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# ON THE SOLVABILITY OF DYNAMIC ELASTIC-VISCO-PLASTIC CONTACT PROBLEMS WITH ADHESION*

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**Abstract**

We consider a dynamic contact problem between an elastic-visco-plastic body and an obstacle, the so-called foundation. The contact is frictionless and is modelled with normal compliance of such a type that the penetration is restricted with unilateral constraint. The adhesion of contact surfaces is taken into account and the evolution of the bonding field is described by a first-order differential equation. We provide a weak formulation of the contact problem in the form of an integro-differential system in which the unknowns are the displacement, the stress and the bonding fields, then we present an existence result for the solution. We consider a sequence of penalized problems which have a unique solution, derive *a priori* estimates and use compactness properties to obtain a solution to the original model, by passing to the limit as the penalization parameter converges to zero.

**MSC**: 74M15, 74H20, 49J40

**keywords:** elastic-visco-plastic material, dynamic process, frictionless contact, normal compliance, Signorini condition, adhesion, variational formulation, weak solution, *a priori* estimates

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# 1 Introduction

Processes of adhesion are important in many industrial settings where parts, usually nonmetallic, are glued together. For this reason, adhesive contact between bodies, when a glue is added to prevent the surfaces from relative motion, has recently received increased attention in the literature. General models with adhesion can be found in [3, 4, 5, 12]. The new idea in these models is the introduction of a surface internal variable, the bonding field $\beta \in [0,1]$, which describes the fractional density of active bonds on the contact surface. At a point on the bonding contact surface $\Gamma_3$, when $\beta = 1$, the adhesion is complete and all the bonds are active; when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion; when $0 < \beta < 1$ the adhesion is partial and only a fraction $\beta$ of the bonds is active. Results on the mathematical analysis of various adhesive contact problems can be found in [1, 13, 14].

In this paper we study a dynamic frictionless contact problem with adhesion for elastic-visco-plastic materials with a constitutive law of the form

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds, \tag{1}$$

where $\boldsymbol{u}$ denotes the displacement field while $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ represent the stress and the linearized strain tensor, respectively. Here $\mathcal{A}$ and $\mathcal{E}$ are linear operators describing the purely viscous and the elastic properties of the material, respectively, and $\mathcal{G}$ is a nonlinear constitutive function which describes the visco-plastic behaviour of the material. In (1) and everywhere in this paper the dot above a variable represents derivative with respect to the time variable $t$.

Examples and mechanical interpretation of constitutive laws of the form (1) can be found in [11]. Here we restrict ourselves to note that for $\mathcal{G} = \mathbf{0}$ the constitutive law (1) reduces to the well-known Kelvin-Voigt viscoelastic constitutive law

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}). \tag{2}$$

Quasistatic contact problems for materials of the form and (2) were investigated in a large number of papers, see e.g. [6] for a survey. There, both the variational analysis and the numerical approach of the problems, including the study of semi-discrete and fully discrete schemes, were provided. Existence results in the study of dynamic problems with Kelvin-Voigt materials of the form (2) can be found in in [8, 9, 10] and, for more details, we send the reader to the monograph [2]. Finally, note that existence results for adhesive contact processes with materials of the form (2) were obtained in [14].

Dynamic frictionless contact problem for rate-type materials of the form (1) were studied in [7, 11]. In [11] we assumed that the contact is frictionless and it is modelled with normal compliance of such a type that the penetration is limited and associated to a unilateral constraint for the displacement field. The novelty of the method used in that paper was in the treatment of the compliance term, which do not necessarily represent a compact perturbation of the original problem, without contact. For the problem in [7] the contact was modelled with normal compliance of such type that the penetration is not limited and the adhesion of the contact surfaces was taken into account. The results in [7, 11] concern the solvability of the corresponding dynamic problems. In addition, the results in [7] concern also the study a fully discrete scheme for solving the problem, including convergence and optimal order error estimates results.

The present paper represents a continuation of [7, 11]. Its novelty arises in the fact there here we use the contact condition in [11] combined with the boundary conditions in [7], which describe the adhesion of the contact surfaces. As a result, we arrive to a new mathematical model, different of those in [7, 11], for which we study the problem of weak solvability.

The paper is organized as follows. In Section 2 we describe the contact problem, list assumptions on the data and provide its variational formulation; then we state an existence result for the weak solution of the problem, Theorem 1. In Section 3 we consider a sequence of penalized problems and state their unique solvability. Then, in Section 4, we provide the proof of Theorem 1; to this end we use compactness properties and a limit procedure as the penalization parameter converges to zero.

We end this introductory section by presenting the notation we shall use in the rest of the paper. We denote by $r_+$ and $r_-$ the positive and negative part of $r$, i.e. $r_+ = max\,\{0, r\}$, $r_- = max\,\{0, -r\}$. We also denote by $\mathbb{S}^N$ the space of second order symmetric tensors on $\mathbb{R}^N$ ($N = 2, 3$), while " $\cdot$ " and $\|\cdot\|$ will represent the inner product and the Euclidean norm on $\mathbb{S}^N$ and $\mathbb{R}^N$. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary $\Gamma$ and let $\boldsymbol{\nu}$ denote the unit outer normal on $\Gamma$. We assume that $\Gamma$ is partitioned into three disjoint measurable parts $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$. Everywhere in what follows the index $i$ and $j$ run from 1 to $N$, summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the independent spatial variable. By $\mathbb{R}_+$ we denote the interval $[0,+\infty)$.

We use the standard notation for Lebesgue ($L_p, \boldsymbol{L}_p \equiv (L_p)^N, p \in [1,\infty]$) and Sobolev-Slobodetskii spaces $W_p^k$, $H^k \equiv W_2^k$, $\boldsymbol{H}^k \equiv (H^k)^N$, $k \geq 0$, $p \in [1,\infty]$) associated to $\Omega$ and $\Gamma$ and their duals. For the spaces with zero traces $\mathring{H}^k$, $\mathring{\boldsymbol{H}}^k = (\mathring{H}^k)^N$ is used if $\frac{1}{2} < k \notin \frac{1}{2} + \mathbb{N}$. Moreover, for a domain $M \subset \mathbb{R}^d$ ($d \in \mathbb{N}$), a Banach space $X$, $k \in \mathbb{N}$ and $p \in [1,\infty]$, we use the standard notation for the Lebesgue spaces $L_p(M;X)$ and for the Sobolev spaces $W_p^k(M;X)$. If $d = 1$ and $M = (0,T)$ is a time interval we shall write $L_p(0,T;X)$ and $W_p^k(M;X)$. For $k \equiv (k_1,k_2) \in \mathbb{R}_+^2$ and some domains $M_i \in \mathbb{R}^{N_i}, i = 1,2$ and $M \equiv M_1 \times M_2$, $H^k(M) \equiv L_2(M_1;H^{k_2}(M_2)) \cap L_2(M_2;H^{k_1}(M_1))$ is the corresponding anisotropic Sobolev-Slobodetskii space. For us $M_1$ will be a time interval $J$ and $M_2$ the domain $\Omega$, its boundary or its parts. The inverse Fourier transform immediately proves that the space $H^{1/2,1}(J\times\Omega)$ takes its (lateral) traces in $H^{1/4,1/2}(J\times\Gamma)$. For details see [2]. Moreover, we use also the spaces $\mathcal{H} = \boldsymbol{L}_2(\Omega;\mathbb{S}^N)$, $\boldsymbol{H}^1(\Omega) = (H^1(\Omega))^N$ and

$$\mathcal{H}_1 = \{\ \boldsymbol{\sigma} \in \mathcal{H}\ ;\ \mathrm{Div}\,\boldsymbol{\sigma} \in \boldsymbol{L}_2(\Omega)\ \}.$$

Here and below $\boldsymbol{\varepsilon}$ and Div are the deformation and the divergence operators, respectively, defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\,(u_{i,j} + u_{j,i}), \quad \mathrm{Div}\,\boldsymbol{\sigma} = (\sigma_{ij,j}).$$

The spaces $\mathcal{H}$, $\boldsymbol{H}^1(\Omega)$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij}\,dx = (\boldsymbol{\sigma},\boldsymbol{\tau})_{\boldsymbol{L}_2(\Omega;\mathbb{S}^N)},$$

$$(\boldsymbol{u},\boldsymbol{v})_1 = (\boldsymbol{u},\boldsymbol{v})_{\boldsymbol{L}_2(\Omega)} + (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}},$$

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} + (\mathrm{Div}\,\boldsymbol{\sigma},\mathrm{Div}\,\boldsymbol{\tau})_{\boldsymbol{L}_2(\Omega)}.$$

In general, we denote by $\|\cdot\|_X$ the norm on a Banach space $X$, this holds, in particular, for the associated norms on the spaces $\mathcal{H}$, $\boldsymbol{H}^1(\Omega)$ and $\mathcal{H}_1$.

