the open interval of endpoints

$$\min\left\{x_j, \frac{1}{x_j} : 1 \le j \le 7\right\} \text{ and } \max\left\{x_j, \frac{1}{x_j} : 1 \le j \le 7\right\}.$$

By Lemma 2, the components of $T_w^8(X) = (x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15})$ verify the double inequality

$$\min\left\{x_j, \frac{1}{x_j} : 1 \le j \le 7\right\} < x_i < \max\left\{x_j, \frac{1}{x_j} : 1 \le j \le 7\right\}, \quad 9 \le i \le 15,$$

which yields

$$\min\left\{x_j, \frac{1}{x_j} : 9 \le j \le 15\right\} > \min\left\{x_j, \frac{1}{x_j} : 1 \le j \le 7\right\}.$$

Consequently

$$\begin{aligned} d_\star(T_w^8(X), C) &= -\log\min\left\{x_i, \frac{1}{x_i} : 9 \le i \le 15\right\} \\ &< -\log\min\left\{x_i, \frac{1}{x_i} : 1 \le i \le 7\right\} = d_\star(X, C) \end{aligned}$$

and the assertion of Lemma 1 is now clear.

The proof of Lemma 2 needs some preliminaries based on an old result due to A.-L. Cauchy (see [9], p. 204):

Lemma 3. *Let $a_1, ..., a_n, b_1, ..., b_n$ be positive numbers. Then*

$$\min\left\{\frac{a_i}{b_i} : 1 \le i \le n\right\} \le \frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} \le \max\left\{\frac{a_i}{b_i} : 1 \le i \le n\right\}.$$

Moreover, both inequalities are strict except the case where $\frac{a_1}{b_1} = ... = \frac{a_n}{b_n}$.

Proof. Letting $m = \min_i \frac{a_i}{b_i}$ and $M = \max_i \frac{a_i}{b_i}$ we get $m \le \frac{a_i}{b_i} \le M$, whence $mb_i \le a_i \le Mb_i$ for all indices i. Summing up we get the result. □

Corollary 1. *Let $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ be positive numbers and*

$$A = \frac{4a_1a_2a_3a_4 + a_5a_6 + a_5a_7 + a_6a_7}{a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4 + 3a_5a_6a_7}.$$

Then,

$$\min\left\{a_1, a_2, a_3, a_4, \frac{1}{a_5}, \frac{1}{a_6}, \frac{1}{a_7}\right\} \le A \le \max\left\{a_1, a_2, a_3, a_4, \frac{1}{a_5}, \frac{1}{a_6}, \frac{1}{a_7}\right\}.$$

Moreover, both inequalities are strict except the case where

$$a_1 = a_2 = a_3 = a_4 = \frac{1}{a_5} = \frac{1}{a_6} = \frac{1}{a_7}.$$

Proof. Apply Lemma 3 to the ratio representing A. □

Lemma 4. *Let $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ be positive numbers not all 1 and*

$$B = \frac{3a_1a_2a_3a_4 + a_5a_6 + a_5a_7 + a_6a_7}{a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4 + 2a_5a_6a_7}.$$

Then

$$\min\left\{a_i, \frac{1}{a_i} : 1 \le i \le 7\right\} < B < \max\left\{a_i, \frac{1}{a_i} : 1 \le i \le 7\right\}. \tag{2}$$

Proof. We shall prove here the second inequality. The first one can be treated similarly.

The basic remark is the invariance of $B = B(a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ under the permutations of indices within the sets $\{1, 2, 3, 4\}$ and $\{5, 6, 7\}$. Consequently, we may assume without loss of generality that

$$a_1 \le a_2 \le a_3 \le a_4 \text{ and } a_5 \le a_6 \le a_7.$$

If $a_5a_6 < a_3a_4$, then the right-hand side of (2) can be easily obtained via Lemma 3.

