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COMPACTNESS METHODS FOR AN INTEGRO-DIFFERENTIAL EQUATION WITH MEASURES[*][†]

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Abstract

In this paper, using some compactness arguments, we prove some local or even global existence results for the $\mathcal{L}^\infty$-solution to an integro-differential Cauchy problem with distributed measures in a real Banach space. An example involving the Dirac measure concentrated at point is included.

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1 Introduction

The main goal of the present paper is to prove some sufficient conditions for the local, or global existence of the $\mathcal{L}^\infty$-solution for the Cauchy problem

$$\begin{cases} du = \left(Au + \int_a^t k\,(t, \tau, u\,(\tau))\,d\tau\right) dt + dg \\ u\,(a) = \xi, \end{cases} \tag{1}$$

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where X is a real Banach space, $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a C_0-semigroup of contractions $\{S(t); t \geq 0\}$, with $R(\lambda, A) = (\lambda I - A)^{-1}$ compact for each $\lambda > 0$, $\xi \in X$, $g \in BV([a, b]; X)$ and $k : \Delta_{[a,b]} \times U \to X$ is a continuous function, while $\Delta_{[a,b]} = \{(s, \tau) \in \mathbb{R}^2; a \leq \tau \leq s \leq b\}$ and U is a nonempty and open subset in X. To this aim, we prove a necessary and sufficient condition for the compactness of the $\mathcal{L}^\infty$-solution operator $(\xi, g) \mapsto u$ associated to the nonhomogeneous linear Cauchy problem of the type

$$\begin{cases} du = (Au)\, dt + dg \\ u(a) = \xi. \end{cases} \tag{2}$$

Also, we prove some results concerning saturated $\mathcal{L}^\infty$ -solution for (1). An example of integro-differential Cauchy problem involving the Dirac measure concentrated at point is included.

The compactness of the $\mathcal{L}^\infty$-solution operator from $X \times BV([a, b]; X)$ to $L^p(a, b; X)$ was studied by Vrabie [35]. Our result, in Section 2, refers to the case $R(\lambda, A) = (\lambda I - A)^{-1}$ compact, for each $\lambda > 0$. Further, when the $\mathcal{L}^\infty$-solution operator applies $X \times BV([a, b]; X) \cap C([a, b]; X)$ into $C([a, b]; X)$, the problem was studied by Grosu [16], assuming that the C_0-semigroup of contractions is compact; by Grosu [18] when the C_0-semigroup of contractions is not necessarily compact (by imposing some conditions on X and g); by Grosu [19] assuming that, for each $\lambda > 0$, the operator $R(\lambda, A) = (\lambda I - A)^{-1}$ is compact.

In the classical case when g is defined by a density, i.e. there exists $f \in L^1(a, b; X)$ such that $dg(s) = f(s)\, ds$, similar compactness results for the mild-solution operator are due to Baras, Hassan, Veron [10], Pazy [28] and Vrabie [36]. The corresponding nonlinear case has been considered by Baras [9], Mitidieri, Vrabie [24] and Vrabie [30] , [32]. This kind of compactness properties are useful in establishing existence results for both Cauchy and periodic problems (see Ahmed [1], Amann [6], Grosu [15], [17], [20], Izsák [21], Mitidieri, Vrabie [24], [25], [26], Paicu [27], Vrabie [30], [31], [33], [34]), as well for optimal control problems with state constraints (see Ahmed [2], Amann [7], Barbu, Precupanu [12], Fattorini [14], Vrabie [36] and the references therein).

Krasnosel'skii *et.al* [22] were the first to investigate the solvability of an integro-differential equation (possible with measures, but non-distributed measures) using compactness arguments and various kind of fixed-point theorems. The study of the existence of various kinds of solutions for integro-differential problems corresponding to (1), but governed by m-accretive operators and with $g \equiv 0$, was treated by compactness arguments in Mitidieri, Vrabie [24], [25], [26], Vrabie [31], [34]. See also Aizicovici, Hannsgen [3], Aizicovici, Staicu [4], Izsák [21]. By using the Leray-Schauder alternative, Lamb, Dhakne [23] study the local and global existence of mild solution of a nonlinear integro-differential equation in Banach space, with A the infinitesimal generator of a strongly continuous semigroup of bounded linear operators. Bahuguna [8] establishes the existence, uniqueness, regularity and continuation of mild solutions for a class of integro-differential equations in an arbitrary Banach space, by using the theory of analytic semigroups combined with compactness arguments.

The paper is divided into five sections, the first being concerned with the introduction of the $\mathcal{L}^\infty$-solution for linear Cauchy problem involving measures (2) and such basic properties of the $\mathcal{L}^\infty$-solution as boundedness, regularity. The results are from Vrabie [35], [36]. Section 2 contains the statement and proof of our first main result, related to the compactness of the $\mathcal{L}^\infty$-solution operator $(\xi, g) \mapsto u$ associated to (2). In Section 3 we establish a theorem referring to local existence for $\mathcal{L}^\infty$-solution of (1). Section 4 includes some facts referring to saturated $\mathcal{L}^\infty$-solution for (1). Section 5 presents an significant example of a integro-differential Cauchy problem in which the Dirac measure is concentrated at point. We notice that the existence result in Section 3 cannot be obtained as a particular case of the existence results for de semilinear evolution equation involving measures in Grosu [15] and Vrabie [36], Chapter 12, where the semigroup generated by A is continuous from $]0, \infty[$ to $\mathcal{L}(X)$ in the uniform operator topology or, more than this, is a compact semigroup.

We assume familiarity with the basic concepts and results concerning C_0-semigroups and infinite-dimensional vector-valued functions of bounded variation and we refer to Barbu and Precupanu [12], Pazy [29] and Vrabie [36] for details. First, we recall for easy reference some results established in Vrabie [35], [36].

Let $g : [a, b] \to X$. Let $\mathcal{P}([a, b])$ be the set of all partitions of the interval $[a, b]$ and $\mathcal{P} \in \mathcal{P}([a, t])$, $\mathcal{P} : a = t_0 < t_1 < ... < t_k = b$. The number

$$Var_{\mathcal{P}}(g, [a, b]) = \sum_{i=0}^{k-1} \|g(t_{i+1}) - g(t_i)\|$$

is called the *variation of the function g relatively to the partition $\mathcal{P}$*. If

$$\sup_{\mathcal{P}\in\mathcal{P}([a,b])} Var_{\mathcal{P}}(g,[a,b]) < +\infty,$$

then g is said to be *of bounded variation*, and the number

$$Var(g,[a,b]) = \sup_{\mathcal{P}\in\mathcal{P}([a,b])} Var_{\mathcal{P}}(g,[a,b])$$

is called *the variation* of the function g on the interval $[a,b]$. We denote by $BV([a,b];X)$ the vector space of all functions of bounded variation from $[a,b]$ to X. Also, we denote by $BV(\mathbb{R};X)$ the space of all functions $g:\mathbb{R}\to X$ whose restrictions to any interval $[a,b]$ belong to $BV([a,b];X)$.

Proposition 1. *If $g\in BV([a,b];X)$, then g is piecewise continuous on $[a,b]$, i.e. there exists an at most countable subset E of $[a,b]$, such that g is continuous on $[a,b]\setminus E$ and, at each $t\in E\cap[a,b[$ (at each $s\in E\cap]a,b]$), there exists the one-sided limit $g(t+0)$ $(g(s-0))$.*

See Vrabie [36], Proposition 1.4.2, p. 14.

Definition 1. A family $\mathcal{G}$ in $BV([a,b];X)$ is *of equibounded variation on* $[a,b]$ if there exists $m_{\mathcal{G}}>0$ such that, for each $g\in\mathcal{G}$, we have

$$Var(g,[a,b]) \le m_{\mathcal{G}}.$$

Let $g\in BV([a,b];X)$ and let $\{S(t);t\ge 0\}$ be a C_0-semigroup of contractions[1] in a Banach space X. Let $\{S(t)^*;t\ge 0\}$ be the dual semigroup defined on X^*, and $\left\{S(t)^{\odot};t\ge 0\right\}$ the *sun dual* semigroup. We notice that $S(t)^{\odot}:X^{\odot}\to X^{\odot}$ is defined by

$$\begin{cases} X^{\odot} = \left\{x^*\in X^*; \lim\limits_{t\uparrow 0} S(t)^* x^* = x^*\right\} \text{ and} \\ S(t)^{\odot}x^{\odot} = S(t)^* x^{\odot}, \text{ for each } x^{\odot}\in X^{\odot}. \end{cases}$$

Let $X_A = (X^{\odot})^*$. We have that $X^{**}\subseteq X_A$. Since X_A depend of A, we call it *the space of admissible measures for A*. There exists a unique element $\int_a^t S(t-s)\,dg(s)\in X_A$ such that

$$\int_a^t S(t-s)\,dg(s) = \lim_{\mu(\mathcal{P})\downarrow 0}\sum_{i=0}^{k-1} S(t-\tau_i)\left(g(t_{i+1})-g(t_i)\right) \tag{3}$$

[1] All the results which will follows hold true also for the general case of C_0-semigroups not necessarily of contractions. However, for simplicity reasons, we preferred to consider only C_0-semigroups of contractions.

weakly-$\odot$ and it is called *the Riemann-Stieltjes integral* on $[a,t]$ of the operator-valued function $\tau \mapsto S(t-\tau)$ with respect to the vector-valued function g. See Vrabie [36], p. $205-206$.