For every element $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ we also use the notation $\boldsymbol{v}$ to denote the trace of $\boldsymbol{v}$ on $\Gamma$ and we denote by $v_\nu$ and $\boldsymbol{v}_\tau$ the normal and the tangential components of $\boldsymbol{v}$ on $\Gamma$ given by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$, $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$. We also denote by $\sigma_\nu$ and $\boldsymbol{\sigma}_\tau$ the normal and the tangential traces of a function $\boldsymbol{\sigma} \in \mathcal{H}_1$, and we note that when $\boldsymbol{\sigma}$ is a regular function then $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$, $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$, and the following Green's formula holds:

$$(\boldsymbol{\sigma}, \varepsilon(\boldsymbol{v}))_{\mathcal{H}} + (\mathrm{Div}\, \boldsymbol{\sigma}, \boldsymbol{v})_{\boldsymbol{L}_2(\Omega)} = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{v} \, da \qquad \forall \boldsymbol{v} \in \boldsymbol{H}^1(\Omega). \tag{3}$$

We introduce the closed subspace of $\boldsymbol{H}^1(\Omega)$ given by

$$V = \{ \ \boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \ ; \ \boldsymbol{v} = \boldsymbol{0} \ \text{ on } \ \Gamma_1 \ \}$$

and let $T > 0$. For each $t \in [0, T]$ we use the notation $Q_t = (0, t) \times \Omega$, $S_{ti} = (0, t) \times \Gamma_i$ and, if $t = T$ we write $Q \equiv Q_T = (0, T) \times \Omega$, $S_i \equiv S_{Ti} = (0, T) \times \Gamma_i$. Also, for every real Banach space $X$ we use the notation $C([0, T]; X)$ and $C^1([0, T]; X)$ for the space of continuous and continuously differentiable functions from $[0, T]$ to $X$, respectively, with their standard norms.

## 2 Problem statement

The physical setting is as follows. An elastic-visco-plastic body occupies a bounded domain $\Omega \subset \mathbb{R}^N$ $(N = 2, 3)$ with a regular boundary $\Gamma$ that is partitioned into three disjoint measurable parts $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$. Let $T > 0$ and let $[0, T]$ denote the time interval of interest. The body is clamped on $S_1 = (0, T) \times \Gamma_1$ and thus the displacement field vanishes there. A volume force of density $\boldsymbol{f}_0$ acts in $Q = (0, T) \times \Omega$ and a surface traction of density $\boldsymbol{f}_2$ acts on $S_2 = (0, T) \times \Gamma_2$. In the reference configuration the body is in adhesion frictionless contact on $S_3 = (0, T) \times \Gamma_3$ with a foundation. The contact is modelled with normal compliance in such a way that the penetration is limited and the evolution of the bonding field is given by a differential equation of the first order. Under these conditions, the classical formulation of the problem is the following.

**Problem $\mathcal{P}$.** *Find a displacement field $\boldsymbol{u} : \Omega \times [0, T] \to \mathbb{R}^N$, a stress field $\boldsymbol{\sigma} : \Omega \times [0, T] \to \mathbb{S}^N$ and a bonding field $\beta : \Gamma_3 \times [0, T] \to [0, 1]$ such that*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds \quad \text{in } Q, \tag{4}$$

$$\rho\ddot{\boldsymbol{u}} = \operatorname{Div}\boldsymbol{\sigma} + \boldsymbol{f}_0 \quad \text{in } Q, \tag{5}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } S_1, \tag{6}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on } S_2, \tag{7}$$

$$u_\nu \le g, \quad \sigma_\nu + p_\nu(u_\nu) - \gamma_\nu R_\nu(u_\nu)\beta^2 \le 0, \\ (\sigma_\nu + p_\nu(u_\nu) - \gamma_\nu R_\nu(u_\nu)\beta^2)(u_\nu - g) = 0 \quad \text{on } S_3, \tag{8}$$

$$-\boldsymbol{\sigma}_\tau = p_\tau(\beta)\boldsymbol{R}_\tau(u_\tau) \quad \text{on } S_3, \tag{9}$$

$$\dot{\beta} = -\big(\beta\,(\gamma_\nu R_\nu(u_\nu)^2 + \gamma_\tau\|\boldsymbol{R}_\tau(\boldsymbol{u}_\tau)\|^2) - \epsilon_a\big)_+ \quad \text{on } S_3, \tag{10}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \ \ \dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1 \quad \text{in } \Omega. \tag{11}$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_3. \tag{12}$$

We briefly describe problem (4)–(12) and provide explanation of the equations and the boundary conditions. Note that here and below $p_\nu$ and $p_\tau$ are given functions, $\gamma_\nu$, $\gamma_\tau$ and $\epsilon_a$ are given positive material parameters and

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \le s \le 0, \\ 0 & \text{if } s > 0; \end{cases} \tag{13}$$

$$\boldsymbol{R}_\tau(\boldsymbol{v}) = \begin{cases} \boldsymbol{v} & \text{if } \|\boldsymbol{v}\| \le L, \\ L\,\dfrac{\boldsymbol{v}}{\|\boldsymbol{v}\|} & \text{if } \|\boldsymbol{v}\| > L \end{cases} \tag{14}$$

with $L > 0$ being a characteristic length of the bond, beyond which there is no any additional traction (see, e.g., [12]).

Equation (4) is the elastic-visco-plastic constitutive law already presented in Section 1, (5) represents the equation of motion in which $\rho$ denotes the density of mass, (6) and (7) are the displacement and traction boundary conditions, respectively. Condition (9) is the tangential boundary condition on the contact surface $\Gamma_3$, and equation (10) describes the evolution of the bonding field, see [7, 14] for details. The functions $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$ in (11) denote the initial displacement and velocity, respectively, and the function $\beta_0$ in (12) represents the initial bonding field.

Using (10) it is easy to see that if $0 \leq \beta_0 \leq 1$ a.e. on $\Gamma_3$, then $0 \leq \beta \leq 1$ a.e. on $\Gamma_3$ during the process. Indeed, let $\boldsymbol{x} \in \Gamma_3$; the equation (10) guarantees that $t \mapsto \beta(\boldsymbol{x}, t)$ is a decreasing function and, therefore, $\beta(\boldsymbol{x}, t) \leq \beta(\boldsymbol{x}, 0) = \beta_0(\boldsymbol{x}) \leq 1$ for all $t \geq 0$, i.e. $\beta \leq 1$. On the other hand, if there exists $t_1 > 0$ such that $\beta(\boldsymbol{x}, t_1) < 0$, then there exists $0 \leq t_0 < t_1$ such that $\beta(\boldsymbol{x}, t_0) = 0$. It follows that $\beta(\boldsymbol{x}, t) \leq 0$ for all $t \geq t_0$ and (10) shows that $\dot{\beta}(\boldsymbol{x}, t) = 0$ for all $t \geq t_0$ which implies that $\beta(\boldsymbol{x}, t) = 0$ for all $t \geq t_0$. We deduce that $\beta(\boldsymbol{x}, t_1) = 0$ which is in contradiction with the assumption $\beta(\boldsymbol{x}, t_1) < 0$. We conclude that $\beta(\boldsymbol{x}, t) \geq 0$ for all $t \in [0, T]$, i.e. $\beta \geq 0$.

Our main interest is on the contact condition (8). Here $\sigma_\nu$ denotes the normal stress, $u_\nu$ is the normal displacement, $g \geq 0$ is given and $p_\nu$ is a function which satisfies

$$\left.\begin{array}{l}
\text{(a) } p_\nu : ]-\infty, g] \to \mathbb{R}. \\
\text{(b) There exists } \ell_\nu > 0 \text{ such that} \\
\qquad |p_\nu(r_1) - p_\nu(r_2)| \leq \ell_\nu |r_1 - r_2| \quad \forall r_1, r_2 \leq g. \\
\text{(c) } (p_\nu(r_1) - p_\nu(r_2))(r_1 - r_2) \geq 0 \quad \forall r_1, r_2 \leq g. \\
\text{(d) } p_\nu(r) = 0 \ \text{ for all } r < 0.
\end{array}\right\} \tag{15}$$

Condition (8) combined with assumption (15) and definition (13) of $R_\nu$ shows that when there is separation between the body and the obstacle (i.e. when $u_\nu < 0$), then $\sigma_\nu = \gamma_\nu R_\nu(u_\nu)\beta^2$, i.e. the normal stress reduces to its adhesive component; it is tensile and proportional to the square of the bonding field and to the normal displacement, as long as it does not exceed the bond length $L$. When $0 \leq u_\nu < g$ then $-\sigma_\nu = p_\nu(u_\nu)$, i.e. normal stress reduces to the reaction of the foundation, and is uniquely determined by the normal displacement; finally, when $u_\nu = g$, the normal stress is not uniquely determined but is submitted to the restriction $-\sigma_\nu \geq p_\nu(g)$. We conclude from above that the contact follows a normal compliance condition with adhesion but up to the limit $g$ and then, when this limit is reached, the contact follows a Signorini-type unilateral condition with the gap $g$. For this reason we refer to the contact condition (8) as a normal compliance contact condition with adhesion, finite penetration and unilateral constraint. Also, note that when $g = 0$ and $p_\nu \equiv 0$, condition (8) becomes Signorini contact condition with adhesion used in [12, 14].