If $a_5a_6 \ge a_3a_4$, then

$$a_6 \ge a_3,$$

which yields

$$a_1 \le a_7.$$

Case 1: $a_1 = a_7$.

Then necessarily $a_1 = a_2 = a_3 = a_6 = a_7$ and $a_5 \geq a_4$, so that $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7$. In this case

$$B = \frac{3a_1^4 + 3a_1^2}{6a_1^3} = \frac{1}{2}\left(a_1 + \frac{1}{a_1}\right) < \max\left\{a_i, \frac{1}{a_i} : 1 \leq i \leq 7\right\},$$

except the case where $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = 1$.

Case 2: $a_7 > a_1$.

In this case we have to consider the monotonic auxiliary function

$$f(x) = \frac{3a_1a_2a_3a_4 + a_5a_6 + a_6a_7 + xa_5}{a_1a_3a_4 + a_1a_2a_4 + a_2a_3a_4 + a_1a_2a_3 + a_5a_6a_7 + xa_5a_6} \quad \text{for } x > 0.$$

Its monotonicity is a consequence of its structure:

$$f(x) = \frac{\alpha + \beta x}{\gamma + \delta x} = \frac{\beta}{\delta} + \frac{\alpha\delta - \beta\gamma}{\delta} \cdot \frac{1}{\gamma + \delta x},$$

where $\alpha, \beta, \delta, \gamma$ are positive parameters.

If f is constant, then it equals its limit at infinity, $\frac{1}{a_6}$. If $a_1 < a_6$, then $B = \frac{1}{a_6} < \frac{1}{a_1}$ and a fortiori

$$B < \max\left\{a_i, \frac{1}{a_i} : 1 \leq i \leq 7\right\}. \tag{3}$$

If $a_1 = a_6$, then $a_1 = a_2 = a_3 = a_6 < a_7$ and from the inequality $a_5a_6 \geq a_3a_4$ we infer that $a_5 \geq a_4$. Consequently $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 < a_7$ and from the condition

$$B = \frac{1}{a_6}$$

we infer that $a_1 = 1$. Then

$$B = 1 < a_7 = \max\left\{a_i, \frac{1}{a_i} : 1 \leq i \leq 7\right\}.$$

If f is strictly monotone, then

$$\begin{aligned} B &= f(a_7) < \max\{f(a_1), \lim_{x\to\infty} f(x)\} \\ &= \max\left\{f(a_1), \frac{1}{a_6}\right\}. \end{aligned}$$

By Lemma 3,

$$f(a_1) = \frac{a_1a_2a_3a_4 + a_1a_2a_3a_4 + a_1a_2a_3a_4 + a_5a_6 + a_1a_5 + a_6a_7}{a_1a_2a_3 + a_1a_2a_4 + a_2a_3a_4 + a_1a_5a_6 + a_1a_3a_4 + a_5a_6a_7}$$
$$\leq \max\left\{a_4, a_3, a_1, \frac{1}{a_1}, \frac{a_5}{a_3a_4}, \frac{1}{a_5}\right\}.$$

so the proof is done if $a_3a_4 \geq 1$ *or* $a_4 \geq a_5$.

To end the proof in the Case 2 it suffices to consider the situation where

$$a_2a_3 < 1,\ a_2a_4 < 1,\ a_3a_4 < 1,$$

and

$$a_4 < a_5.$$

Necessarily, $a_1, a_2, a_3 < 1$.

Under these assumptions we will show that

$$f(a_1) < \frac{1}{a_1}, \tag{4}$$

which yields the inequality (3). In fact (4) is equivalent to

$$3a_1^2a_2a_3a_4 + a_1^2a_5 + a_1a_6a_7 \\ - a_1a_2a_4 - a_1a_3a_4 - a_2a_3a_4 - a_1a_2a_3 - a_5a_6a_7 < 0,$$

a fact which follows by adding side by side the following four inequalities:

$$\begin{aligned} a_1a_2a_4\,(a_1a_3 - 1) &< 0 \\ a_1a_3a_4\,(a_1a_2 - 1) &< 0 \\ a_2a_3a_4\left(a_1^2 - 1\right) &< 0 \end{aligned}$$

and

$$a_1^2a_5 - a_1a_2a_3 + (a_1 - a_5)\,a_6a_7 < a_1^2(a_5 - a_1) - (a_5 - a_1)\,a_6a_7 < 0.$$

This ends the proof of Lemma 2. □

We pass now to the *Proof of Lemma* 2.

