CONTROL OF DETERMINISTIC AND STOCHASTIC SYSTEMS WITH SEVERAL SMALL PARAMETERS – A SURVEY[*]

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Abstract

The past three decades of research on multiparameter singularly perturbed systems are reviewed, including recent results. Particular attention is paid to stability analysis, control, filtering problems and dynamic games. First, a parameter-independent design methodology is summarized, which employs a two-time-scale and descriptor systems approach without information on the small parameters. Further, various computational algorithms are included to avoid ill-conditioned systems: the exact slow-fast decomposition method, the recursive algorithm and Newton's method are considered in particular. Convergence results are presented and the existence and uniqueness of the solutions are discussed. Second, the new results obtained via the stochastic approach are presented. Finally, the results of a simulation of a practical power system are presented to validate the efficiency of the considered design methods.

keywords: Singular perturbations, several small parameters, deterministic systems, stochastic systems, robust control, Nash games.

1 Introduction

When several small singular perturbation parameters of the same order of magnitude are present in the dynamic model of a physical system, the control problem is usually solved as a single parameter perturbation problem [18, 19, 21]; such a system is called a singularly perturbed system (SPS). Although this is achieved by scaling the coefficients, these parameters are often not known exactly. Thus, it is not applicable to a wider class of problems. One solution is to use the so-called multimodeling systems approach (see e.g. [1, 2, 7, 21, 22]). In addition, a joint multitime scale-multiparameter singularly perturbed system (MSPS) has been formulated [14, 23]. It should be noted that these small parameters are of different orders of magnitude.

Stability analysis, control and filtering problems in MSPSs have been thoroughly investigated. Multiarea power systems [1, 7] and passenger cars [15, 17, 29] can be modelled as MSPSs, which are widely used to represent system dynamics.

Since the investigations into the stability for the multimodel situation in [3, 4, 6], much of the interest in linear quadratic (LQ) control has been motivated by applications of the theory to multimodeling systems [1, 2, 12]. These interests in extending LQ control to dynamic games [5, 8, 9, 10, 13] were revealed. An overview of multimodeling control may be found in [11]. The recent theoretical advances in multimodeling techniques allow a revisiting of LQ control [49, 50, 52], the filtering problem [51, 54], the H_∞ control problem [48, 59], guaranteed cost control [56] and Nash games [53, 55, 57, 58]. A direct approach to the Lur'e problem for MSPSs has been proposed [27]. To extend the validity of continuous MSPSs, stability analysis, composite state feedback control and Nash games have been considered for discrete MSPSs [24, 25, 26].

In this paper, we present a survey of MSPSs in various control problems. Although many of the references consider deterministic problems, stochastic cases are also reviewed here. First, the results of stability analysis and the important related tests are given. After introducing the feature of the multiparameter algebraic Riccati equations (MARE) that is based on the LQ control for MSPSs, we discuss the two-time-scale design method for cases where the singular perturbation parameters are sufficiently small or unknown. However, iterative methods for finding the desired solutions are discussed when such parameters are known. In particular, to avoid ill-conditioned systems, the exact slow-fast decomposition method, recursive

computation and Newton's method are surveyed. It is shown that these results are also valid for the filtering problem, H_∞ control problem, guaranteed cost control and Nash games. Moreover, some new results for stochastic systems that are governed by Itô differential equations are also discussed. Finally, it is shown that the concepts and methods surveyed in this paper can be exploited to solve the stochastic H_∞ control problem for an actual MSPS.

Notation: The notations used in this paper are fairly standard. **block diag** denotes the block diagonal matrix. $\det M$ denotes the determinant of M. $\text{vec} M$ denotes an ordered stack of the columns of M. $\otimes$ denotes Kronecker product. $\text{Re}\lambda(M)$ denotes a real part of $\lambda \in \mathbf{C}$ of M. $E[\cdot]$ denotes the expectation operator. The space of the $\Re^k$-valued functions that are quadratically integrable on $(0, \infty)$ are denoted by $L_2^k(0, \infty)$.