We now describe the assumptions on the data we consider in the study of the mechanical problem (4)–(12). We assume that the operators $\mathcal{A}$ and $\mathcal{E}$ are linear and, moreover, the following condition is satisfied for $\mathcal{T} = \mathcal{A}$, $\mathcal{E}$:

$$\left.\begin{array}{l}\text{(a) } \mathcal{T} = (\mathcal{T}_{ijk\ell}) : \varOmega \times \mathbb{S}^N \to \mathbb{S}^N.\\ \text{(b) } \mathcal{T}_{ijk\ell} \in L_\infty(\varOmega),\ 1 \le i, j, k, \ell \le N.\\ \text{(c) } \mathcal{T}\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \mathcal{T}\boldsymbol{\tau},\ \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^N,\ \text{a.e. in } \varOmega.\\ \text{(d) There exists } a_0 > 0 \text{ such that}\\ \qquad \mathcal{T}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \ge a_0 \|\boldsymbol{\tau}\|^2\ \ \forall\, \boldsymbol{\tau} \in \mathbb{S}^N,\ \text{a.e. in } \varOmega.\end{array}\right\} \tag{16}$$

The operator $\mathcal{G}$ may be nonlinear and satisfies

$$\left.\begin{array}{l}\text{(a) } \mathcal{G} : \varOmega \times \mathbb{S}^N \times \mathbb{S}^N \to \mathbb{S}^N.\\ \text{(b) There exists } \ell_{\mathcal{G}} > 0 \text{ such that}\\ \qquad \|\mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1) - \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2)\|\\ \qquad\quad \le \ell_{\mathcal{G}}\, (\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|)\\ \qquad\qquad \forall\, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^N,\ \text{a.e. } \boldsymbol{x} \in \varOmega.\\ \text{(c) For any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^N,\ \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon})\\ \qquad \text{is measurable on } \varOmega.\\ \text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \mathbf{0}, \mathbf{0}) \text{ belongs to } \mathcal{H}.\end{array}\right\} \tag{17}$$

The tangential function $p_\tau$ is such that

$$\left.\begin{array}{l}\text{(a) } p_\tau : \varGamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}_+.\\ \text{(b) There exists } \ell_\tau > 0 \text{ such that}\\ \qquad |p_\tau(\boldsymbol{x}, \beta_1) - p_\tau(\boldsymbol{x}, \beta_2)| \le \ell_\tau\, |\beta_1 - \beta_2|\\ \qquad \forall\, \beta_1,\ \beta_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \varGamma_3.\\ \text{(c) There exists } M_\tau > 0 \text{ such that } |p_\tau(\boldsymbol{x}, \beta)| \le M_\tau\\ \qquad \forall\, \beta \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \varGamma_3.\\ \text{(d) For any } \beta \in \mathbb{R},\ \boldsymbol{x} \mapsto p_\tau(\boldsymbol{x}, \beta) \text{ is measurable on } \varGamma_3.\\ \text{(e) The mapping } \boldsymbol{x} \mapsto p_\tau(\boldsymbol{x}, 0) \text{ belongs to } L^2(\varGamma_3).\end{array}\right\} \tag{18}$$

We also suppose that the mass density satisfies

$$\rho \in L_\infty(\varOmega), \quad \text{there exists } \rho^* > 0 \text{ such that } \ \rho(\boldsymbol{x}) \ge \rho^* \text{ a.e. } \boldsymbol{x} \in \varOmega, \tag{19}$$

and the body forces and surface tractions have the regularity

$$\boldsymbol{f}_0 \in L_2(Q), \quad \boldsymbol{f}_2 \in L_2(S_2). \tag{20}$$

We remark that conditions (20) may be weakened.

The adhesion coefficients $\gamma_\nu$, $\gamma_\tau$ and $\epsilon_a$ satisfy the conditions

$$\gamma_\nu,\ \gamma_\tau \in L_\infty(\Gamma_3), \quad \epsilon_a \in L_2(\Gamma_3), \quad \gamma_\nu,\, \gamma_\tau,\, \epsilon_a \geq 0 \quad \text{a.e. on } \Gamma_3 \tag{21}$$

and, finally, the initial data satisfy

$$\boldsymbol{u}_0 \in V, \quad \boldsymbol{v}_0 \in \boldsymbol{L}_2(\Omega), \quad \beta_0 \in L_2(\Gamma_3). \tag{22}$$

In the rest of the paper we will use a modified inner product on the Hilbert space $H = \boldsymbol{L}_2(\Omega)$, given by

$$(\boldsymbol{u}, \boldsymbol{v})_H = (\rho\, \boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{L}_2(\Omega)} \quad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in H, \tag{23}$$

that is, it is weighted with $\rho$, and we let $\|\cdot\|_H$ be the associated norm, i.e.,

$$\|\boldsymbol{v}\|_H = (\rho\, \boldsymbol{v}, \boldsymbol{v})^{1/2}_{\boldsymbol{L}_2(\Omega)} \quad \forall\, \boldsymbol{v} \in H.$$

It follows from assumption (19) that $\|\cdot\|_H$ and $\|\cdot\|_{\boldsymbol{L}_2(\Omega)}$ are equivalent norms on $H$, and also the inclusion mapping of $(V, \|\cdot\|_V)$ into $(H, \|\cdot\|_H)$ is continuous and dense. We denote by $V'$ the dual space of $V$ and we use the notation $\langle\cdot,\cdot\rangle_{V'\times V}$ to represent the duality pairing between $V'$ and $V$ and we recall that

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle_{V'\times V} = (\boldsymbol{u}, \boldsymbol{v})_H \qquad \forall\, \boldsymbol{u} \in H,\, \boldsymbol{v} \in V. \tag{24}$$

Finally, we denote by $\|\cdot\|_{V'}$ the norm on $V'$.

Assumptions (20) allow us, for a.e. $t \in (0, T)$, to define $\boldsymbol{f}(t) \in V'$ by

$$\langle \boldsymbol{f}(t), \boldsymbol{v}\rangle_{V'\times V} \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}\, da \quad \forall\, \boldsymbol{v} \in V, \tag{25}$$

and note that

$$\boldsymbol{f} \in L_2(0, T; V'). \tag{26}$$

Also, define $j : L_\infty(\Gamma_3) \times V \times V \to \mathbb{R}$ by the formula

$$j(\beta, \boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} \big[\big(p_\nu(u_\nu) - \gamma_\nu R_\nu(u_\nu)\beta^2\big)\, v_\nu + p_\tau(\beta)\, \boldsymbol{R}_\tau(\boldsymbol{u}_\tau) \cdot \boldsymbol{v}_\tau\big]\, da. \tag{27}$$

and note that the integral is well-defined due to the assumptions (15), (18) and (21).