If $\alpha : [a,b] \to \mathbb{R}$ is a given function, we define

$$\int_a^t \alpha(s) S(t-s)\, dg(s) = \lim_{\mu(\mathcal{P})\downarrow 0} \sum_{i=0}^{k-1} \alpha(\tau_i) S(t-\tau_i)(g(t_{i+1}) - g(t_i))$$

whenever the limit on the right-hand side exists in the weak-$\odot$ topology on X. This happens, for instance, if α is the characteristic function of a proper subinterval of $[a,t]$.

Remark 1. Since $\{S(t)\,;t\ge 0\}$ is a C_0-semigroup of contractions, it readily follows that, whenever $\int_a^t S(t-s)\,dg(s) \in X$, we have

$$\left\| \int_a^t S(t-s)\, dg(s) \right\| \le Var(g,[a,t]),$$

for each $t \in\,]a,b]$.

Remark 2.

(i) For each $c \in [a,b[$, and each $\delta > 0$ such that $c+\delta \in [a,b]$, and each $t \in [c+\delta, b]$, we have

$$\int_c^{c+\delta} S(t-s)\, dg(s) = \int_c^{c+\delta} \chi_{]c,c+\delta]} S(t-s)\, dg(s) + S(t-c)(g(c+0) - g(c)),$$

where $\aleph_{]c,c+\delta]}$ denotes the characteristic function of $]c,c+\delta]$. Similarly, for each $c \in\,]a,b]$, and each $\delta > 0$ such that $c-\delta \in [a,b]$, and each $t \in [c,b]$, we have

$$\int_{c-\delta}^{c} S(t-s)\, dg(s) = \int_{c-\delta}^{c} \chi_{[c-\delta,c[}(s) S(t-s)\, dg(s) +$$

$$+ S(t-c)(g(c) - g(c-0)).$$

See Vrabie [36], Remark 9.1.1, p. 207.

(ii) For each $g \in BV([a,b];X)$ and each $h \in]0, b-a[$ we have

$$\int_a^{b-h} \left\| \int_t^{t+h} S(t+h-s)\, dg(s) \right\| dt \le h\, Var(g,[a,b])$$

and

$$\int_{a+h}^{b} \left\| \int_{t-h}^{t} S(t-s)\, dg(s) \right\| dt \le h\, Var(g,[a,b]).$$

See Vrabie [36], Lemma 9.4.1, p.216.

Remark 3.

(i) If X is reflexive, the weak-$\odot$ topology on X is the weak topology on X, and therefore $X_A = X$.

See Vrabie [36], Remark 9.1.2, p. 207.

(ii) If g is *defined by a density*, i.e. there exists $f \in L^1(a,b;X)$ such that $dg(s) = f(s)\, ds$, then, for each $t \in [a,b]$, the limit in (3) exists in the norm topology of X and $\int_a^t S(t-s)\, dg(s) = \int_a^t S(t-s)\, f(s)\, ds \in X$. This happens, for example, whenever X has the Radon-Nicodym property and g is absolutely continuous on $[a,b]$, case in which $f = g'$ a.e. on $[a,b]$. Some specific but important such instances are those in which X is either reflexive, or have a separable dual.

See Vrabie [36], Remark 9.1.2, p. 207.

(iii) If $\{S(t)\,; t \ge 0\}$ is *continuous from* $]0,\infty[$ *to* $\mathcal{L}(X)$ *in the uniform operator topology*, then, for each $g \in BV([a,b];X)$ and $t \in]a,b]$, the limit in (3) exists in the norm topology of X and we have $\int_a^t S(t-s)dg(s) \in X$.

See Vrabie [36], Theorem 9.1.1, p. 208.

Assume that

(H_0) The Banach space X, the infinitesimal generator $A : D(A) \subseteq X \to X$ of a C_0-semigroup of contractions $\{S(t)\,; t \ge 0\}$ and the function $g \in BV([a,b];X)$ are such that $\int_a^t S(t-s)\, dg(s) \in X$. See Remark 3.

Definition 2. Assume that (H_0) is satisfied. The function $u : [a,b] \to X$ defined by

$$u(t) = S(t-a)\,\xi + \int_a^t S(t-s)\, dg(s), \tag{4}$$

for each $t \in [a,b]$, is called an $\mathcal{L}^\infty$*-solution on* $[a,b]$ of the problem (2).

See Vrabie [36], Definition 9.1.1, p. 209.

A similar concept was introduced by Amann [5] in the case in which A generates an analytic C_0-semigroup, by using a "transposition argument".

Remark 4. We notice that each $\mathcal{L}^\infty$-solution u satisfies

$$\|u(t)\| \le \|\xi\| + Var(g, [a,b]),$$

for each $t \in [a,b]$.

Theorem 1. (Regularity of $\mathcal{L}^\infty$-solutions) *Assume that* (H_0) *is satisfied. Let* $(a,\xi) \in \mathbb{R} \times X$. *Let* u *be the* $\mathcal{L}^\infty$*-solution of* (2) *corresponding to* ξ *and* g. *Then, for each* $t \in [a,b[$ *and each* $s \in]a,b]$, *there exists* $u(t+0)$ *and* $u^*(s-0) = \lim_{h\downarrow 0} S(h)\,u(s-h)$ *and*

$$\begin{cases} u(t+0) - u(t) = g(t+0) - g(t) \\ u(s) - u^*(s-0) = g(s) - g(s-0)\,. \end{cases}$$

If, in addition, either the semigroup generated by A *is continuous from* $]0,\infty[$ *to* $\mathcal{L}(X)$ *in the uniform operator topology, or it can be imbedded into a group, then for each* $s \in]a,b]$, *there exists* $u(s-0) = u^*(s-0)$. *So, in this case,* u *is continuous from the right* (*left*) *at* $t \in [a,b[$ $(t \in]a,b])$ *if and only if* g *is continuous from the right* (*left*) *at* t. *In particular,* u *is continuous at any point at which* g *is continuous and thus* u *is piecewise continuous on* $[a,b]$.

See Vrabie [36], Theorem 9.2.1, p. 210.

2 Compactness of the solution operator in $L^p(a,b;X)$ for $p \in [1,+\infty[$. The case when $(\lambda I - A)^{-1}$ is compact

Our goal here is to prove a necessary and sufficient condition in order that the family of all $\mathcal{L}^\infty$-solution of the problem (2), when ξ ranges in a given subset in X and g ranges in a subset of equibounded variation in $BV([a,b];X)$, be relatively compact in $L^p(a,b;X)$ for $p \in [1,+\infty[$.

We begin with some fundamental compactness result in $L^p(a,b;X)$.

Definition 3. A subset $\mathcal{F}$ in $L^p(a,b;X)$ is *p-equiintegrable* if

$$\lim_{h\downarrow 0}\int_a^{b-h}\|f(t+h)-f(t)\|^p\,dt=0,$$

uniformly for $f\in\mathcal{F}$.

Theorem 2. (Kolmogorov-Riesz-Weil) *A subset $\mathcal{F}$ in $L^p(a,b;X)$ is relatively compact if and only if*

(i) *$\mathcal{F}$ is p-equiintegrable ;*

(ii) *for each $[\alpha,\beta]\subset[a,b]$ the set*

$$\left\{\int_\alpha^\beta f(t)\,dt;f\in\mathcal{F}\right\}$$

is relatively compact in X.

See Vrabie [36], Theorem A.4.1, p. 305.