2   Stability

A general frame-work for the stability of a MSPS is formulated in [1, 3, 4, 6, 7, 21, 22]. Stability is very important for a linear or nonlinear MSPS when capturing the behaviour of the closed-loop MSPS. For a linear MSPS, the sufficient conditions for uniform asymptotic stability have been derived, and the asymptotic behaviour of the solution has also been investigated by using the transformation [1] and the D-stability [3]. In contrast, it is known that the Lyapunov method can be used to estimate the stability of a system by using a Lyapunov function without solving the nonlinear differential equations [4, 6]. The purpose of this section is to review the asymptotic stability for several sufficiently small parameters. These results are based on the asymptotic stability of a reduced-order slow system and fast subsystems.

A linear system of strongly coupled slow subsystem and weakly coupled fast subsystems is considered by (1).

$$\dot{x}(t) = A_0 x(t) + \sum_{j=1}^{N} A_{0j} z_j(t), \ x(0) = x^0, \tag{1a}$$

$$\varepsilon_i \dot{z}_i(t) = A_{i0} x(t) + A_{ii} z_i(t) + \sum_{j=1, \ j \neq i}^{N} \varepsilon_{ij} A_{ij} z_j(t), \ z_i(0) = z_i^0, \tag{1b}$$

where $x(t) \in \Re^{n_0}$ means the slow state vector. $z_i(t) \in \Re^{n_i}$, $i = 1, \ldots, N$ mean the fast state vectors. All matrices above are of appropriate dimensions. The small singular perturbation parameters $\varepsilon_i > 0$, one per subsystem, represent time constant, inertias, masses etc., while the small regular perturbation parameters ε_{ij}, $i \neq j$ represent weak coupling between the subsystems.

The following result is well known for the stability of linear MSPS.

Lemma 1. *[1] If* $\mathrm{Re}\lambda(A_{ii}) < 0$, $i = 1, \ldots, N$ *and* $\mathrm{Re}\lambda(A_s) < 0$, *then there exists a positive scalar* σ_1 *such that*

$$x(t) = x_s(t) + O(\|\varepsilon\|), \tag{2a}$$

$$z_i(t) = -A_{ii}^{-1} A_{i0} x_s(t) + z_{if}\left(\frac{t}{\varepsilon_i}\right) + O(\|\varepsilon\|), \tag{2b}$$

hold for all $t \in [0, \infty)$ *and all* $\varepsilon \in H$, $0 < \|\varepsilon\| \leq \sigma_1$, *where*

$$\varepsilon := \begin{bmatrix} \varepsilon_1 & \cdots & \varepsilon_N & \varepsilon_{12} & \cdots & \varepsilon_{N(N-1)} \end{bmatrix} \in \Re^{N^2},$$

$$H := \left\{ \varepsilon \in \Re^{N^2} \,\middle|\, m_{ij} \leq \frac{\varepsilon_j}{\varepsilon_i} \leq M_{ij},\ \bar{m}_{ij} \leq \frac{\varepsilon_{ij}}{\varepsilon_i} \leq \bar{M}_{ij}, \right.$$

$$\left. m_{ij} > 0,\ \bar{m}_{ij} > 0,\ M_{ij} < \infty,\ \bar{M}_{ij} < \infty \right\},$$

$$\dot{x}_s(t) := A_s x_s(t), A_s := A_0 - \sum_{j=1}^{N} A_{0j} A_{jj}^{-1} A_{j0}, \dot{z}_{if}(t) := A_{ii} z_{if}(t), i = 1, \ldots, N.$$

As an important implication, the following result is given for the stability of an uncertain MSPS.