The basic remark is the monotonicity of the iterate

$$x_8 = x_8(w) = \frac{(1+w)x_1x_2x_3x_4 + x_5x_6 + x_5x_7 + x_6x_7}{x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 + wx_5x_6x_7}$$

as a function of $w \in [2,3]$ (when $x_1, \ldots x_7 > 0$ are kept fixed). According to Lemma 4 and Corollary 1,

$$\min\{x_i, \frac{1}{x_i} : 1 \leq i \leq 7\} \leq \min\{x_8(2), x_8(3)\} \leq x_8$$
$$\leq \max\{x_8(2), x_8(3)\} \leq \max\{x_i, \frac{1}{x_i} : 1 \leq i \leq 7\},$$

a fact that can be extended by mathematical induction to

$$\min\left\{x_i, \frac{1}{x_i} : 1 \leq i \leq 7\right\} \leq x_k \leq \max\left\{x_i, \frac{1}{x_i} : 1 \leq i \leq 7\right\}, \tag{5}$$

for all $k = 8, 9, \ldots$. It remains to prove that both inequalities in (5) are strict for $k \geq 9$, except the case where $x_1 = \cdots = x_7 = 1$.

In fact, by an argument above we know that

$$\min\{x_i, \frac{1}{x_i} : 2 \leq i \leq 8\} \leq \min\{x_9(2), x_9(3)\}$$
$$\leq x_9$$
$$\leq \max\{x_9(2), x_9(3)\} \leq \max\{x_i, \frac{1}{x_i} : 2 \leq i \leq 8\},$$

where

$$x_9 = x_9(w) = \frac{(1+w)x_2x_3x_4x_5 + x_6x_7 + x_6x_8 + x_7x_8}{x_2x_3x_4 + x_2x_3x_5 + x_2x_4x_5 + x_3x_4x_5 + wx_6x_7x_8}.$$

If (for some w) $x_9 = \max\{x_i, \frac{1}{x_i} : 1 \leq i \leq 7\}$, then necessarily

$$\max\{x_9(2), x_9(3)\} = \max\{x_i, \frac{1}{x_i} : 2 \leq i \leq 8\},$$

and the following two possibilities occur:

Case a: $x_9(2) = \max\{x_i, \frac{1}{x_i} : 2 \leq i \leq 8\}$.

By Lemma 4 we get $x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = x_8 = 1$, which yields $x_9 = 1$. So $1 = \max\{1, x_1, \frac{1}{x_1}\}$, whence $x_1 = 1$.

Case b: $x_9(3) = \max\{x_i, \frac{1}{x_i} : 2 \leq i \leq 8\}$.

By Corollary 1, we get

$$x_2 = x_3 = x_4 = x_5 = \frac{1}{x_6} = \frac{1}{x_7} = \frac{1}{x_8},$$

and thus

$$\begin{aligned} \max\{x_i, \frac{1}{x_i} : 1 \leq i \leq 7\} &= x_9 \leq x_9(3) = \frac{1}{x_8} \\ &\leq \frac{1}{\min\{x_9(2), x_9(3)\}} \leq \max\{x_i, \frac{1}{x_i} : 1 \leq i \leq 7\}. \end{aligned}$$