Next, we need the set of admissible displacements field and the set of admissible bonding fields defined by

$$K = \{\ \boldsymbol{v} \in V\ ;\ v_\nu \le g \quad \text{a.e. on } \Gamma_3\ \}, \tag{28}$$

$$Q = \{\ \beta \in L_2(\Gamma_3)\ ;\ 0 \le \beta \le 1 \ \text{ a.e. on } \Gamma_3\ \}, \tag{29}$$

respectively. Finally, we reinforce assumption (22) with

$$\boldsymbol{u}_0 \in K, \tag{30}$$

$$\beta_0 \in Q. \tag{31}$$

We continue with a brief description of the steps in the derivation of a variational formulation for this mechanical problem. To this end, assume that $(\boldsymbol{u}, \boldsymbol{\sigma}, \beta)$ are smooth functions satisfying (4)–(12). We use the set of admissible displacements fields, (28), as well as the functional $j$, (27). Also, we introduce the sets

$$\mathcal{K} = \{\ \boldsymbol{v} \in H^1(0,T;V)\ ;\ \boldsymbol{v}(t) \in K \quad \forall\, t \in [0,T]\ \}, \tag{32}$$

$$\mathcal{Q} = \{\ \beta \in W^1_\infty(0,T;L_2(\Gamma_3))\ ;\ \beta(t) \in Q \quad \forall\, t \in [0,T]\ \}. \tag{33}$$

Let $t \in [0,T]$ and let $\boldsymbol{w} \in \mathcal{K}$. We take the dot product of equation (5) with $\boldsymbol{w}(t) - \boldsymbol{u}(t)$, integrate the result over $\Omega$ and use Green's formula (3) to obtain

$$\begin{aligned}(\rho\, \ddot{\boldsymbol{u}}(t), \boldsymbol{w}(t) - \boldsymbol{u}(t))_{\boldsymbol{L}_2(\Omega)} + (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{w}(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathcal{H}} \\ = \int_\Omega \boldsymbol{f}_0(t) \cdot (\boldsymbol{w}(t) - \boldsymbol{u}(t))\, dx + \int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{w} - \boldsymbol{u}(t))\, da.\end{aligned} \tag{34}$$

Applying the boundary conditions (7) and (9) and noting that $\boldsymbol{w}(t) = \boldsymbol{0}$ on $\Gamma_1$, we have

$$\begin{aligned}\int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{w}(t) - \boldsymbol{u}(t))\, da = \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot (\boldsymbol{w}(t) - \boldsymbol{u}(t))\, da \\ + \int_{\Gamma_3} \sigma_\nu(t)\, (w_\nu(t) - u_\nu(t))\, da - \int_{\Gamma_3} p_\tau(\beta(t))\boldsymbol{R}_\tau(\boldsymbol{u}_\tau(t)) \cdot \boldsymbol{w}_\tau(t)\, da.\end{aligned} \tag{35}$$

Moreover, (8) yields

$$\begin{aligned}\int_{\Gamma_3} (\sigma_\nu(t) - \gamma_\nu R_\nu(u_\nu(t))\beta^2(t))\, (w_\nu(t) - u_\nu(t))\, da \\ \ge \int_{\Gamma_3} p_\nu(u_\nu(t))(u_\nu(t) - w_\nu(t))\, da.\end{aligned} \tag{36}$$

We combine now (34)–(36) and use (23)–(25) and (27) to find that

$$\begin{aligned}&\langle \ddot{\boldsymbol{u}}(t), \boldsymbol{w}(t)-\boldsymbol{u}(t)\rangle_{V'\times V} + (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{w}(t))-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathcal{H}} \\ &\quad + j(\beta(t), \boldsymbol{u}(t), \boldsymbol{w}(t)-\boldsymbol{u}(t)) \geq \langle \boldsymbol{f}(t), \boldsymbol{w}-\boldsymbol{u}(t)\rangle_{V'\times V}.\end{aligned} \tag{37}$$

Then, we integrate (37) on $[0,T]$, perform an integration by part, use the initial conditions (11), and combine the resulting inequality with the constitutive law (4), the differential equation (10), the initial condition (12) and the unilateral constraint in (8). As a result we obtain the following variational formulation of Problem $\mathcal{P}$.

**Problem $\mathcal{P}^V$.** *Find a displacement field $\boldsymbol{u} : [0,T] \to V$, a stress field $\boldsymbol{\sigma} : [0,T] \to \mathcal{H}$ and a bonding field $\beta : [0,T] \to L_\infty(\Gamma_3)$ such that $\boldsymbol{u} \in \mathcal{K}$, $\beta \in \mathcal{Q}$,*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds \quad \text{a.e. } t \in (0,T), \tag{38}$$

$$\begin{aligned}&\int_0^T (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{w}(t)-\boldsymbol{u}(t)))_{\mathcal{H}}\, dt - \int_0^T (\dot{\boldsymbol{u}}(t), \dot{\boldsymbol{w}}(t)-\dot{\boldsymbol{u}}(t))_H\, dt \\ &\quad + \int_0^T j(\beta(t), \boldsymbol{u}(t), \boldsymbol{w}(t)-\boldsymbol{u}(t))\, dt + (\dot{\boldsymbol{u}}(T), \boldsymbol{w}(T)-\boldsymbol{u}(T))_H \\ &\quad \geq \int_0^T \langle \boldsymbol{f}(t), \boldsymbol{w}(t)-\boldsymbol{u}(t)\rangle_{V'\times V}\, dt + (\boldsymbol{u}_1, \boldsymbol{w}(0)-\boldsymbol{u}_0)_H \quad \forall\, \boldsymbol{w} \in \mathcal{K},\end{aligned} \tag{39}$$

$$\dot{\beta}(t) - \big(\beta\,(\gamma_\nu R_\nu(u_\nu)^2 + \gamma_\tau \|\boldsymbol{R}(\boldsymbol{u}_\tau)\|^2) - \epsilon_a\big)_+ \quad \text{a.e. } t \in (0,T), \tag{40}$$

$$\beta(0) = \beta_0. \tag{41}$$

The main result of this section concerns the solvability of Problem $\mathcal{P}^V$ and can be stated as follows.

**Theorem 1.** *Assume that conditions* (15)–(22), (30) *and* (31) *hold. Then Problem $\mathcal{P}^V$ has at least a solution. Moreover, the solution satisfies*

$$\boldsymbol{u} \in \mathcal{K}, \quad \dot{\boldsymbol{u}} \in \boldsymbol{H}^{1/2,1}(Q), \tag{42}$$

$$\boldsymbol{\sigma} \in L_2(0,T;\mathcal{H}), \tag{43}$$

$$\beta \in \mathcal{Q}. \tag{44}$$

We conclude by Theorem 1 that the frictionless contact problem with normal compliance, adhesion and unilateral constraint, (4)–(12), has at least a *weak solution* and it satisfies (42)–(48). The question of the uniqueness of the solution is left open.

## 3 Penalized problems

We turn now to the proof of Theorem 1 which will be carried out in several steps and it is based on a limit procedure on estimates for the solutions of a sequence of regularized problems, similar to that used in [2, Ch. 4] and [11]. Since the modifications are straightforward, sometimes we omit the details.

We start with the construction of the penalized problems. To this end, for every $\lambda > 0$ we consider the function $p_{\nu\lambda} : \mathbb{R} \to \mathbb{R}$ defined by

$$p_{\nu\lambda}(r) = \begin{cases} p_\nu(r) & \text{if } r \le g, \\ \frac{1}{\lambda}(r-g) + p_\nu(g) & \text{if } r > g, \end{cases} \tag{45}$$

and let $P_{\nu\lambda} : \mathbb{R} \to \mathbb{R}$ be the function defined by $P_{\nu\lambda}(r) = \displaystyle\int_0^r p_{\nu\lambda}(s)\,ds$, i.e.

$$P_{\nu\lambda}(r) = \begin{cases} \displaystyle\int_0^r p_\nu(s)\,ds & \text{if } r \le g, \\ \frac{1}{2\lambda}(r-g)^2 + p_\nu(g)(r-g) + \displaystyle\int_0^g p_\nu(s)\,ds & \text{if } r > g. \end{cases} \tag{46}$$

We also consider the functional $j_\lambda : L_\infty(\Gamma_3) \times V \times V \to \mathbb{R}$ given by

$$\begin{aligned} j_\lambda(\beta, \boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} \big[ \big(p_{\nu\lambda}(u_\nu) - \gamma_\nu R_\nu(u_\nu)\beta^2\big) v_\nu \\ + p_\tau(\beta)\, \boldsymbol{R}_\tau(\boldsymbol{u}_\tau) \cdot \boldsymbol{v}_\tau \big]\, da. \end{aligned} \tag{47}$$

We use the notation above to define the following penalized frictionless contact problems.