Lemma 2. *[52] Let us consider uncertain MSPS*

$$\dot{x}(t) = [F_0 + O(\|\varepsilon\|)]x(t) + [F_{0f} + O(\|\varepsilon\|)]z(t),\ x(0) = x^0, \tag{3a}$$

$$\Pi_\varepsilon \dot{z}(t) = [F_{f0} + O(\|\varepsilon\|)]x(t) + [F_f + O(\|\varepsilon\|)]z(t),\ z(0) = z^0, \tag{3b}$$

where

$$\Pi_\varepsilon := \mathbf{block\ diag}\left(\varepsilon_1 I_{n_1} \quad \cdots \quad \varepsilon_N I_{n_N} \right),\ z(t) := \begin{bmatrix} z_1^T(t) & \cdots & z_N^T(t) \end{bmatrix}^T,$$

$$F_{0f} := \begin{bmatrix} F_{01} \cdots F_{0N} \end{bmatrix}, F_{f0} := \begin{bmatrix} F_{10}^T \cdots F_{N0}^T \end{bmatrix}^T, F_f := \mathbf{block\ diag}\left(F_{11} \cdots F_{NN} \right),$$

$x(t) \in \Re^{n_0}$ and $z_i(t) \in \Re^{n_i}$, $i = 1, \ ... \ , N$ are the state vectors. All matrices above are of appropriate dimensions.

If F_{ii}, $i = 1, \ ... \ , N$ and $\bar{F} = F_0 - \sum_{j=1}^{N} F_{0j} F_{jj}^{-1} F_{j0}$ are stable, then there exists a positive scalar σ_2 such that for all $t \in [0, \ \infty)$ and all $\varepsilon \in H$, $0 < \|\varepsilon\| \leq \sigma_2$, uncertain MSPS (3) is asymptotically stable.

Asymptotic expansions of the solutions as well as the problem of exponential stability of the zero state equilibrium of a singularly perturbed linear system with several small parameters of different orders of magnitude may be found in [39], see also Chapter 3 in [40].

At the end of this section, sufficient conditions are stated to guarantee the asymptotic stability of a class of nonlinear SPS with several perturbation parameters of the same order. Now, let us consider the nonlinear MSPS given by (4).

$$\dot{x}(t) = f(t, \ x) + F(t, \ x)z(t), \tag{4a}$$

$$\Pi_\varepsilon \dot{z}(t) = g(t, \ x) + G(t, \ x)z(t). \tag{4b}$$

We assume that the following conditions are satisfied for all $x(t) \in S_x$, where S_x is a closed set in $\Re^{n_0}$ containing the origin and for all $t \geq t_0$.

(a) $x(t) = 0$ is the unique point in S_x for which $f(t, \ 0) = 0$ and $g(t, \ 0) = 0$.

(b) f, g, F, G and $h := G^{-1}(t, \ x)g(t, \ x)$ are bounded and satisfy the necessary smoothness requirements for existence, uniqueness and continuity of the solution of (4). Moreover, $G(t, \ x)$ and $h(t, \ x)$ have bounded first partial derivatives with respect to t and $x(t)$.

(c) There exists a positive definite Lyapunov function $V(t, \ x)$ such that

$$V_t + V_x f_0(t, \ x) \leq -\kappa_1 \psi^2(x), \ \|V_x F(t, \ x)\| \leq \kappa_2 \psi(x),$$
$$\|h_t + h_x f_0(t, \ x)\| \leq \kappa_3 \psi(x),$$
$$f_0(t, \ x) := f(t, \ x) - F(t, \ x)h(t, \ x), \ V_t := \frac{\partial V}{\partial t}, \ V_x := \frac{\partial V}{\partial x},$$
$$h_t := \frac{\partial h}{\partial t}, \ h_x := \frac{\partial h}{\partial x},$$

where $\psi(x)$ is a positive definite function of $x(t)$, κ_1, κ_2 and κ_3 are positive scalars.