Therefore

$$\min\{x_8(2), x_8(3)\} = \min\{x_i, \frac{1}{x_i} : 1 \leq i \leq 7\}.$$

If $x_8(2) = \min\{x_i, \frac{1}{x_i} : 1 \leq i \leq 7\}$, then Lemma 4 yields $x_1 = x_2 = \ldots = x_7 = 1$, a fact that contradicts our hypotheses. If $x_8(3) = \min\{x_i, \frac{1}{x_i} : 1 \leq i \leq 7\}$, then the same conclusion can be derived from Corollary 1. Consequently, $x_9 \neq \max\{x_i, \frac{1}{x_i} : 1 \leq i \leq 7\}$. In a similar manner one can conclude that $x_9 \neq \min\{x_i, \frac{1}{x_i} : 1 \leq i \leq 7\}$, and all inequalities in (5) are strict. This ends the proof of Lemma 2.

3 More on the dynamics of T_w

What can be said about the dynamics of T_w for the other values of w? The linearization of T_w at the equilibrium point C is given by the matrix

$$A_w = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{w-1}{w+4} & \frac{w-2}{w+4} & \frac{w-2}{w+4} & \frac{w-2}{w+4} & \frac{2-w}{w+4} & \frac{2-w}{w+4} & \frac{2-w}{w+4} \end{pmatrix}$$

with the characteristic polynomial

$$P_w(x) = x^7 + \frac{w-2}{w+4}(x^6 + x^5 + x^4) - \frac{w-2}{w+4}(x^3 + x^2 + x) - \frac{w-1}{w+4}.$$

The nature of the roots of P_w will be revealed via the Schur-Cohn Theorem [1]. This theorem concerns the principal minors of the Hermitian Schur-Cohn matrix, which (modulo a positive factor) equals

$$\begin{pmatrix} 2w+3 & w-2 & w-2 & w-2 & 2-w & 2-w & 2-w \\ w-2 & 2w+3 & w-2 & w-2 & w-2 & 2-w & 2-w \\ w-2 & w-2 & 2w+3 & w-2 & w-2 & w-2 & 2-w \\ w-2 & w-2 & w-2 & 2w+3 & w-2 & w-2 & w-2 \\ 2-w & w-2 & w-2 & w-2 & 2w+3 & w-2 & w-2 \\ 2-w & 2-w & w-2 & w-2 & w-2 & 2w+3 & w-2 \\ 2-w & 2-w & 2-w & w-2 & w-2 & w-2 & 2w+3 \end{pmatrix}.$$

These minors are:

$$D_1 = 2w+3$$
$$D_2 = 3w^2+16w+5$$
$$D_3 = 4w^3+39w^2+90w-25$$
$$D_4 = 5w^4+72w^3+330w^2+400w-375$$
$$D_5 = -6w^5+67w^4+1076w^3+2706w^2-1390w-725$$
$$D_6 = 7w^6-280w^5+1351w^4+15\,736w^3+1925w^2-17\,024w+5005$$
$$D_7 = 343w^6-8918w^5+51\,793w^4+89\,180w^3-88\,151w^2-80\,262w+57\,967.$$

When all principal minors are nonzero, then there are no roots on the unit circle. Moreover, the number ν of sign variations in the sequence

$$1, D_1, D_2, D_3, D_4, D_5, D_6, D_7 \tag{6}$$

gives us the numbers of roots outside unit disk.

The positive roots of $D_1, D_2, D_3, D_4, D_5, D_6, D_7$ form the set

$$\mathcal{W} = \{0.25000,\ 0.347\,52,\ 0.60000,\ 0.697\,01,\ 0.725\,87,\ 13.591\,79,\ 20.802\,98,\ 31.652\,47\}$$

For $w > 0$, not in the set $\mathcal{W}$, the dynamical system associated to T_w is hyperbolic (and thus topologically equivalent to the system associated to A_w). This is a consequence of the Grobman-Hartman theorem, [11], p. 119. Computing the number of sign variations in the Schur-Cohn sequence (6), we can conclude that the point $C = (1, 1, 1, 1, 1, 1, 1)$ is a global attractor when $w \in (0.725\,87, 13.591\,79)$, and a saddle point when $w \in (0, \infty)\backslash\mathcal{W}$. At the moment we lack an argument to decide whether the zone of *global asymptotic stability* is the entire interval $(0.725\,87, 13.591\,79)$ or not.