In that follows, we shall assume that, for each $(\xi,g)\in X\times BV([a,b];X)$, the Cauchy problem (2) has a unique $\mathcal{L}^\infty$-solution u and $u\in L^p(a,b;X)$ (see Definition 2, Remarks 3 and 4). Furthermore, for $p\in[1,+\infty[$, let us define the $\mathcal{L}^\infty$-solution operator, $Q:X\times BV([a,b];X)\to L^p(a,b;X)$, by

$$Q(\xi,g)=u.$$

The main result in this section is

Theorem 3. *Let $A:D(A)\subseteq X\to X$ the infinitesimal generator of a C_0-semigroup of contractions $\{S(t);t\geq 0\}$, with $R(\lambda,A)=(\lambda I-A)^{-1}$ compact for some $\lambda>0$. Let $\mathcal{D}$ be a bounded subset in X and $\mathcal{G}$ a subset in $BV([a,b];X)$ of equibounded variation. Then $Q(\mathcal{D},\mathcal{G})$ is relatively compact in $L^p(a,b;X)$ for each $p\in[1,+\infty[$ if and only if $Q(\mathcal{D},\mathcal{G})$ is 1 -equiintegrable.*

Proof. Necessity. We assume that $Q(\mathcal{D},\mathcal{G})$ is relatively compact subset in $L^p(a,b;X)$, for each $p\in[1,+\infty[$. By virtue of Theorem 2, for $\mathcal{F}=Q(\mathcal{D},\mathcal{G})$, the necessity is obvious, namely $Q(\mathcal{D},\mathcal{G})$ is 1-equiintegrable.

Sufficiency. To prove the sufficiency we also make use of the same Theorem 2 and Lemma A.1.2, p. 293 in Vrabie [36]. Indeed, let us assume that $Q(\mathcal{D},\mathcal{G})$ is 1-equiintegrable. First, let us observe that, by virtue of the Lebesque Dominated Convergence Theorem, it suffices to show that $Q(\mathcal{D}, \mathcal{G})$ is relatively compact in $L^1(a, b; X)$ and bounded in $\mathcal{L}^\infty(a, b; X)$.

To this aim, let us observe that, by hypothesis, there exists $m_{\mathcal{D}} > 0$ and $m_{\mathcal{G}} > 0$ such that

$$\|\xi\| \leq m_{\mathcal{D}} \text{ and } Var(g, [a, b]) \leq m_{\mathcal{G}},$$

for each $(\xi, g) \in \mathcal{D} \times \mathcal{G}$. Then, by virtue of Remark 4, for $M = m_{\mathcal{D}} + m_{\mathcal{G}} > 0$, we have that

$$\|Q(\xi, g)(t)\| \leq M, \tag{5}$$

for each $(\xi, g) \in \mathcal{D} \times \mathcal{G}$ and $t \in [a, b]$. So $Q(\mathcal{D}, \mathcal{G})$ is bounded in $\mathcal{L}^\infty(a, b; X)$.

Next, we will show that $Q(\mathcal{D}, \mathcal{G})$ is relatively compact in $L^1(a, b; X)$.

(i) $Q(\mathcal{D}, \mathcal{G})$ is 1-equiintegrable by hypothesis.

(ii) We prove that, for each $\alpha, \beta > 0$ such that $[\alpha, \beta] \subset [a, b]$, the set

$$\left\{ \int_\alpha^\beta Q(\xi, g)(t)\, dt; Q(\xi, g) \in Q(\mathcal{D}, \mathcal{G}) \right\} \tag{6}$$

is relatively compact in X. Indeed, let $\alpha, \beta > 0$ be such that $[\alpha, \beta] \subset [a, b]$ and let $\lambda > 0$. Let $I_\lambda : Q(\mathcal{D}, \mathcal{G}) \to X$ be defined by

$$I_\lambda(Q(\xi, g)) = \lambda R(\lambda, A) \int_\alpha^\beta Q(\xi, g)(t)\, dt. \tag{7}$$

Since $R(\lambda, A)$ is compact, an appeal to (5) shows that $I_\lambda(Q(\mathcal{D}, \mathcal{G}))$ is relatively compact in X.

To complete the proof, by virtue of Lemma $A.1.2$, p. 293 in Vrabie [36], it suffices to show that

$$\lim_{\lambda \to \infty} \|I_\lambda Q(\xi, g) - I(Q(\xi, g))\| = 0, \tag{8}$$

uniformly for $(\xi, g) \in \mathcal{D} \times \mathcal{G}$, where $I : Q(\mathcal{D}, \mathcal{G}) \to X$ is defined by

$$I(Q(\xi, g)) = \int_\alpha^\beta Q(\xi, g)(t)\, dt. \tag{9}$$

We recall that, if $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a C_0-semigroup of contraction $\{S(t); t \geq 0\}$, we have

$$R(\lambda; A)x = \int_0^\infty e^{-\lambda\tau} S(\tau) x d\tau,$$

for each $x \in X$. See Vrabie [36], Theorem 3.1.1 (Hille-Yosida), p. 52.

A computational argument shows that

$$\begin{aligned}
&\left\| I_\lambda \left(Q\left(\xi, g\right)\right) - I\left(Q\left(\xi, g\right)\right)\right\| \\
&= \left\| \lambda \int_0^\infty e^{-\lambda\tau} S\left(\tau\right) \left(\int_\alpha^\beta Q\left(\xi, g\right)\left(t\right) dt \right) d\tau - \int_\alpha^\beta Q\left(\xi, g\right)\left(t\right) dt \right\| \\
&= \left\| \lambda \int_0^\infty e^{-\lambda\tau} \left[S\left(\tau\right) \int_\alpha^\beta Q\left(\xi, g\right)\left(t\right) dt - \int_\alpha^\beta Q\left(\xi, g\right)\left(t\right) dt \right] d\tau \right\| \\
&\le \lambda \int_0^\infty e^{-\lambda\tau} \left\| S\left(\tau\right) \int_\alpha^\beta Q\left(\xi, g\right)\left(t\right) dt - \int_\alpha^\beta Q\left(\xi, g\right)\left(t\right) dt \right\| d\tau, \qquad (10)
\end{aligned}$$

for each $(\xi, g) \in \mathcal{D} \times \mathcal{G}$.

Case 1. $[\alpha, \beta] \subset [a, b[$ and $\alpha = a$. There exists $\eta = \eta(\alpha, \beta) > 0$ such that, for each $\tau \in]0, \eta]$, we have $[\alpha, \beta] \subset [a, b - \tau]$. Since $Q(\mathcal{D}, \mathcal{G})$ is 1-equiintegrable, we deduce that for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that, for each $\tau \in]0, \delta(\varepsilon)]$, we have

$$\int_a^{b-\tau} \left\| Q\left(\xi, g\right)\left(t + \tau\right) - Q\left(\xi, g\right)\left(t\right)\right\| dt < \varepsilon, \qquad (11)$$

for each $(\xi, g) \in \mathcal{D} \times \mathcal{G}$. Then, for $\gamma(\varepsilon) = \min(\eta, \delta(\varepsilon), b - a)$ and for each $\tau \in]0, \gamma(\varepsilon)]$, we have

$$\begin{aligned}
&\left\| S\left(\tau\right) \int_\alpha^\beta Q\left(\xi, g\right)\left(t\right) dt - \int_\alpha^\beta Q\left(\xi, g\right)\left(t\right) dt \right\| \\
&\le \left\| S\left(\tau\right) \int_a^{b-\tau} Q\left(\xi, g\right)\left(t\right) dt - \int_a^{b-\tau} Q\left(\xi, g\right)\left(t + \tau\right) dt \right\| \\
&\quad + \int_a^{b-\tau} \left\| Q\left(\xi, g\right)\left(t + \tau\right) - Q\left(\xi, g\right)\left(t\right)\right\| dt, \qquad (12)
\end{aligned}$$

for each $(\xi, g) \in \mathcal{D} \times \mathcal{G}$. Using Definition 2 and (ii) in Remark 2, we deduce that

$$\begin{aligned}
&\left\| S\left(\tau\right) \int_a^{b-\tau} Q\left(\xi, g\right)\left(t\right) dt - \int_a^{b-\tau} Q\left(\xi, g\right)\left(t + \tau\right) dt \right\| \\
&= \left\| \int_a^{b-\tau} \left[S\left(\tau\right) \left(S\left(t - a\right)\xi + \int_a^t S\left(t - s\right) dg\left(s\right) \right) \right.\right.
\end{aligned}$$

$$-\left(S\left(t+\tau-a\right)\xi+\int_a^t S\left(t+\tau-s\right)dg\left(s\right)\right)\Big]\,dt\Big\|$$

$$\leq \int_a^{b-\tau}\left\|\int_t^{t+\tau} S\left(t+\tau-s\right)dg\left(s\right)\right\|dt \leq \tau\,Var\left(g,[a,b]\right)\leq \tau m_{\mathcal{G}}, \tag{13}$$

for each $(\xi, g) \in \mathcal{D}\times\mathcal{G}$ and $\tau \in \left]0,\gamma\left(\varepsilon\right)\right]$.