(d) The real parts of the eigenvalues of $\Pi_\varepsilon^{-1} G$ are strictly negative, that is $\operatorname{Re}\lambda(\Pi_\varepsilon^{-1} G) \leq -\tau < 0$ for all $\varepsilon \in H$, where τ is a positive scalar independent of t, x and ε.

The asymptotic stability of equation (4) is established in the following basic lemma.

Lemma 3. *[6] Under conditions* (a)-(d), *there exists a positive scalar* σ_3 *such that for all* $\varepsilon \in H$ *with* $0 < \|\varepsilon\| \leq \sigma_3$, *the origin* $x = 0$, $z = 0$ *is an asymptotically stable equilibrium point of* (4).

It should be observed that in practice, Lemma 1 is included in Lemma 3 as a special case.

For the problem of exponential stability of a singularly perturbed linear system with state delays we refer to [16] and [41].

3   Linear Quadratic Regulator (LQR) Problem

The solution of a LQ regulator (LQR) problem is usually given in the form of state feedback control. Indeed, the LQR technique was used to solve the active suspension control problem [29]. In this section, we discuss the LQR problems from the point of view of the reduced-order technique and numerical aspects. These results will be covered as the extension of SPS [18, 19, 21].

3.1   Two-Time-Scale Decomposition

When the small perturbation parameters ε_i are not known, a popular approach to deal with the MSPS is the two-time-scale decomposition method (see e.g. [1, 21]). In practice, since ε_i is very small or unknown, the previous technique is very efficient. First, the LQ control problem for the MSPS was studied by using composite controller design [1, 2]. In [2], the resulting near-optimal controller has been proven to have a performance level, i.e. $O(\|\varepsilon\|)$, where $\|\varepsilon\|$ denotes the norm of the vector $\varepsilon := [\varepsilon_1 \ \cdots \ \varepsilon_N]$, close to the optimal performance level for the standard and nonstandard MSPS. However, one major drawback of this method is that the fast state matrices A_{ii} are invertible. Indeed, if this condition holds, we cannot obtain the reduced-order slow subsystems. To avoid the invertibility assumptions, the descriptor systems approach [28] can be used. The descriptor systems approach will

be discussed later as a nonstandard MSPS. Although the descriptor systems approach can still be used for general MSPSs, the two-time-scale decomposition method is recommended in this case because the fast state matrices are invertible in most practical systems. Some properties of the two-time-scale decomposition method are described next.

We consider a specific structure of N-lower level multi-fast subsystems interconnected through the dynamics of a higher level slow subsystem [1, 7, 52].

$$\dot{x}(t) = A_0 x(t) + \sum_{j=1}^{N} A_{0j} z_j(t) + \sum_{j=1}^{N} B_{0j} u_j(t), \; x(0) = x^0, \tag{5a}$$

$$\varepsilon_i \dot{z}_i(t) = A_{i0} x(t) + A_{ii} z_i(t) + B_{ii} u_i(t), \; z_i(0) = z_i^0, \; i = 1, \; \dots, N, \tag{5b}$$

where $u_i(t) \in \Re^{m_i}$, $i = 1, \; \dots, N$ are the control inputs.

It should be noted that all fast state matrices A_{ii}, $i = 1, \; \dots, N$ are invertible. The performance criterion is given by

$$J = \frac{1}{2} \int_0^\infty \left(\xi^T(t) Q \xi(t) + \sum_{j=1}^{N} u_j^T(t) R_j u_j(t) \right) dt, \tag{6}$$

where

$$\xi(t) := \begin{bmatrix} x^T(t) & z_1^T(t) & \cdots & z_N^T(t) \end{bmatrix}^T \in \mathbf{R}^{\bar{n}}, \; \bar{n} := \sum_{j=0}^{N} n_j,$$

$$Q := C^T C = \begin{bmatrix} Q_{00} & Q_{0f} \\ Q_{0f}^T & Q_f \end{bmatrix}, \; Q_{00} := C_0^T C_0 = \sum_{j=0}^{N} C_{j0}^T C_{j0},$$