4 An application to interlacing sequences

Suppose there are given two families of real numbers, $x_1 \le x_2 \le x_3 \le x_4$ and $y_1 \le y_2 \le y_3$, which belong to the interval $(0, 1)$ and verify respectively the equations

$$x^4 - A_1x^3 + A_2x^2 - A_3x + A_4 = 0,$$

and

$$y^3 - B_1y^2 + B_2y - B_3 = 0.$$

What can be said about x_1 when the two sequences verify an interlacing condition of the following form

$$x_1 \le y_1 \le x_2 \le y_2 \le x_3 \le y_3 \le x_4? \tag{7}$$

For example, the condition (7) is verified by the roots of any quartic polynomial (with all roots real) and the roots of its derivative.

An alternative way to describe the phenomenon of interlacing sequences (of any finite length) is provided by the spectral theory. Precisely, if $A \in M_n(\mathbb{R})$ is a self-adjoint matrix and A_{n-1} is its $(n-1) \times (n-1)$ submatrix, obtained by deleting the last row and column, then their eigenvalues interlace. Moreover, all pairs of interlacing sequences can be obtained this way. See [4], Theorem 4.3.8 and Theorem 4.3.10.

According to Lemma 4,

$$x_1 < \min\left\{\frac{3A_4 + B_2}{A_3 + 2B_3}, \frac{A_3 + 2B_3}{3A_4 + B_2}\right\}.$$

This fact (and its higher order analogues) proves useful in the numerical computation of the smallest eigenvalue of certain symmetric matrices.

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References

[1] Y. Bistritz, Reflections on Schur-Cohn Matrices and Jury-Marden Tables and Classification of Related Unit Circle Zero Location Criteria, *Circuits Systems Signal Processing* **15** (1996), 111-136.

[2] E. Camouzis and G. Ladas, *Dynamics of Third Order Rational Difference Equations with Open Problems and Conjectures* (*Discrete Mathematics and Applications*), Chapman & Hall/CRC, Boca Raton, Fl., U.S.A., 2007.

[3] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities,* Cambridge Mathematical Library, 2nd Ed., 1952, Reprinted 1988.

[4] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge Univ. Press, 1993.

[5] N. Kruse and T. Nesermann, Global asymptotic stability in some discrete dynamical systems, *J. Math. Anal. Appl.* **235** (1999), 151-158.

[6] V. L. Kocić and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Application,* vol. **256** of *Mathematics and Its Applications,* Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993.

[7] M. R. S. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations, with Open Problems and Conjectures,* Chapman and Hall/CRC Press, Boca Raton, Fl., U.S.A., 2002.

[8] X. Li and D. Zhu, Global asymptotic stability in a rational equation, *J. of Difference Equations and Applications*, **9** (2003), 833-839.

[9] D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.

[10] T. Nesermann, Positive nonlinear difference equations: some results and applications, *Nonlinear Analysis,* **47** (2001), 4707-4717.

[11] L. Perko, *Differential Equations and Dynamical Systems*, Springer-Verlag, Berlin, 1991.

[12] M. Vidyasagar, *Nonlinear Systems Analysis*, SIAM Classics in Applied Mathematics, SIAM Press, 2002.

[13] X. Yang, Global asymptotic stability in a class of generalized Putnam equations, *J. Math. Anal. Appl.,* **322** (2006), 693-698.

[14] X. Yang, M. Yang and H. Liu, A part-metric-related inequality chain and application to the stability analysis of difference equations, *Journal of Inequalities and Applications,* vol. 2007, Article ID 196618, 9 pp, doi:10.1155/2007/19618.