Then, taking into account of (13) and (11), from (12) and (10), it follows that, for each μ, $0<\mu<\gamma\left(\varepsilon\right)$, we have

$$\left\|I_\lambda\left(Q\left(\xi,g\right)\right)-I\left(Q\left(\xi,g\right)\right)\right\|$$

$$\leq \lambda\int_0^{\mu} e^{-\lambda\tau}\left\|S\left(\tau\right)\int_a^{b-\tau}Q\left(\xi,g\right)\left(t\right)dt-\int_a^{b-\tau}Q\left(\xi,g\right)\left(t+\tau\right)dt\right\|d\tau$$

$$+\lambda\int_0^{\mu} e^{-\lambda\tau}\left(\int_a^{b-\tau}\left\|Q\left(\xi,g\right)\left(t+\tau\right)-Q\left(\xi,g\right)\left(t\right)dt\right\|\right)d\tau$$

$$+\lambda\int_\mu^{\infty} e^{-\lambda\tau}\left\|S\left(\tau\right)\int_a^{b-\tau}Q\left(\xi,g\right)\left(t\right)dt-\int_a^{b-\tau}Q\left(\xi,g\right)\left(t\right)dt\right\|d\tau$$

$$\leq \lambda\int_0^{\mu} e^{-\lambda\tau}\tau m_{\mathcal{G}}d\tau+\lambda\int_0^{\mu}e^{-\lambda\tau}\varepsilon d\tau+\lambda\int_\mu^\infty e^{-\lambda\tau}2M\left(b-\tau-a\right)d\tau$$

$$=m_{\mathcal{G}}\left(\lambda^{-1}-\mu e^{-\lambda\mu}-\lambda^{-1}e^{-\lambda\mu}\right)$$

$$+\varepsilon\left(1-e^{-\lambda\mu}\right)+2Me^{-\lambda\mu}\left(b-a-\mu-\lambda^{-1}\right),$$

for each $(\xi, g) \in \mathcal{D}\times\mathcal{G}$. Since

$$\lim_{\lambda\to\infty}\left[m_{\mathcal{G}}\left(\frac{1}{\lambda}-\mu\frac{1}{e^{\lambda\mu}}-\frac{1}{\lambda e^{\lambda\mu}}\right)+\varepsilon\left(1-\frac{1}{e^{\lambda\mu}}\right)+2M\frac{b-a-\mu-\frac{1}{\lambda}}{e^{\lambda\mu}}\right]=\varepsilon,$$

we obtain

$$\limsup_{\lambda\to\infty}\left\|I_\lambda\left(Q\left(\xi,g\right)\right)-I\left(Q\left(\xi,g\right)\right)\right\|\leq\varepsilon,$$

uniformly for $(\xi, g) \in \mathcal{D}\times\mathcal{G}$. Since $\varepsilon>0$ was arbitrary, then (8) holds true.

Case 2. $[\alpha,\beta]\subset\left]a,b\right]$. There exists $\eta=\eta\left(\alpha,\beta\right)>0$ such that for each $\tau\in\left]0,\eta\right[$ we have $[\alpha,\beta]\subset[a-\tau,b]$. Using the very same arguments as in Case 1, based on the Definition 2 and the Remark 2, we deduce (8).

Case 3. $[\alpha,\beta]=[a,b]$. Since $Q(\mathcal{D},\mathcal{G})$ is 1-equiintegrable, we deduce that, for each $\varepsilon>0$, there exists $\delta(\varepsilon)>0$ such that, for each $\tau\in\,]0,\delta(\varepsilon)]$, we have

$$\int_a^{b-\tau}\|Q(\xi,g)(t+\tau)-Q(\xi,g)(t)\|\,dt<\varepsilon.$$

By virtue of the same arguments as those used in Case 1, we deduce that

$$\begin{aligned}&\left\|S(\tau)\int_a^b Q(\xi,g)(t)\,dt-\int_a^b Q(\xi,g)(t)\,dt\right\|\\ &\le\left\|S(\tau)\int_a^{b-\tau}Q(\xi,g)(t)\,dt-\int_a^{b-\tau}Q(\xi,g)(t)\,dt\right\|\\ &+\left\|S(\tau)\int_{b-\tau}^{b}Q(\xi,g)(t)\,dt-\int_{b-\tau}^{b}Q(\xi,g)(t)\,dt\right\|\\ &\le\tau\cdot m_{\mathcal{G}}+2M\tau=\tau(m_{\mathcal{G}}+2M),\end{aligned}$$

for each $(\xi,g)\in\mathcal{D}\times\mathcal{G}$. So (8) holds true and this completes the proof. $\square$

Remark 5. If X is finite dimensional then, for each bounded subset $\mathcal{D}$ in X and each $\mathcal{G}$ in $BV([a,b];X)$ of equibounded variation, $Q(\mathcal{D},\mathcal{G})$ is relatively compact subset in $L^p(a,b;X)$ for each $p\in[1,+\infty[$ and thus p -equiintegrable. This follows from the observation that $\{Q(\xi,g)(t)\,;\,(\xi,g)\in\mathcal{D}\times\mathcal{G},t\in[a,b]\}$ is bounded (see Remark 4) and, inasmuch as X is finite dimensional, the set above is relatively compact. The conclusion follows from Theorem 9.4.1, p. 217, in Vrabie [36].

In infinite dimensional real Banach spaces the p-equiintegrability condition is not always an intrinsic property of the set $Q(\mathcal{D},\mathcal{G})$, with $\mathcal{D}$ bounded subset in X and $\mathcal{G}$ of equibounded variation in $BV([a,b];X)$. It is of interest to study this problem in the three cases of Remark 3. If $\{S(t)\,;t\ge 0\}$ is a *compact* C_0-semigroup of contractions (and thus is continuous from $]0,\infty[$ *to* $\mathcal{L}(X)$ in the uniform operator topology), then $Q(\mathcal{D},\mathcal{G})$ is relatively compact subset in $L^p(a,b;X)$, for each $p\in[1,+\infty)$, and thus p-equiintegrable. See Vrabie [36], Theorem 9.4.2, p. 219. But, if $A:D(A)\subseteq X\to X$ is the infinitesimal generator of a C_0-semigroup of contractions $\{S(t)\,;t\ge 0\}$ which is not compact and $R(\lambda,A)=(\lambda I-A)^{-1}$ is compact for some $\lambda>0$, even if X is a reflexive real Banach space, then the answer to this problem is in the negative, as we can see from the next example, which is a simple adaptation from Vrabie [36], Example 9.4.1, p. 219.

Example 1. Take $X = L^2_{2\pi}(\mathbb{R})$ the space of all equivalence classes, with respect to the almost everywhere equality on $\mathbb{R}$, of measurable and 2π-periodic function from $\mathbb{R}$ to $\mathbb{R}$. Endowed with the $L^2(0, 2\pi; \mathbb{R})$-scalar product, this is a real Hilbert space, and thus is a real reflexive Banach space. Let us define $A : D(A) \subseteq X \to X$ by

$$\begin{cases} D(A) = \left\{ u \in L^2_{2\pi}(\mathbb{R}) ; \dfrac{du}{dx} \in L^2_{2\pi}(\mathbb{R}) \right\} \text{ and} \\ Au = \dfrac{du}{dx}, \text{ for each } u \in D(A) . \end{cases}$$

The C_0-semigroup generated by A on $L^2_{2\pi}(\mathbb{R})$ is defined by

$$(S(t)u)(x) = u(x - t),$$

for each $t \geq 0$, $u \in L^2_{2\pi}(\mathbb{R})$ and for a.e. $x \in \mathbb{R}$.

Clearly $S(t)$ is not compact. Nevertheless, for each $\lambda > 0$, $R(\lambda, A)$ is a compact operator from $R(\lambda I - A)$ in $L^2_{2\pi}(\mathbb{R})$.

Next, let $\mathcal{D} = \{0\}$ and $\mathcal{G} = \{g_n; n \in \mathbb{N}^*\}$, where, for each $n \in \mathbb{N}^*$, the function $g_n : [0, 1] \to L^2_{2\pi}(\mathbb{R})$ is defined by

$$(g_n(t))(x) = -\frac{1}{n} \cos n(t + x),$$

a.e. $(t, x) \in]0, 1[\times \mathbb{R}$. For each $n \in \mathbb{N}^*$, g_n is in $BV\left([0, 1] ; L^2_{2\pi}(\mathbb{R})\right)$ and $\mathcal{G}$ is of equibounded variation on $[0, 1]$. A simple computation shows that

$$Q(\mathcal{D}, \mathcal{G}) = \{u_n; n \in \mathbb{N}^*\},$$

where, for each $n \in \mathbb{N}^*$, the function $u_n : [0, 1] \to L^2_{2\pi}(\mathbb{R})$ is defined by

$$(u_n(t))(x) = t \sin n(t + x),$$

for each $(t, x) \in [0, 1] \times R$. In this case $Q(\mathcal{D}, \mathcal{G})$ is not p-equiintegrable on $[0, 1]$ because the family is not relatively compact in $L^p\left(0, 1; L^2(0, 2\pi; \mathbb{R})\right)$.