$$Q_{0f} := C_0^T C_f = \begin{bmatrix} Q_{01} & \cdots & Q_{0N} \end{bmatrix} = \begin{bmatrix} C_{10}^T C_{11} & \cdots & C_{N0}^T C_{NN} \end{bmatrix},$$

$$Q_f := C_f^T C_f = \mathbf{block\ diag} \begin{pmatrix} Q_{11} & \cdots & Q_{NN} \end{pmatrix} =$$

$$\mathbf{block\ diag} \begin{pmatrix} C_{11}^T C_{11} & \cdots & C_{NN}^T C_{NN} \end{pmatrix},$$

$$C := \begin{bmatrix} C_0 & C_f \end{bmatrix}, \ C_0 := \begin{bmatrix} C_{00} \\ C_{10} \\ \vdots \\ C_{N0} \end{bmatrix},$$

$$C_f := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ C_{11} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & C_{NN} \end{bmatrix},$$

$$R := \textbf{block diag} \begin{pmatrix} R_1 & \cdots & R_N \end{pmatrix}.$$

Let the optimal control for the LQ control problem (5) and (6) be

$$u_{\text{opt}}(t) = K_{\text{opt}}\xi(t) = -R^{-1}B_\varepsilon^T P_\varepsilon \xi(t), \tag{7}$$

where P_ε satisfies the MARE

$$P_\varepsilon A_\varepsilon + A_\varepsilon^T P_\varepsilon - P_\varepsilon S_\varepsilon P_\varepsilon + Q = 0, \tag{8}$$

with

$$A_\varepsilon := \begin{bmatrix} A_0 & A_{0f} \\ \Pi_\varepsilon^{-1} A_{f0} & \Pi_\varepsilon^{-1} A_f \end{bmatrix},$$

$$A_{0f} := \begin{bmatrix} A_{01} & \cdots & A_{0N} \end{bmatrix}, \ A_{f0} := \begin{bmatrix} A_{10}^T & \cdots & A_{N0}^T \end{bmatrix}^T,$$

$$A_f := \textbf{block diag} \begin{pmatrix} A_{11} & \cdots & A_{NN} \end{pmatrix},$$

$$S_\varepsilon := B_\varepsilon R^{-1} B_\varepsilon^T = \begin{bmatrix} S_{00} & S_{0f}\Pi_\varepsilon^{-1} \\ \Pi_\varepsilon^{-1} S_{0f}^T & \Pi_\varepsilon^{-1} S_f \Pi_\varepsilon^{-1} \end{bmatrix},$$

$$S_{00} := B_0 R^{-1} B_0^T = \sum_{j=1}^{N} B_{0j} R_j^{-1} B_{0j}^T,$$

$$S_{0f} := B_0 R^{-1} B_f^T = \begin{bmatrix} S_{01} & \cdots & S_{0N} \end{bmatrix} =$$
$$\begin{bmatrix} B_{01} R_1^{-1} B_{11}^T \cdots & B_{0N} R_N^{-1} B_{NN}^T \end{bmatrix},$$

$$S_f := B_f R^{-1} B_f^T = \textbf{block diag} \begin{pmatrix} S_{11} & \cdots & S_{NN} \end{pmatrix} =$$
$$\textbf{block diag} \begin{pmatrix} B_{11} R_1^{-1} B_{11}^T & \cdots & B_{NN} R_N^{-1} B_{NN}^T \end{pmatrix},$$

$$B_\varepsilon := \begin{bmatrix} B_0 \\ \Pi_\varepsilon^{-1} B_f \end{bmatrix}, \ B_0 := \begin{bmatrix} B_{01} & \cdots & B_{0N} \end{bmatrix},$$

$$B_f := \textbf{block diag} \begin{pmatrix} B_{11} & \cdots & B_{NN} \end{pmatrix}.$$

However, we cannot solve the MARE (8) without the knowledge of the small perturbation parameters ε_i. When ε_i is very small or unknown, the two-time-scale design method [1, 52] is very efficient.