**Problem** $\mathcal{P}_\lambda^V$ *Find a displacement field* $\boldsymbol{u}_\lambda : [0,T] \to V$*, a stress field* $\boldsymbol{\sigma}_\lambda : [0,T] \to \mathcal{H}$ *and a bonding field* $\beta_\lambda : [0,T] \to L_\infty(\Gamma_3)$ *such that for almost every* $t \in (0,T)$

$$\begin{aligned} \boldsymbol{\sigma}_\lambda(t) =& \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\lambda(t)) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\lambda(t)) \\ &+ \int_0^t \mathcal{G}(\boldsymbol{\sigma}_\lambda(s) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\lambda(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}_\lambda(s)))\, ds, \end{aligned} \tag{48}$$

$$\langle \ddot{\boldsymbol{u}}_\lambda(t), \boldsymbol{w}\rangle_{V'\times V} + (\boldsymbol{\sigma}_\lambda(t), \boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}} + j_\lambda(\beta_\lambda(t)\boldsymbol{u}_\lambda(t), \boldsymbol{w}) \\ = \langle \boldsymbol{f}(t), \boldsymbol{w}\rangle_{V'\times V} \quad \forall\, \boldsymbol{w} \in V, \tag{49}$$

$$\dot{\beta}_\lambda(t) - \left(\beta_\lambda \left(\gamma_\nu R_\nu(u_{\lambda\nu})^2 + \gamma_\tau \|\boldsymbol{R}(\boldsymbol{u}_{\lambda\tau}\|^2\right) - \epsilon_a\right)_+, \tag{50}$$

$$\boldsymbol{u}_\lambda(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}_\lambda(0) = \boldsymbol{u}_1, \quad \beta_\lambda(0) = \beta_0. \tag{51}$$

Clearly, Problem $\mathcal{P}^V_\lambda$ represents the variational formulation of an adhesive contact problem similar to that studied in [7], in which the penetration is allowed and unlimited. Moreover, keeping in mind the definition of the function $p_{\nu\lambda}$, we formally recover condition (8) in the limit as $\lambda \to 0$. For this reason we refer to Problem $\mathcal{P}^V_\lambda$ as a *penalized* of the original frictionless contact problem $\mathcal{P}^V$.

Note that the function $p_{\nu\lambda}$ defined in (45) is monotone and Lipschitz continuous. This allows to obtain the following existence and uniqueness result.

**Theorem 2.** *Under the assumptions* (15)–(22) *and* (31)*, Problem* $\mathcal{P}^V_\lambda$ *has a unique solution* $(\boldsymbol{u}_\lambda, \boldsymbol{\sigma}_\lambda, \beta_\lambda)$*, and*

$$\boldsymbol{u}_\lambda \in W^1_2(0,T;V) \cap C^1([0,T];H), \quad \ddot{\boldsymbol{u}}_\lambda \in L_2(0,T;V'), \tag{52}$$

$$\boldsymbol{\sigma}_\lambda \in L_2(0,T;\mathcal{H}), \quad \operatorname{Div} \boldsymbol{\sigma}_\lambda \in L_2(0,T;V'), \tag{53}$$

$$\beta_\lambda \in \mathcal{Q}. \tag{54}$$

The proof of Theorem 2 is based on arguments similar to those presented in [7, 14] and, therefore, is omitted. Nevertheless, we note that the regularity $\beta_\lambda \in \mathcal{Q}$ of the bonding field follows from the differential equation (50), the initial condition $\beta_\lambda(0) = \beta_0$ and assumption (31).

# 4 Proof of Theorem 1

We now proceed to *a priori* estimates. Everywhere below we assume that (15)–(22), (30) and (31) hold. Also, below $c$ will represent a generic positive constant which may depend on the problem data but does not depend on $\lambda$ or $T$, nor on the positive numbers $k$ and $T_0$ which will be specified later; also its value may change from line to line.

(i) ***A priori* estimates.** Let $\lambda > 0$. We put $\boldsymbol{w} = \dot{\boldsymbol{u}}_\lambda(t)$ in (49) to obtain

$$\langle \ddot{\boldsymbol{u}}_\lambda(t), \dot{\boldsymbol{u}}_\lambda(t)\rangle_{V'\times V} + (\boldsymbol{\sigma}_\lambda(t), \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\lambda(t)))_{\mathcal{H}} + j_\lambda(\beta_\lambda(t), \boldsymbol{u}(t), \dot{\boldsymbol{u}}_\lambda(t)) \\ = \langle \boldsymbol{f}(t), \dot{\boldsymbol{u}}_\lambda(t)\rangle_{V'\times V}, \quad \text{a.e. } t \in (0,T). \tag{55}$$

We integrate equation (55) with respect to time, use (13)–(18), the definition (46) of the function $P_{\nu\lambda}$, and the regularity (30) of the initial data $\boldsymbol{u}_0$. After some calculation, based on monotonicity arguments, we obtain that there exists $T_0 \in (0,T]$ such that

$$\|\dot{\boldsymbol{u}}_\lambda\|^2_{L_\infty(0,T_0;\boldsymbol{L}_2(\Omega))} + \|\dot{\boldsymbol{u}}_\lambda\|^2_{L_2(0,T_0;V)} \\ +\|\boldsymbol{u}_\lambda\|^2_{L_\infty(0,T_0;V)} + \|P_\lambda(u_{\lambda\nu})\|_{L_\infty(0,T_0;L_1(\Gamma_3))} \le c. \tag{56}$$

Here and below $u_{\lambda\nu}$ and $\sigma_{\lambda\nu}$ represent the normal trace of $\boldsymbol{u}_\lambda$ and $\boldsymbol{\sigma}_\lambda$, respectively. Also, note that the restriction of the length of the interval of time arise from the need to obtain a convenient estimate involving the integral term in (48); a similar argument will be used in the step (v) of the proof which we present below.

(ii) ***Dual* estimate.** To obtain the *dual* estimate we test in (49) with an arbitrary element $\boldsymbol{w} \in L_2(0,T_0;\mathring{\boldsymbol{H}}^1(\Omega))$. This together with (56) yields

$$\|\ddot{\boldsymbol{u}}_\lambda\|^2_{L_2(0,T_0;\boldsymbol{H}^{-1}(\Omega))} \le c. \tag{57}$$

Interpolating (56) and (57) we finally arrive at

$$\|\dot{\boldsymbol{u}}_\lambda\|^2_{\boldsymbol{H}^{1/2,1}(Q_{T_0})} + \|\dot{\boldsymbol{u}}_\lambda\|^2_{L_\infty(0,T_0;\boldsymbol{L}_2(\Omega))} + \|P_\lambda(u_{\lambda\nu})\|_{L_\infty(0,T_0,L_1(\Gamma_3))} \le c. \tag{58}$$

Hence by the standard use of the extension operator and Fourier transform, we can prove that $\ddot{\boldsymbol{u}}$ belongs to the dual of the space $H^{1/2,1}(Q_{T_0})$ and (49) has a sense from test functions from $H^{1/2,1}(Q_{T_0})$. For details cf. [2], Chapter 2. Moreover, since $-\sigma_{\lambda\nu} = p_{\nu\lambda}(u_{\lambda\nu}) - \gamma_\nu R_\nu(u_{\lambda\nu})\beta_\lambda^2$ on $S_3$, using the Green formula and standard trace theorem we have

$$\|p_{\nu\lambda}(u_{\lambda\nu})\|_{H^{-1/4,-1/2}(S_{T_0 3})} \le c. \tag{59}$$

(iii) **First convergence results as $\lambda \to 0$.** We prove now some convergence results involving the approximate displacement field $\boldsymbol{u}_\lambda$. To this end, consider a sequence of positive numbers $\{\lambda_n\}$ converging to zero as $n \to \infty$. The validity of (56)–(58) shows that there exists an element $\boldsymbol{u}$ such that

$$\dot{\boldsymbol{u}} \in \boldsymbol{H}^{1/2,1}(Q_{T_0}) \cap L_\infty(0, T_0; \boldsymbol{L}_2(\Omega)) \tag{60}$$

and, for a subsequence $\{\lambda_{n_k}\} \subset \{\lambda_n\}$, the following convergences hold as $k \to \infty$:

$$\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_k) \rightharpoonup \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) \text{ in } L_2(0, T_0; \mathcal{H}), \tag{61}$$

$$\ddot{\boldsymbol{u}}_k \rightharpoonup \ddot{\boldsymbol{u}} \quad \text{ in } L_2(0, T_0; \boldsymbol{H}^{-1}(\Omega)), \tag{62}$$

$$\dot{\boldsymbol{u}}_k \rightharpoonup \dot{\boldsymbol{u}} \quad \text{ in } \boldsymbol{H}^{1/2,1}(Q_{T_0}), \tag{63}$$

$$\dot{\boldsymbol{u}}_k \to \dot{\boldsymbol{u}} \quad \text{ in } \boldsymbol{L}_2(Q_{T_0}), \tag{64}$$

$$\boldsymbol{u}_k \to \boldsymbol{u} \quad \text{ in } \boldsymbol{L}_2(S_{T_0}), \tag{65}$$

$$\dot{\boldsymbol{u}}_k \to \dot{\boldsymbol{u}} \quad \text{ in } \boldsymbol{L}_2(S_{T_0}). \tag{66}$$

Here and below we use the notation $\boldsymbol{u}_k = \boldsymbol{u}_{\lambda_{n_k}}$ and $\lambda_k = \lambda_{n_k}$. Indeed, (64) follows from (63) by the standard compact imbedding theorem. An analogous argument works also for (65), and it is based on the convergence in the space $H^1(0, T_0; \boldsymbol{L}_2(\Omega)) \cap L_2(0, T_0; \boldsymbol{H}^{1/2}(\Gamma))$.