3 A local existence theorem

Let X be a real Banach space, $A : D(A) \subseteq X \to X$ the infinitesimal generator of a C_0-semigroup of contractions $\{S(t); t \geq 0\}$, $\xi \in X$ and $g \in BV([a, b]; X)$. Let $k : \Delta_{[a,b]} \times U \to X$ be a continuous function, where

$\Delta_{[a,b]} = \left\{(s,\tau) \in \mathbb{R}^2; a \leq \tau \leq s \leq b\right\}$ and U is a nonempty and open subset in X. Let us consider the Cauchy problem (1). The aim of this section is to prove a local existence result concerning $\mathfrak{L}^\infty$-solutions for (1), by assuming that g, A, X satisfy

(H_0) For each $t \in [a,b]$, $\int_a^t S(t-s)\, dg(s) \in X$. See Remark 3.

Definition 4. Let us assume that (H_0) holds true. A function $u : [a,c] \to X$, $a < c \leq b$, is called an *$\mathfrak{L}^\infty$-solution of the problem* (1) *on* $[a,c]$ if

(i) for each $t \in [a,c[$ there exists $u(t+0)$;

(ii) for each $t \in [a,c[$, $(t,\tau,u(\tau+0)) \in \Delta_{[a,c[} \times U$;

(iii) $t \mapsto \int_a^t k(t,\tau,u(\tau+0))\, d\tau$ is in $L^1(a,c;X)$ and u is an $\mathfrak{L}^\infty$-solution on $[a,c]$ in the sense of Definition 2 for the following Cauchy problem

$$\begin{cases} du = (Au)\, dt + dh \\ u(a) = \xi, \end{cases}$$

where $h : [a,c] \to X$ is defined by

$$h(t) = \int_a^t \left(\int_a^s k(s,\tau,u(\tau+0))\, d\tau\right) ds + g(t),$$

for all $t \in [a,c]$.

We define the *$\mathfrak{L}^\infty$-solution of* (1) only on a semi-open interval $[a,c[$ by requiring (i) , (ii) (iii), except for the condition "$t \mapsto \int_a^t k(t,\tau,u(\tau+0))\, d\tau$ is in $L^1([a,c[;X)$" which should be relaxed to "$t \mapsto \int_a^t k(t,\tau,u(\tau+0))\, d\tau$ is in $L^1_{\mathrm{loc}}([a,c[;X)$".

Remark 6. We observe that, if $u : [a,c] \to X, a < c \leq b$ is an $\mathfrak{L}^\infty$-solution of the problem (1) on $[a,c]$, then u satisfies hypothesis of Theorem 1. Since $k : \Delta_{[a,b]} \times X \to X$ is continuous, then $k(s,\tau,u(\tau+0)) = k(s,\tau,u(\tau))$ a.e. on $\Delta_{[a,c]}$. Thus, in Definition 4, h is given by

$$h(t) = \int_a^t \left(\int_a^s k(s,\tau,u(\tau))\, d\tau\right) ds + g(t),$$

for all $t \in [a, c]$. That is, for each $t \in [a, c]$, we have that

$$u(t) = S(t-a)\,\xi + \int_a^t S(t-s)\left(\int_a^s k(s, \tau, u(\tau))\,d\tau\right) ds + \\ + \int_a^t S(t-s)\,dg(s) \tag{14}$$

First, we shall prove:

Lemma 1. *Let X be a real Banach space, $A : D(A) \subseteq X \to X$ the infinitesimal generator of a C_0 -semigroup of contractions $\{S(t); t \geq 0\}$, and let us assume that $R(\lambda, A) = (\lambda I - A)^{-1}$ compact for each $\lambda > 0$ (or equivalently for some $\lambda > 0$). Let $g \in BV([a, b]; X)$ and let us assume that (H_0) holds true. Let us assume further that $k : \Delta_{[a,b]} \times X \to X$ is continuous and bounded and, for each $(s, \tau) \in \Delta_{[a,b]}$, the function $u \mapsto k(s, \tau, u)$ is uniformly continuous on X. Then, for each $[a, c] \subset [a, b]$ and each $\xi \in X$, the problem (1) has at least one $\mathfrak{L}^\infty$-solution on $[a, c]$.*

Proof. Let $[a, c] \subset [a, b]$, $\xi \in X$, $\lambda > 0$ and let us consider the delay equation

$$\begin{cases} u_\lambda(t) = \xi & \text{for } t \in [a-\lambda, a] \\ du_\lambda = \left(Au_\lambda + \left(\int_a^t k(t, \tau, u_\lambda(\tau - \lambda))\,d\tau\right)\right) dt + dg & \text{for } t \in [a, b]. \end{cases} \tag{15}$$

From (H_0) and taking into account that k is continuous, by virtue of Remark 3, we obtain that (15) has a unique $\mathfrak{L}^\infty$-solution defined successively on $[a, a+\lambda]$, $[a+\lambda, a+2\lambda]$, and so on. For each $n \in \mathbb{N}^*$, let us denote by u_n the unique $\mathfrak{L}^\infty$ -solution of the problem (15) corresponding to $\lambda = \dfrac{1}{n}$, that is

$$u_n(t) = S(t-a)\,\xi + \int_a^t S(t-s)\left(\int_a^s k\left(s, \tau, u_n\left(\tau - \frac{1}{n}\right)\right) d\tau\right) ds \\ + \int_a^t S(t-s)\,dg(s), \tag{16}$$

for each $t \in [a, c]$. As k is bounded, it follows that the family

$$\mathcal{G} = \left\{ t \mapsto \int_a^t \left(\int_a^s k\left(s, \tau, u_n\left(\tau - \frac{1}{n}\right)\right) d\tau\right) ds + g(t); n \in \mathbb{N}^* \right\}$$

is of equibounded variation.

We will prove that the set $\mathcal{F} = \{u_n; n \in \mathbb{N}^*\}$ is 1-equiintegrable in $L^p(a,c;X)$, i.e.

$$\lim_{h\downarrow 0}\int_a^{c-h}\|u_n(t+h)-u_n(t)\|\,dt=0, \tag{17}$$

uniformly for $u_n \in \mathcal{F}$. Indeed, from (16) we deduce that

$$\begin{aligned}u_n(t+h)-u_n(t)&=S(t+h-a)\xi-S(t-a)\xi\\+\int_a^{t+h}S(t+h-s)&\left(\int_a^s k\left(s,\tau,u_n\left(\tau-\frac{1}{n}\right)\right)d\tau\right)ds\\-\int_a^t S(t-s)&\left(\int_a^s k\left(s,\tau,u_n\left(\tau-\frac{1}{n}\right)\right)d\tau\right)ds\\+\int_a^{t+h}S(t+h-s)\,dg(s)&-\int_a^t S(t-s)\,dg(s),\end{aligned}$$

for each $t\in[a,c]$ and for each $h\in\,]0,c-a]$. A simple computational argument shows that

$$\begin{aligned}\|u_n(t+h)-u_n(t)\|&\le\|S(h)S(t-a)\xi-S(t-a)\xi\|\\+\left\|\int_a^{t+h}S(t+h-s)\right.&\left(\int_a^s k\left(s,\tau,u_n\left(\tau-\frac{1}{n}\right)\right)d\tau\right)ds\\-\int_a^t S(t-s)&\left.\left(\int_a^s k\left(s,\tau,u_n\left(\tau-\frac{1}{n}\right)\right)d\tau\right)ds\right\|\\+\left\|S(h)\int_a^t S(t-s)\,dg(s)\right.&\left.-\int_a^t S(t-s)\,dg(s)\right\|\\&+\left\|\int_t^{t+h}S(t+h-s)\,dg(s)\right\|,\end{aligned}\tag{18}$$

for each $t\in[a,c]$ and $h\in\,]0,c-a]$.