According to [1, 7], the near-optimal closed-loop control is given by

$$
\begin{aligned}
u_{icom}(t) \;=\; & -[(I_{m_i} - R_i^{-1}B_{ii}^T X_{ii} A_{ii}^{-1} B_{ii})\tilde{R}_i^{-1}(\tilde{D}_i^T \tilde{C}_{i0} + \tilde{B}_{0i}^T X_{00}) \\
& + R_i^{-1}B_{ii}^T X_{ii} A_{ii}^{-1} A_{i0}]x(t) - R_i^{-1}B_{ii}^T X_{ii} z_i(t), \; i = 1, \ldots, N, (9)
\end{aligned}
$$

where $\tilde{B}_{0i} = B_{0i} - A_{0i}A_{ii}^{-1}B_{ii}$, $\tilde{C}_{i0} = C_{i0} - C_{ii}A_{ii}^{-1}A_{i0}$, $\tilde{R}_i = R_i + \tilde{D}_i^T \tilde{D}_i$, $\tilde{D}_i = -C_{ii}A_{ii}^{-1}B_{ii}$.

In the above, X_{00} is the unique stabilizing positive semidefinite symmetric solution of the following algebraic Riccati equation (ARE)

$$
\begin{aligned}
X_{00}(A_s - B_s R_s^{-1} D_s^T C_s) + (A_s - B_s R_s^{-1} D_s^T C_s)^T X_{00} - \\
-X_{00}B_s R_s^{-1}B_s^T X_{00} + C_s^T(I_{\bar{l}} - D_s R_s^{-1}D_s^T)C_s = 0, \quad (10)
\end{aligned}
$$

where

$$
\begin{aligned}
R_s &= R + D_s^T D_s, \; B_s = B_0 - A_{0f}A_f^{-1}B_f = \\
&\begin{bmatrix} B_{01} - A_{01}A_{11}^{-1}B_{11} & \cdots & B_{0N} - A_{0N}A_{NN}^{-1}B_{NN} \end{bmatrix}, \\
C_s &= C_0 - C_f A_f^{-1} A_{f0} = \begin{bmatrix} C_{00}^T & (C_{10} - C_{11}A_{11}^{-1}A_{10})^T & \cdots \end{bmatrix} \\
&\qquad\qquad (C_{N0} - C_{NN}A_{NN}^{-1}A_{N0})^T \end{bmatrix}^T, \\
D_s &= -C_f A_f^{-1}B_f = - \begin{bmatrix} 0 & \cdots & 0 \\ C_{11}A_{11}^{-1}B_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & C_{NN}A_{NN}^{-1}B_{NN} \end{bmatrix}.
\end{aligned}
$$

$X_{ii}, \; i = 1, \ldots, N$ are the unique stabilizing positive semidefinite solution of the following AREs

$$
X_{ii}A_{ii} + A_{ii}^T X_{ii} - X_{ii}S_{ii}X_{ii} + Q_{ii} = 0. \quad (11)
$$

It is well known from [1] that the controller (9) is identical with the following controller

$$
u_{icom}(t) = -R_i^{-1}B_{i0}^T X_{00}x(t) - R_i^{-1}B_{ii}^T X_{i0}x(t) - R_i^{-1}B_{ii}^T X_{ii}z_i(t), \quad (12)
$$

where $X_{i0}, \; i = 1, \ldots, N$ are

$$
X_{i0}^T = [X_{00}(S_{0i}X_{ii} - A_{0i}) - (A_{i0}^T X_{ii} + Q_{0i})](A_{ii} - S_{ii}X_{ii})^{-1}. \quad (13)
$$