(iv) **$\boldsymbol{u}$ is an locally admissible displacement field.** We use the notation $p_k = p_{\nu\lambda_{n_k}}$ and we denote by $u_{k\nu}$ the normal trace of $\boldsymbol{u}_k$. Let $k \in \mathbb{N}$. It follows from (59) that

$$\int_0^{T_0} \int_{\Gamma_3} p_k(u_{k\nu})(u_{k\nu} - g)\, da\, dt \le c \tag{67}$$

which implies that

$$\int_0^{T_0} \int_{\Gamma_3 \cap \{u_{k\nu} \le g\}} p_k(u_{k\nu}) u_{k\nu}\, da\, dt - \int_0^{T_0} \int_{\Gamma_3 \cap \{u_{k\nu} \le g\}} p_k(u_{k\nu}) g\, da\, dt$$
$$+ \int_0^{T_0} \int_{\Gamma_3 \cap \{u_{k\nu} > g\}} p_k(u_{k\nu})(u_{k\nu} - g)\, da\, dt \le c.$$

We neglect the first term in the left hand side of the previous inequality and note that $p_k(u_{k\nu}) \le p(g)$ on $\Gamma_3 \cap \{u_{k\nu} \le g\}$. As a result we obtain

$$\int_0^{T_0} \int_{\Gamma_3 \cap \{u_{k\nu} > g\}} p_k(u_{k\nu})(u_{k\nu} - g)\, da\, dt \le c. \tag{68}$$

We use in (68) the definition of the function $p_k$, (45), and elementary manipulations to see that

$$\frac{1}{\lambda_k}\int_0^{T_0}\int_{\Gamma_3\cap\{u_{k\nu}>g\}}(u_{k\nu}-g)^2\,da\,dt\le c.$$

This last inequality shows that

$$\int_0^{T_0}\int_{\Gamma_3}\big[(u_{k\nu}-g)_+\big]^2\,da\,dt\le c\lambda_k. \tag{69}$$

We pass now to the limit in (69) as $k\to\infty$ and use (65) to see that

$$\int_0^{T_0}\int_{\Gamma_3}\big[(u_\nu-g)_+\big]^2\,da\,dt\le 0,$$

which shows that $(u_\nu(t)-g)_+=0$ a.e. on $\Gamma_3$, for all $t\in[0,T_0]$. We conclude that

$$\boldsymbol{u}(t)\in K \qquad \forall\, t\in[0,T_0], \tag{70}$$

i.e. $\boldsymbol{u}$ is an locally admissible displacement field.

(v) **A strong convergence result.** Let $k\in\mathbb{N}$. Consider the functions $\boldsymbol{\sigma}_k^I$, $\boldsymbol{\sigma}^I$ and $\beta$ defined by the equalities

$$\boldsymbol{\sigma}_k^I(t)=\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_k(t))+\int_0^t\mathcal{G}(\boldsymbol{\sigma}_k^I(s),\boldsymbol{\varepsilon}(\boldsymbol{u}_k(s)))\,ds, \tag{71}$$

$$\boldsymbol{\sigma}^I(t)=\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))+\int_0^t\mathcal{G}(\boldsymbol{\sigma}^I(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,ds, \tag{72}$$

$$\beta(t)=-\int_0^t\big(\beta(s)\,(\gamma_\nu R_\nu(u_\nu(s))^2+\gamma_\tau\|\boldsymbol{R}(\boldsymbol{u}_\tau(s)\|^2)-\epsilon_a\big)_+\,ds+\beta_0, \tag{73}$$

for all $t\in[0,T_0]$. The definition of these functions is based on the Banach fixed point theorem, which show that the integral equations (71), (72) and (74) have a unique solution. In addition we denote $\beta_k=\beta_{\lambda_{n_k}}$. It follows from (50) and (51) that $\beta_k$ satisfies

$$\beta_k(t)=-\int_0^t\big(\beta_k(s)\,(\gamma_\nu R_\nu(u_{k\nu}(s))^2+\gamma_\tau\|\boldsymbol{R}(\boldsymbol{u}_{k\tau}(s)\|^2)-\epsilon_a\big)_+\,ds+\beta_0 \tag{74}$$

for all $t \in [0, T_0]$ where, here and below, $\boldsymbol{u}_{k\tau}$ represents the tangential trace of $\boldsymbol{u}_k$.

We write (49) for $\lambda = \lambda_k$, take $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{u}_k$ and use (71) to obtain

$$\begin{aligned}
&\langle \ddot{\boldsymbol{u}}_k, \boldsymbol{u} - \boldsymbol{u}_k \rangle_{V' \times V} + (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_k), \boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_k))_{\mathcal{H}} + (\boldsymbol{\sigma}_k^I, \boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_k))_{\mathcal{H}} \\
&\quad + \int_{\Gamma_3} p_k(u_{k\nu})(u_\nu - u_{k\nu})\, da - \int_{\Gamma_3} \gamma_\nu R_\nu(u_{k\nu}) \beta_k^2 (u_\nu - u_{k\nu})\, da \\
&\quad + \int_{\Gamma_3} p_\tau(\beta_k) \boldsymbol{R}(\boldsymbol{u}_{k\tau}) \cdot (\boldsymbol{u}_\tau - \boldsymbol{u}_{k\tau})\, da \langle \boldsymbol{f}, \boldsymbol{u} - \boldsymbol{u}_k \rangle_{V' \times V}
\end{aligned}$$

a.e. on $(0, T_0)$. Next, using the monotonicity of the function $p_k$, (70) and the properties of the operators $R_\nu$ and $\boldsymbol{R}_\tau$ we obtain

$$\begin{aligned}
&\langle \ddot{\boldsymbol{u}}_k, \boldsymbol{u} - \boldsymbol{u}_k \rangle_{V' \times V} + (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_k), \boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_k))_{\mathcal{H}} \\
&\qquad + (\boldsymbol{\sigma}_k^I, \boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_k))_{\mathcal{H}} + \int_{\Gamma_3} p_\nu(u_\nu)(u_\nu - u_{k\nu})\, da \\
&\qquad + c \int_{\Gamma_3} \|\boldsymbol{u} - \boldsymbol{u}_k\|\, da \geq \langle \boldsymbol{f}, \boldsymbol{u} - \boldsymbol{u}_k \rangle_{V' \times V},
\end{aligned}$$

a.e. on $(0, T_0)$, which shows that

$$\begin{aligned}
&(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_k - \dot{\boldsymbol{u}}), \boldsymbol{\varepsilon}(\boldsymbol{u}_k - \boldsymbol{u}))_{\mathcal{H}} + (\boldsymbol{\sigma}_k^I - \boldsymbol{\sigma}^I, \boldsymbol{\varepsilon}(\boldsymbol{u}_k - \boldsymbol{u}))_{\mathcal{H}} \\
&\quad \leq (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}), \boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_k))_{\mathcal{H}} + (\boldsymbol{\sigma}^I, \boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_k))_{\mathcal{H}} + \langle \ddot{\boldsymbol{u}}_k, \boldsymbol{u} - \boldsymbol{u}_k \rangle_{V' \times V} \\
&\qquad + \int_{\Gamma_3} p_\nu(u_\nu)(u_\nu - u_{k\nu})\, da + c \int_{\Gamma_3} \|\boldsymbol{u} - \boldsymbol{u}_k\|\, da + \langle \boldsymbol{f}, \boldsymbol{u}_k - \boldsymbol{u} \rangle_{V' \times V}.
\end{aligned}$$

a.e. on $(0, T_0)$. Let $t \in [0, T_0]$. We integrate the previous inequality over $[0, t]$, use standard integration by parts and the initial conditions to find that

$$\begin{aligned}
&(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_k(t)-\boldsymbol{u}(t)),\boldsymbol{\varepsilon}(\boldsymbol{u}_k(t)-\boldsymbol{u}(t)))_{\mathcal{H}} \\
&\qquad+\int_0^t(\boldsymbol{\sigma}_k^I(s)-\boldsymbol{\sigma}^I(s),\boldsymbol{\varepsilon}(\boldsymbol{u}_k(s)-\boldsymbol{u}(s)))_{\mathcal{H}}\,ds \\
&\le\int_0^t(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(s)),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)-\boldsymbol{u}_k(s)))_{\mathcal{H}}\,ds \\
&\qquad+\int_0^t(\boldsymbol{\sigma}^I(s),\boldsymbol{\varepsilon}(\boldsymbol{u}_k(s)-\boldsymbol{u}(s)))_{\mathcal{H}}\,ds \\
&\qquad+\int_0^t\langle\dot{\boldsymbol{u}}_k(s),\dot{\boldsymbol{u}}_k(s)-\dot{\boldsymbol{u}}(s)\rangle_{V'\times V}\,ds-(\dot{\boldsymbol{u}}_k(t),\boldsymbol{u}_k(t)-\boldsymbol{u}(t))_H \\
&\qquad+\int_0^t\int_{\Gamma_3}p_\nu(u_\nu)(u_\nu-u_{k\nu})\,da\,dt+c\int_0^t\int_{\Gamma_3}\|\boldsymbol{u}-\boldsymbol{u}_k\|\,da \\
&\qquad+\int_0^t\langle\boldsymbol{f},\boldsymbol{u}_k-\boldsymbol{u}\rangle_{V'\times V}\equiv C_t(k).
\end{aligned}\tag{75}$$