We recall that, for each $x\in X$, $\lim_{h\downarrow 0}S(h)x=x$. Then, for ξ fixed in X and t fixed in $[a,c]$, we deduce

$$\lim_{h\downarrow 0}S(h)S(t-a)\xi=S(t-a)\xi \tag{19}$$

and

$$\lim_{h\downarrow 0} S(h)\int_a^t S(t-s)\,dg(s) = \int_a^t S(t-s)\,dg(s). \tag{20}$$

Let us denote by

$$T_1(t,h) = \|S(h)S(t-a)\xi - S(t-a)\xi\|$$

$$+\left\|S(h)\int_a^t S(t-s)\,dg(s) - \int_a^t S(t-s)\,dg(s)\right\|,$$

for each $t\in[a,c]$ and $h\in\,]0,c-a]$. Then, by virtue of (19), (20) and the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{h\downarrow 0}\int_a^{c-h} T_1(t,h)\,dt = 0. \tag{21}$$

Next, let us denote by

$$T_2(t,h) = \int_a^{t+h} S(t+h-s)\left(\int_a^s k\left(s,\tau,u_n\left(\tau-\frac{1}{n}\right)\right)d\tau\right)ds$$

$$-\int_a^t S(t-s)\left(\int_a^s k\left(s,\tau,u_n\left(\tau-\frac{1}{n}\right)\right)d\tau\right)ds.$$

The change of variable $h-s=-\theta$ in the first term of $T_2(t,h)$ leads to

$$T_2(t,h) = \int_{a-h}^t S(t-\theta)\left(\int_a^{h+\theta} k\left(h+\theta,\tau,u_n\left(\tau-\frac{1}{n}\right)\right)d\tau\right)d\theta$$

$$-\int_a^t S(t-s)\left(\int_a^s k\left(s,\tau,u_n\left(\tau-\frac{1}{n}\right)\right)d\tau\right)ds$$

$$=\int_{a-h}^a S(t-s)\left(\int_a^{h+s} k\left(h+s,\tau,u_n\left(\tau-\frac{1}{n}\right)\right)d\tau\right)ds$$

$$+\int_a^t S(t-s)\left(\int_a^s k\left(h+s,\tau,u_n\left(\tau-\frac{1}{n}\right)\right)d\tau-\int_a^s k\left(s,\tau,u_n\left(\tau-\frac{1}{n}\right)\right)d\tau\right)ds$$

$$+\int_a^t S(t-s)\left(\int_s^{s+h} k\left(h+s,\tau,u_n\left(\tau-\frac{1}{n}\right)\right)d\tau\right)ds,$$

and, since $\|S(t-s)\|_{\mathcal{L}(X)} \leq 1$, then

$$\|T_2(t,h)\| \leq \int_{a-h}^{a}\left(\int_{a}^{h+s}\left\|k\left(h+s,\tau,u_n\left(\tau-\frac{1}{n}\right)\right)\right\|d\tau\right)ds$$

$$+\int_{a}^{t}\left(\int_{a}^{s}\left\|k\left(h+s,\tau,u_n\left(\tau-\frac{1}{n}\right)\right)-k\left(s,\tau,u_n\left(\tau-\frac{1}{n}\right)\right)\right\|d\tau\right)ds$$

$$+\int_{a}^{t}\int_{s}^{s+h}\left\|k\left(h+s,\tau,u_n\left(\tau-\frac{1}{n}\right)\right)\right\|d\tau ds. \tag{22}$$

Finally, let us denote by

$T_3(t,h) =$ the last term of the right-hand side of inequality above.

Since $\{S(t); t \geq 0\}$ is a C_0 -semigroup of contractions, $k : \Delta_{[a,b]} \times X \to X$ is uniformly continuous in the last variable and bounded, and $\left\{t \mapsto u_n\left(t-\frac{1}{n}\right);\ n \in \mathbb{N}^*\right\}$ is bounded (see Remark 4), by virtue of the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{h\downarrow 0}\int_{a}^{c-h} T_3(t,h)\,dt = 0, \tag{23}$$

uniformly for $u_n \in \mathcal{F}$. Using (ii) in Remark 2, we also deduce that

$$\int_{a}^{c-h}\left\|\int_{t}^{t+h} S(t+h-s)\,dg(s)\right\|dt \leq h\,Var(g,[a,c]), \tag{24}$$

for each $h \in\,]0, c-a]$.

Then, integrating both sides from a to $c-h$ in the inequality (18), passing to the limit for $h \downarrow 0$, and taking into account of (21), (23), (24), we obtain

$$\limsup_{h\downarrow 0}\int_{a}^{c-h}\|u_n(t+h)-u_n(t)\| \leq 0, \tag{25}$$

uniformly for $u_n \in \mathcal{F}$. We conclude that $\mathcal{F}$ satisfies (17) and then is 1-equiintegrable in $L^p(a,c;X)$. From Theorem 3, we deduce that, for each $p \in [1,+\infty[$, the set $\{u_n; n \in \mathbb{N}^*\}$ is relatively compact in $L^p(a,c;X)$. So, we may assume, with no loss of generality, that there exists $\lim\limits_{n\to\infty} u_n = u$ in $L^1(a,c;X)$. On the other hand, we also have $\lim\limits_{n\to\infty} u_n\left(\tau-\frac{1}{n}\right) = u(\tau)$

a.e. for $\tau \in [a, c]$. Since k is continuous, and $\left\{t \mapsto u_n\left(\tau - \frac{1}{n}\right); n \in \mathbb{N}^*\right\}$ is bounded (see Remark 4) by the Lebesgue Dominated Convergence Theorem, we deduce that

$$\lim_{n\to\infty} \int_a^s k\left(s, \tau, u_n\left(\tau - \frac{1}{n}\right)\right) d\tau = \int_a^s k\left(s, \tau, u\left(\tau\right)\right) d\tau, \tag{26}$$

for each $s \in [a, c]$. Accordingly, passing to the limit for n tending to $+\infty$ in the equality (16), we conclude that u satisfies (6), and thus it is an $\mathfrak{L}^\infty$ -solution of the problem (1) on $[a, c]$. The proof is complete. □

Theorem 4. *Let X be a real Banach space, U a nonempty and open subset in X, $A : D(A) \subseteq X \to X$ the infinitesimal generator of a C_0-semigroup of contractions $\{S(t); t \geq 0\}$, with $R(\lambda, A) = (\lambda I - A)^{-1}$ compact for each $\lambda > 0$ (equivalently, for some $\lambda > 0$) and let $g \in BV([a, b]; X)$. Let us assume that (H_0) holds true. Let us further assume that $k : \Delta_{[a,b]} \times X \to X$ is continuous, bounded and, for each $(s, \tau) \in \Delta_{[a,b]}$, the function $u \mapsto k(s, \tau, u)$ is uniformly continuous on X. Then, for each $\xi \in X$ with*

$$g(a + 0) - g(a) + \xi \in U,$$

there exists $c_1 > a$ with $[a, c_1] \subset [a, b]$ such that the problem (1) has at least one $\mathfrak{L}^\infty$-solution on $[a, c_1]$.

Proof. Let $\xi \in X$ with $g(a + 0) - g(a) + \xi \in U$, and let us denote by $\eta = g(a + 0) - g(a) + \xi$. Since U is open and k is continuous, there exist $r > 0$ and $M > 0$ such that $B(\eta, r) \subset U$ and

$$\|k(s, \tau, u)\| \leq M, \tag{27}$$

for each $(s, \tau, u) \in \Delta_{[a,b]} \times B(\eta, r)$. Let us define $\rho : X \to X$ by

$$\rho(y) = \begin{cases} y & \text{for } y \in B(\eta, r) \\ \dfrac{r}{\|y - \eta\|}(y - \eta) + \eta & \text{for } y \in X \setminus B(\eta, r). \end{cases}$$

Clearly ρ maps X to $B(\eta, r)$ and is continuous. Now, let us define the mapping $k_r : \Delta_{[a,b]} \times X \to X$ by

$$k_r(s, \tau, u) = k(s, \tau, \rho(u)), \tag{28}$$

for each $(s,\tau) \in \Delta_{[a,b]}$ and $u \in X$. From (27), we conclude that k_r is bounded. Moreover, since k is a continuous function which is uniformly continuous in the last variable it follows that k_r is a continuous function which is uniformly continuous in the last variable. From Lemma 1, we know that the Cauchy problem

$$\begin{cases} du = \left(Au + \int_a^t k_r\left(t,\tau,u\left(\tau\right)\right)d\tau\right)dt + dg \\ u\left(a\right) = \xi \end{cases} \tag{29}$$

has at least one $\mathfrak{L}^\infty$-solution, $u : [a,b] \to X$.

We will prove that this $\mathfrak{L}^\infty$-solutions is in fact an $\mathfrak{L}^\infty$-solution on $[a,c_1]$, with $c_1 < b$ of the problem (1) in the sense of Definition 4. Indeed, by virtue of Theorem 1, it follows that $u\left(t+0\right) = u\left(t\right)$ a.e. on $[a,b]$. But k_r is continuous and accordingly

$$\int_a^t \left(\int_a^s k\left(s,\tau,u\left(\tau+0\right)\right)d\tau\right)ds = \int_a^t \left(\int_a^s k\left(s,\tau,u\left(\tau\right)\right)d\tau\right)ds,$$

for each $t \in [a,b]$. Since $u\left(a\right) = \xi$, by Theorem 1, we have that

$$u\left(\tau\right) - \eta = u\left(\tau\right) - g\left(a+0\right) + g\left(a\right) - \xi = u\left(\tau\right) - u\left(a+0\right).$$

Then, taking into account that $\lim\limits_{\tau\downarrow a} u\left(\tau\right) = u\left(a+0\right)$, it follows that there exists $c_1 \in [a,b]$ such that, for each $\tau \in \left]a,c_1\right[$, we have

$$\left\|u\left(\tau\right) - \eta\right\| < r,$$

i.e. $\left(t,\tau,u\left(\tau\right)\right) \in \Delta_{]a,c_1[} \times B\left(\eta,r\right) \subset \Delta_{]a,c_1[} \times U$. Since u is piecewise continuous at least to the right on $[a,c_1]$ and $\left(a,a,g\left(a+0\right) - g\left(a\right) + \xi\right) \in \Delta_{[a,c_1[} \times U$, it follows that $\left(t,\tau,u\left(\tau\right)\right) \in \Delta_{[a,c_1[} \times U$. But in this case $\rho\left(u\left(\tau+0\right)\right) = u\left(\tau+0\right)$ for each $t \in \left[a,c_1\right[$, and consequently $k_r\left(s,\tau,u\left(\tau+0\right)\right)$ must coincide with $k\left(s,\tau,u\left(\tau+0\right)\right)$, for each $(s,\tau) \in \Delta_{[a,c_1[}$. Hence the function $u : [a,c_1] \to X$ is an $\mathfrak{L}^\infty$-solution of the problem (1) in the sense of Definition 4, as claimed. $\square$

4 Continuation of $\mathfrak{L}^\infty$-Solutions

In this section we study some problems concerning noncontinuable $\mathfrak{L}^\infty$-solutions for the integro-differential equation with measures (1), by assuming that

(H_1) X is a real Banach space, $(a,\xi) \in \mathbb{R} \times X$ and $g \in BV(\mathbb{R}; X)$;

(H_2) $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a C_0-semigroup of contraction $\{S(t); t \geq 0\}$, with $R(\lambda, A) = (\lambda I - A)^{-1}$ compact operator for each $\lambda > 0$ (equivalently, for some $\lambda > 0$) ;

(H_3) $k : \Delta_{[a,b]} \times U \to X$ is a continuous function, where

$$\Delta_{[a,b]} = \left\{(s,\tau) \in \mathbb{R}^2; a \leq \tau \leq s \leq b\right\},$$

and U is a nonempty and open subset in X. Moreover, for each $(s,\tau) \in \Delta_{[a,b]}$, the function $u \mapsto k(s,\tau,u)$ is uniformly continuous on X.

We assume that g, A, X satisfy, in addition, the hypothesis (H_0).

Unlike the classical case of mild, or C^0-solutions, in this framework there are two concepts of non-continuable solutions. We begin with the definition of the corresponding two types of continuable $\mathfrak{L}^\infty$-solutions.

Definition 5. An $\mathfrak{L}^\infty$-solution $u : \mathbb{I} \to X$ of (1), with $\mathbb{I} = [a, c[$ $(\mathbb{I} = [a,c])$, $a < c \leq b$ $(a < c < b)$ is *continuable* if there exists another $\mathfrak{L}^\infty$ -solution of (1), $v : [a, c_1] \to X$, with $c_1 \geq c$ $(c_1 > c)$, such that $u(t) = v(t)$, for each $t \in \mathbb{I}$. If $b > c$, the $\mathfrak{L}^\infty$-solution u is called *strictly continuable*. An $\mathfrak{L}^\infty$-solution is called *saturated* (*noncontinuable*) if it is not continuable. If the projection of $\Delta_{[a,b]} \times U$ on $\mathbb{R}$ contains $\mathbb{R}_+$, an $\mathfrak{L}^\infty$-solution u is called *global* if it is defined on $[a, +\infty[$.

Using a very similar proof with that of Lemma 4.1 in Grosu [15], we deduce:

Lemma 2. *Assume that* (H_0), (H_1), (H_2), *and* (H_3) *are satisfied. An* $\mathfrak{L}^\infty$-*solution,* $u : [a, c[\to X$, *of* (1) *is*

(i) continuable with $c_1 = c$ *if and only if there exists*

$$u^*(c-0) = \lim_{h \downarrow 0} S(h)\, u(c-h)$$

and

$$g(c+0) - g(c-0) + u^*(c-0) \notin U;$$

(ii) strictly continuable if and only if there exists

$$u^*(c-0) = \lim_{h \downarrow 0} S(h)\, u(c-h)$$

and

$$g(c+0) - g(c-0) + u^*(c-0) \in U;$$

Remark 7. By Lemma 2, we observe that, in contrast with the case of the strong solutions or $\mathcal{C}_0$-solutions, where each saturated solution were necessarily defined on an interval of the form $[a, c[$, here there exists saturated $\mathcal{L}^\infty$ - solutions of the problem (1) which are defined on a closed interval $[a, c]$. This happens if $g(c+0) - g(c-0) + u^*(c-0) \notin U$.

By virtue of Theorem 1, if, in addition, either the semigroup generated by A is continuous from $]0, \infty[$ to $\mathcal{L}(X)$ in the uniform operator topology, or it can be imbedded into a group, then there exists $u(c-0) = u^*(c-0)$. Next, we focus our attention on strictly continuable $\mathcal{L}^\infty$-solution.

Proposition 2. *Assume that* (H_0), (H_1), (H_2), *and* (H_3) *are satisfied. Let* $u : [a, c[\to X$ *an* $\mathcal{L}^\infty$*-solution of* (1). *If* $c < +\infty$ *and* $t \mapsto \int_a^t k(t, \tau, u(\tau+0))\, d\tau$ *is in* $L^1(a, c; X)$, *then there exists*

$$u^*(c-0) = \lim_{h \downarrow 0} S(h)\, u(c-h).$$

Proof. We know that the $\mathcal{L}^\infty$-solution u verifies

$$u(t) = S(t-a)\xi + \int_a^t S(t-s)\left(\int_a^s k(s, \tau, u(\tau+0))\, d\tau\right) ds$$
$$+ \int_a^t S(t-s)\, dg(s),$$

for each $t \in [a, c[$, and then

$$S(h)\, u(c-h) = S(c-a)\xi + \int_a^{c-h} S(c-s)\left(\int_a^s k(s, \tau, u(\tau+0))\, d\tau\right) ds$$
$$+ \int_a^{c-h} S(c-s)\, dg(s).$$

The first and the second terms in the right side of the equality above have limit when h tends to 0, because $t \mapsto \int_a^t k(t,\tau,u(\tau+0))\,d\tau$ is in $\mathfrak{L}^1(a,c;X)$. It remains to show that there exists

$$\lim_{h\downarrow 0}\int_a^{c-h} S(c-s)\,dg(s).$$

But this is a direct consequence of Theorem 1. $\square$

Remark 8. The hypothesis "$t \mapsto \int_a^t k(t,\tau,u(\tau+0))\,d\tau$ is in $L^1(a,c;X)$" in Proposition 2 can be replaced with either "$t \mapsto \int_a^t k(t,\tau,u(\tau))\,d\tau$ is in $L^1(a,c;X)$" or "$t \mapsto \int_a^t k(t,\tau,u^*(\tau-0))\,d\tau$ is in $L^1(a,c;X)$". This follows because, by Theorem 1, these three mappings coincide a.e. on $]a,b[$.

Theorem 5. *Assume that* (H_0), (H_1), (H_2), *and* (H_3) *are satisfied. If* $u:\mathbb{J}\to X$ *is an* $\mathfrak{L}^\infty$*-solution of* (1), *with* $\mathbb{J}=[a,c[$ *or* $\mathbb{J}=[a,c]$, *then either* u *is saturated, or it can be extended up to a saturated one.*

The proof is an consequence of Zorn's Lemma and therefore we do not give details.

By Theorem 4 and Theorem 5, it follows that:

Corollary 1. *Assume that* (H_0), (H_1), (H_2), *and* (H_3) *are satisfied. Then, for each* $(a,\xi)\in\mathbb{R}\times X$ *with*

$$g(a+0)-g(a)+\xi\in U,$$

the problem (1) *has at least one saturated* $\mathfrak{L}^\infty$*-solution.*

Next, we assume that

(H_3^∞) $k:\Delta_{[a,+\infty[}\times U\to X$ is a continuous function, where

$$\Delta_{[a,+\infty[}=\left\{(s,\tau)\in\mathbb{R}^2; a\le\tau\le s<+\infty\right\},$$

and U is a nonempty and open subset in X. Moreover, for each $(s,\tau)\in\Delta_{[a,+\infty[}$, the function $u\mapsto k(s,\tau,u)$ is uniformly continuous on X.