With the bound

$$(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_k(t)-\boldsymbol{u}(t)),\boldsymbol{\varepsilon}(\boldsymbol{u}_k(t)-\boldsymbol{u}(t)))_{\mathcal{H}}\ge 0,$$

inequality (75) leads to

$$\int_0^t(\boldsymbol{\sigma}_k^I(s)-\boldsymbol{\sigma}^I(s),\boldsymbol{\varepsilon}(\boldsymbol{u}_k(s)-\boldsymbol{u}(s)))_{\mathcal{H}}\,ds\le C_t(k).\tag{76}$$

On the other hand, it follows from (71) and (72) that

$$\begin{aligned}
&(\boldsymbol{\sigma}_k^I(s)-\boldsymbol{\sigma}^I(s),\boldsymbol{\varepsilon}(\boldsymbol{u}_k(s)-\boldsymbol{u}(s)))_{\mathcal{H}} \\
&=(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)-\boldsymbol{u}_k(s)),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)-\boldsymbol{u}_k(s)))_{\mathcal{H}} \\
&+\Big(\int_0^s\big[\mathcal{G}(\boldsymbol{\sigma}_k^I(r),\boldsymbol{\varepsilon}(\boldsymbol{u}_k(r))-\mathcal{G}(\boldsymbol{\sigma}^I(r),\boldsymbol{\varepsilon}(\boldsymbol{u}(r))\big]\,dr,\ \boldsymbol{\varepsilon}(\boldsymbol{u}_k(s)-\boldsymbol{u}(s))\Big)_{\mathcal{H}}.
\end{aligned}\tag{77}$$

for all $s\in[0,T]$. We combine (76) and (77) and use assumption (16) and

(17) on the operators $\mathcal{E}$ and $\mathcal{G}$ to obtain

$$e_0 \int_0^t \|\boldsymbol{\varepsilon}(\boldsymbol{u}_k(s) - \boldsymbol{u}(s)))\|_{\mathcal{H}}^2\, ds \le C_t(k) + \tag{78}$$
$$c\,T_0\Big(\int_0^t \big[\|\boldsymbol{\sigma}_k^I(r) - \boldsymbol{\sigma}^I(r)\|_{\mathcal{H}} + \|\boldsymbol{\varepsilon}(\boldsymbol{u}_k(r) - \boldsymbol{u}(r))\|_{\mathcal{H}}\big]\, dr\Big)$$
$$\cdot \int_0^t \|\boldsymbol{\varepsilon}(\boldsymbol{u}_k(s) - \boldsymbol{u}(s))\|_{\mathcal{H}}\, ds.$$

We use again (71), (72), (17) and Gronwall's inequality to see that

$$\|\boldsymbol{\sigma}_k^I(r) - \boldsymbol{\sigma}^I(r)\|_{\mathcal{H}} \le c\Big(\|\boldsymbol{\varepsilon}(\boldsymbol{u}_k(r) - \boldsymbol{u}(r))\|_{\mathcal{H}} \tag{79}$$
$$+ \int_0^r \|\boldsymbol{\varepsilon}(\boldsymbol{u}_k(\xi) - \boldsymbol{u}(\xi))\|_{\mathcal{H}}\, d\xi\Big) \quad \forall\, r \in [0, T_0],$$

and using this inequality in (78) we obtain

$$e_0 \int_0^t \|\boldsymbol{\varepsilon}(\boldsymbol{u}_k(s) - \boldsymbol{u}(s))\|_{\mathcal{H}}^2\, ds \le C_t(k) + \tag{80}$$
$$c\,T_0(1 + T_0)\Big(\int_0^t \|\boldsymbol{\varepsilon}(\boldsymbol{u}_k(s) - \boldsymbol{u}(s))\|_{\mathcal{H}}\, ds\Big)^2.$$

Since

$$\Big(\int_0^t \|\boldsymbol{\varepsilon}(\boldsymbol{u}_k(s) - \boldsymbol{u}(s))\|_{\mathcal{H}}\, ds\Big)^2 \le T_0 \int_0^t \|\boldsymbol{\varepsilon}(\boldsymbol{u}_k(s) - \boldsymbol{u}(s))\|_{\mathcal{H}}^2\, ds,$$

it follows from (80) that for $T_0$ small enough we have

$$\int_0^t \|\boldsymbol{\varepsilon}(\boldsymbol{u}_k(s) - \boldsymbol{u}(s))\|_{\mathcal{H}}^2\, ds \le c\,C_t(k). \tag{81}$$

We use now the convergences (61)–(66) and the definition of $C_t(k)$ in (75) to see that

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_k) \to \boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in} \quad L_2(0, T_0; \mathcal{H}), \quad \text{as } k \to \infty. \tag{82}$$

This convergence combined with inequality (79) shows that

$$\boldsymbol{\sigma}_k^I \to \boldsymbol{\sigma}^I \quad \text{in} \quad L_2(0, T_0; \mathcal{H}), \quad \text{as } k \to \infty. \tag{83}$$

Finally, we note that from (74) and (73) it follows that

$$\|\beta_k(t)-\beta(t))\|_{L_2(\Gamma_3)} \le \int_0^t \|\boldsymbol{u}_k(s)-\boldsymbol{u}(s)\|_V\,ds$$

and, combining this inequality with the convergence (65) yields

$$\beta_k \to \beta \quad \text{in} \quad L_2(0,T_0;L_2(\Gamma_3)), \quad \text{as } k\to\infty. \tag{84}$$

(vi) **Existence of the solution.** Let $k\in\mathbb{N}$. We write (49) for $\lambda=\lambda_k$, take $\boldsymbol{w}=\boldsymbol{v}-\boldsymbol{u}_k$ where $\boldsymbol{v}\in\mathcal{K}$ is an arbitrary test function and use (71) to obtain

$$\begin{aligned}&\langle \ddot{\boldsymbol{u}}_k,\boldsymbol{v}-\boldsymbol{u}_k\rangle_{V'\times V}+(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_k),\boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u}_k))_{\mathcal{H}}+(\boldsymbol{\sigma}_k^I,\boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u}_k))_{\mathcal{H}}\\&+\int_{\Gamma_3}p_k(u_{k\nu})(v_\nu-u_{k\nu})\,da-\int_{\Gamma_3}\gamma_\nu R_\nu(u_{k\nu})\beta_k^2(v_\nu-u_{k\nu})\,da\\&+\int_{\Gamma_3}p_\tau(\beta_k)\boldsymbol{R}(\boldsymbol{u}_{k\tau})\cdot(\boldsymbol{v}_\tau-\boldsymbol{u}_{k\tau})\,da=\langle\boldsymbol{f},\boldsymbol{v}-\boldsymbol{u}_k\rangle_{V'\times V}\end{aligned} \tag{85}$$

a.e. on $(0,T_0)$. Now, since the function $p_k$ is nondecreasing and $\boldsymbol{v}\in\mathcal{K}$ we find that

$$\int_{\Gamma_3}p_k(u_{k\nu})(v_\nu-u_{k\nu})\,da \le \int_{\Gamma_3}p_k(v_\nu)(v_\nu-u_{k\nu})\,da$$

and, using this inequality in (85), yields

$$\begin{aligned}&\langle \ddot{\boldsymbol{u}}_k,\boldsymbol{v}-\boldsymbol{u}_k\rangle_{V'\times V}+(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_k),\boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u}_k))_{\mathcal{H}}+(\boldsymbol{\sigma}_k^I,\boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u}_k))_{\mathcal{H}}\\&+\int_{\Gamma_3}p_\nu(v_\nu)(v_\nu-u_{k\nu})\,da-\int_{\Gamma_3}\gamma_\nu R_\nu(u_{k\nu})\beta_k^2(v_\nu-u_{k\nu})\,da\\&+\int_{\Gamma_3}p_\tau(\beta_k)\boldsymbol{R}(\boldsymbol{u}_{k\tau})\cdot(\boldsymbol{v}_\tau-\boldsymbol{u}_{k\tau})\,da\ge\langle\boldsymbol{f},\boldsymbol{v}-\boldsymbol{u}_k\rangle_{V'\times V}\end{aligned}$$