Theorem 6. *Assume that* (H_0), (H_1), (H_2), *and* (H_3^∞) *are satisfied. In addition, let us assume that* k *maps bounded subset in* $\Delta_{[a,+\infty[}\times U$ *into bounded subset in* X. *Let* $u:[a,c[\,\to X$ *be a saturated* $\mathfrak{L}^\infty$*-solution of* (1). *Then either :*

(i) $t \mapsto u(t+0)$ *is unbounded on* $[a,c[$ *and, if* $c<+\infty$*, there exists*

$$\lim_{t\uparrow c}\|u(t+0)\| = +\infty,$$

or

(ii) $t \mapsto u(t+0)$ *is bounded on* $[a,c[$ *and it is global, i.e.* $c=+\infty$*, or*

(iii) $t \mapsto u(t+0)$ *is bounded on* $[a,c[$*, is non-global and, in this case, there exists*

$$u^*(c-0) = \lim_{h\downarrow 0} S(h)\,u(c-h)$$

and

$$g(c+0) - g(c-0) + u^*(c-0) \notin U.$$

The proof follows the very same lines as those in the proof of Theorem 4.2 in, Grosu [15] and therefore we omit it.

Corollary 2. *Assume that* (H_0)*,* (H_1)*,* (H_2)*, and* (H_3^∞) *with* $U=X$ *are satisfied. Further, let us assume that* $k:\Delta_{[a,+\infty[}\times X\to X$ *maps bounded subset in* $\Delta_{[a,+\infty[}\times X$ *into bounded subset in* X*. Then, each saturated* $\mathfrak{L}^\infty$*-solution* $u:[a,c[\,\to X$ *of* (1) *is either global, or it is not global, and in this case there exists*

$$\lim_{t\uparrow c}\|u(t+0)\| = +\infty.$$

We can reformulate

Corollary 3. *Assume that* (H_0)*,* (H_1)*,* (H_2)*, and* (H_3^∞) *with* $U=X$ *are satisfied. Further, let us assume that* $k:\Delta_{[a,+\infty[}\times X\to X$ *maps bounded subset in* $\Delta_{[a,+\infty[}\times X$ *into bounded subset in* X*. Then a necessary and sufficient condition in order that an* $\mathfrak{L}^\infty$ *-solution* $u:[a,c[\,\to X$ *for* (1) *be strictly continuable is that* $c<+\infty$ *and* $t\mapsto u(t+0)$ *be bounded on* $[a,c[$*.*

We conclude this section by noticing that, whenever k is bounded, each saturated solution is global, i.e. defined on $[a,+\infty[$. Namely, we have

Theorem 7. *Assume that* (H_0)*,* (H_1) *with* $a>0$*,* (H_2)*, and* (H_3^∞) *with* $U=X$ *are satisfied. Assume that there exists* $M>0$ *such that*

$$\|k(s,\tau,u)\| \le M,$$

for each $(s, \tau, u) \in \Delta_{[a,+\infty[} \times X$*. Then, for each* $(a, \xi) \in \mathbb{R}_+ \times X$*, the problem* (1) *has at least one global* $\mathfrak{L}^\infty$*-solution* $u : [a, +\infty[\to X$ *(or, for each* $(a, \xi) \in \mathbb{R}_+ \times X$*, we have that each saturated* $\mathfrak{L}^\infty$ *-solution* $u : [a, c[\to X$ *of* (1) *is global, i.e.,* $c = +\infty$*).*

Proof. By Corollary 1, the problem (1) has at least one saturated $\mathfrak{L}^\infty$-solution $u : [a, c[\to X$. To complete the proof, is suffices to show that $c = +\infty$. To this aim, let us assume by contradiction that $c < +\infty$. By Definition 4 of $\mathfrak{L}^\infty$-solution for (1), by Theorem 1 and Remark 1, and by the boundedness hypothesis on k, we deduce

$$\|u\,(t+0)\| \le \|S\,(t-a)\,\xi\|$$

$$+\int_a^t \|S\,(t-s)\|_{\mathfrak{L}(X)} \left(\int_a^s \|k\,(s, \tau, u\,(\tau+0))\|\, d\tau\right) ds + \left\|\int_a^{t+0} S\,(t-s)\, dg\,(s)\right\|$$

$$\le \|\xi\| + \int_a^t \int_a^s M\, d\tau ds + Var(g, [a, c])$$

$$\le \|\xi\| + M\frac{(c-a)^2}{2} + Var(g, [a, c])\,,$$

for each $t \in [a, c[$. That is $t \mapsto u\,(t+0)$ is bounded on $[a, c[$. By Corollary 3, we conclude that u is strictly continuable, thereby contradicting the hypothesis. This contradiction can be eliminated only if $c = +\infty$. This completes the proof. □

5 An Example

Let $X = L^2_{2\pi}(\mathbb{R})$ the space of all equivalence classes, with respect to the almost everywhere equality on $\mathbb{R}$, of measurable and 2π-periodic function from $\mathbb{R}$ to $\mathbb{R}$. Endowed with the $L^2(0, 2\pi; \mathbb{R})$-scalar product, this is a real Hilbert space, and thus is a real reflexive Banach space. Let us define $A : D(A) \subseteq X \to X$ by

$$\begin{cases} D\,(A) = \left\{u \in L^2_{2\pi}(\mathbb{R})\,;\, \dfrac{du}{dx} \in L^2_{2\pi}(\mathbb{R})\right\} \text{ and} \\ Au = \dfrac{du}{dx}, \text{ for each } u \in D\,(A)\,. \end{cases}$$

Let $\xi \in L^2_{2\pi}(\mathbb{R})$.

Let $k : \Delta_{[0,+\infty[} \times U \to X$ be a continuous function, where

$$\Delta_{[0,+\infty[} = \left\{ (s,\tau) \in \mathbb{R}^2; 0 \leq \tau \leq s \leq \infty \right\},$$

and U is a nonempty and open subset in X. Let $t_0 \in]0,+\infty[$ arbitrarily chosen, but fixed, $\delta(t-t_0)$ the Dirac measure concentrated in t_0 and $l : \mathbb{R} \to \mathbb{R}$ a continuous function in $L^2_{2\pi}(\mathbb{R})$. Let us define the function $g_0 : [0,+\infty[\to X$ by

$$g_0(t) = \begin{cases} -\frac{1}{2}l & \text{for } 0 \leq t < t_0 \\ 0 & \text{for } t = t_0 \\ \frac{1}{2}l & \text{for } t_0 < t. \end{cases}$$

Let us consider the integro-differential Cauchy problem

$$\begin{cases} du = \left(Au + \int_a^t k(t,\tau,u(\tau))\, d\tau \right) dt + dg_0 \\ u(0) = \xi. \end{cases} \tag{30}$$

Theorem 8. *Assume that the above hypotheses are satisfied. Assume that $k : \Delta_{[0,+\infty[} \times U \to X$ is continuous and that, for each $(s,\tau) \in \Delta_{[a,+\infty[}$, the function $u \mapsto k(s,\tau,u)$ is uniformly continuous on $L^2_{2\pi}(\mathbb{R})$. Then, for each $\xi \in U$, there exists $T_0 > 0$ such that the Cauchy problem* (30) *have at least one $\mathcal{L}^\infty$-solution on* $[0,T_0]$. *If, in addition, k is bounded, then the $\mathcal{L}^\infty$-solution of* (30) *can be continued to a global $\mathcal{L}^\infty$-solution.*

Proof. First, we will show that the hypotheses of Theorem 4 are satisfied. Let us observe that the C_0-semigroup generated by A on $L^2_{2\pi}(\mathbb{R})$ is defined by

$$(S(t)u)(x) = u(x-t),$$

for each $t \geq 0$, $u \in L^2_{2\pi}(\mathbb{R})$ and for a.e. $x \in \mathbb{R}$. This C_0-semigroup can be imbedded into a group. It is known that, for each $\lambda > 0$, $R(\lambda, A)$ is a compact operator from $R(\lambda I - A)$ into $L^2_{2\pi}(\mathbb{R})$.

Next, we observe that, for each $T > 0$, $g_0 \in BV([0,T];X)$, because $Var(g_0;[0,T]) = \|l\|_X < +\infty$. Then $g_0 \in BV\left([0,+\infty[; L^2_{2\pi}(\mathbb{R})\right)$. Moreover, since $X = L^2_{2\pi}(\mathbb{R})$ is a reflexive Banach space, then the space of admissible measures for A is $X_A = X$. In this case, the integral $\int_0^t S(t-s)\, dg(s)$ exists in the norm topology of X.

We observe that $(dg_0)(t) = l\delta(t-t_0)$ in the sense of distributions, where $\delta(t-t_0)$ is the Dirac measure concentrated in $t_0 \in]0,+\infty[$.

The hypotheses of Theorem 4 are satisfied. Then there exists $T_0 \in [0, +\infty[$, such that the problem (30) has at least one $\mathcal{L}^\infty$-solution on $[0, T_0]$ in the sense of Definition 4.

In addition, if k is bounded, by Theorem 7, we obtain that this $\mathcal{L}^\infty$-solution can be continued on $[0, +\infty[$. The proof is complete. $\square$

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