a.e. on $(0,T_0)$. We integrate the last inequality on $(0,T_0)$, perform integration by parts and use the convergences (61)–(64), (82), (83) and (84) to obtain

$$\begin{aligned}
&\int_0^{T_0} (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}(t) - \boldsymbol{u}(t)))_{\mathcal{H}}\, dt \qquad (86)\\
&\quad + \int_0^{T_0} (\boldsymbol{\sigma}^I(t), \boldsymbol{\varepsilon}(\boldsymbol{v}(t) - \boldsymbol{u}(t)))_{\mathcal{H}}\, dt - \int_0^{T_0} (\dot{\boldsymbol{u}}(t), \dot{\boldsymbol{v}}(t) - \dot{\boldsymbol{u}}(t))_H\, dt\\
&\quad + \int_0^{T_0} \int_{\Gamma_3} p_\nu(v_\nu(t))(v_\nu(t) - u_\nu(t))\, dt\\
&\quad - \int_0^{T_0} \int_{\Gamma_3} \gamma_\nu R_\nu(u_\nu(t))\beta^2(t)(v_\nu(t) - u_\nu(t))\, da\, dt\\
&\quad + \int_0^{T_0} \int_{\Gamma_3} p_\tau(\beta(t))\boldsymbol{R}(\boldsymbol{u}_\tau(t)) \cdot (\boldsymbol{v}_\tau(t) - \boldsymbol{u}_\tau(t))\, da\, dt\\
&\quad + (\dot{\boldsymbol{u}}(T_0), \boldsymbol{v}(T_0) - \boldsymbol{u}(T_0))_H\\
&\geq \int_0^{T_0} \langle \boldsymbol{f}(t), \boldsymbol{v}(t) - \boldsymbol{u}(t)\rangle_{V' \times V}\, dt + (\boldsymbol{u}_1, \boldsymbol{v}(0) - \boldsymbol{u}_0)_H \quad \forall\, \boldsymbol{v} \in \mathcal{K}.
\end{aligned}$$

Next, we take $\boldsymbol{v} = \boldsymbol{u} + \theta(\boldsymbol{w} - \boldsymbol{u})$ in (86), where $\boldsymbol{w}$ is arbitrary in $\mathcal{K}$ and $\theta \in ]0, 1[$, then we divide the resulting inequality by $\theta$. As a result we find

$$\begin{aligned}
&\int_0^{T_0} (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}(t) - \boldsymbol{u}(t)))_{\mathcal{H}}\, dt \qquad (87)\\
&\quad + \int_0^{T_0} (\boldsymbol{\sigma}^I(t), \boldsymbol{\varepsilon}(\boldsymbol{w}(t) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathcal{H}}\, dt - \int_0^{T_0} (\dot{\boldsymbol{u}}(t), \dot{\boldsymbol{w}}(t) - \dot{\boldsymbol{u}}(t))_H\, dt\\
&\quad + \int_0^{T_0} \int_{\Gamma_3} p_\nu(u_\nu(t) + \theta(w_\nu(t) - u_\nu(t)))(w_\nu(t) - u_\nu(t))\, da\, dt\\
&\quad - \int_0^{T_0} \int_{\Gamma_3} \gamma_\nu R_\nu(u_\nu(t) + \theta(w_\nu(t) - u_\nu(t))\beta^2(t)(w_\nu(t) - u_\nu(t))\, da\, dt\\
&\quad + \int_0^{T_0} \int_{\Gamma_3} p_\tau(\beta(s))\boldsymbol{R}(\boldsymbol{u}_\tau(t) + \theta(w_\tau(t) - \boldsymbol{u}_\tau(t))\\
&\qquad\qquad \cdot (\boldsymbol{w}_\tau(t) - \boldsymbol{u}_\tau(t))\, da\, dt + (\dot{\boldsymbol{u}}(T_0), \boldsymbol{w}(T_0) - \boldsymbol{u}(T_0))_H\\
&\geq \int_0^{T_0} \langle \boldsymbol{f}(t), \boldsymbol{w}(t) - \boldsymbol{u}(t)\rangle_{V' \times V}\, dt + (\boldsymbol{u}_1, \boldsymbol{w}(0) - \boldsymbol{u}_0)_H \quad \forall\, \boldsymbol{w} \in \mathcal{K}.
\end{aligned}$$

We now use the properties (15) of the functions $p_\nu$, $R_\nu$ and $R_\tau$ to see that, as $\theta \to 0$, the following convergences hold:

$$\int_0^{T_0}\int_{\Gamma_3} p_\nu(u_\nu(t)+\theta(w_\nu(t)-u_\nu(t)))(w_\nu(t)-u_\nu(t))\,da\,dt \tag{88}$$
$$\to \int_0^{T_0}\int_{\Gamma_3} p_\nu(u_\nu(t))(w_\nu(t)-u_\nu(t))\,da\,dt,$$

$$\int_0^{T_0}\int_{\Gamma_3} \gamma_\nu R_\nu(u_\nu(t)+\theta(w_\nu(t)-u_\nu(t)))\beta^2(t)(v_\nu(t)-u_\nu(t))\,da\,dt \tag{89}$$
$$\to \int_0^{T_0}\int_{\Gamma_3} \gamma_\nu R_\nu(u_\nu(t))\beta^2(t)(v_\nu(t)-u_\nu(t))\,da\,dt,$$

$$\int_0^{T_0}\int_{\Gamma_3} p_\tau(\beta(t))\boldsymbol{R}(\boldsymbol{u}_\tau(t))+\theta(w_\tau(t)-\boldsymbol{u}_\tau(t))\cdot(\boldsymbol{w}_\tau(t)-\boldsymbol{u}_\tau(t))\,da\,dt \tag{90}$$
$$\to \int_0^{T_0}\int_{\Gamma_3} p_\tau(\beta(t))\boldsymbol{R}(\boldsymbol{u}_\tau(t))\cdot(\boldsymbol{w}_\tau(t)-\boldsymbol{u}_\tau(t))\,da\,dt.$$

Therefore, passing to the limit in (87) as $\theta \to 0$ and using (88)–(90) and the definition (27) of the functional $j$ we obtain

$$\begin{aligned}
&\int_0^{T_0}(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)),\boldsymbol{\varepsilon}(\boldsymbol{w}(t)-\boldsymbol{u}(t)))_{\mathcal{H}}\,dt \\
&\quad+\int_0^{T_0}(\boldsymbol{\sigma}^I(t),\boldsymbol{\varepsilon}(\boldsymbol{w}(t)-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathcal{H}}\,dt-\int_0^{T_0}(\dot{\boldsymbol{u}}(t),\dot{\boldsymbol{w}}(t)-\dot{\boldsymbol{u}}(t))_H\,dt \\
&\quad+\int_0^{T_0} j(\beta(t),\boldsymbol{u}(t),\boldsymbol{w}(t)-\boldsymbol{u}(t))\,dt+(\dot{\boldsymbol{u}}(T_0),\boldsymbol{w}(T_0)-\boldsymbol{u}(T_0))_H \\
&\geq \int_0^{T_0}\langle \boldsymbol{f}(t),\boldsymbol{w}(t)-\boldsymbol{u}(t)\rangle_{V'\times V}\,dt+(\boldsymbol{u}_1,\boldsymbol{w}(T_0)-\boldsymbol{u}_0)_H \quad \forall\,\boldsymbol{w}\in\mathcal{K}.
\end{aligned} \tag{91}$$

Let $\boldsymbol{\sigma} : [0,T_0] \to \mathcal{H}$ be the function given by

$$\boldsymbol{\sigma}(t)=\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))+\boldsymbol{\sigma}^I(t) \qquad \text{a.e. } t\in(0,T_0). \tag{92}$$

It follows from (91), (92) and (72) that $(\boldsymbol{u},\boldsymbol{\sigma},\beta)$ satisfy (38), (39) on the interval $(0,T_0)$. Also, (73) implies that (40) and holds on $(0,T_0)$, too, and the initial condition (41) is satisfied. It follows from (60), (72), (92) and

(73) that $(\boldsymbol{u}, \boldsymbol{\sigma}, \beta)$ has the regularity expressed in (42)–(44) on the time interval $(0, T_0)$. We conclude that $(\boldsymbol{u}, \boldsymbol{\sigma}, \beta)$ is a local solution of the Problem $\mathcal{P}^V$. Using now the standard successive approximation argument we obtain a solution on the whole interval $(0, T)$, which concludes the proof. $\diamond$

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