Furthermore, the composite controller $u_{\mathrm{com}}(t) = \begin{bmatrix} u_{1\mathrm{com}}(t)^T & \cdots & u_{N\mathrm{com}}(t)^T \end{bmatrix}^T$ can be rewritten as the following composite controller

$$u_{\mathrm{com}}(t) := K_{\mathrm{com}}\xi(t) = -R^{-1}B^T \begin{bmatrix} X_{00} & 0 & 0 & \cdots & 0 \\ X_{10} & X_{11} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{N0} & 0 & 0 & \cdots & X_{NN} \end{bmatrix} \xi(t). \quad (14)$$

Theorem 1. *[1] There exists a positive scalar $\bar{\sigma}_1$ such that for all $\varepsilon \in H$ with $0 < \|\varepsilon\| \leq \bar{\sigma}_1$ the closed loop MSPS (5) is asymptotically stable. Furthermore, the use of the composite controller (14) results in J_{app} satisfying*

$$\lim_{\|\varepsilon\| \to +0} (J_{\mathrm{com}} - J_{\mathrm{opt}}) = 0, \quad (15)$$

where $J_{\mathrm{opt}} = \xi^T(0)P_\varepsilon\xi(0)$ and $J_{\mathrm{com}} = \xi^T(0)W_\varepsilon\xi(0)$ with

$$W_\varepsilon(A_\varepsilon + B_\varepsilon K_{\mathrm{com}}) + (A_\varepsilon + B_\varepsilon K_{\mathrm{com}})^T W_\varepsilon + K_{\mathrm{com}}^T R K_{\mathrm{com}} + Q = 0.$$

According to Theorem 1, the detailed cost degradation has not been established. This property is described in a subsequent section.

3.2 Matrix Riccati Equations

The multimodel strategies for the LQ control problem are given in terms of Riccati or Riccati-type equations, which are parameterized by several small positive perturbation parameters. The existence of a unique and bounded solution to the MARE (8) was first shown in [13]. This important result is summarized as follows.

Since the matrices A_ε and B_ε contain the term of ε_i^{-1}, a solution P_ε of the MARE (8), if it exists, must contain terms of ε_i. Taking this fact into consideration, we look for a solution P_ε of the MARE (8) with the structure

$$P_\varepsilon := \begin{bmatrix} P_{00} & P_{f0}^T \Pi_\varepsilon \\ \Pi_\varepsilon P_{f0} & \Pi_\varepsilon P_f \end{bmatrix}, \quad P_{00} = P_{00}^T,$$

$$P_{f0} := \begin{bmatrix} P_{10} \\ \vdots \\ P_{N0} \end{bmatrix},$$

$$
P_f := \begin{bmatrix}
P_{11} & \alpha_{12}P_{21}^T & \alpha_{13}P_{31}^T & \cdots & \alpha_{1N}P_{N1}^T \\
P_{21} & P_{22} & \alpha_{23}P_{32}^T & \cdots & \alpha_{2N}P_{N2}^T \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
P_{(N-1)1} & P_{(N-1)2} & P_{(N-1)3} & \cdots & \alpha_{(N-1)N}P_{N(N-1)}^T \\
P_{N1} & P_{N2} & P_{N3} & \cdots & P_{NN}
\end{bmatrix},
$$

$$
\Pi_\varepsilon P_f = P_f^T \Pi_\varepsilon.
$$

In order to guarantee the existence of the reduced-order ARE and its standard stabilizability and the detectability conditions when $\|\varepsilon\| \to +0$, Assumptions 1 and 2 are needed.

Assumption 1. *The triples* (A_{ii}, B_{ii}, C_{ii}), $i = 1, \ldots, N$ *are stabilizable and detectable.*

Assumption 2.

$$
\text{rank} \begin{bmatrix} sI_{n_0} - A_0 & -A_{0f} & B_0 \\ -A_{f0} & -A_f & B_f \end{bmatrix} = \bar{n}, \tag{16a}
$$

$$
\text{rank} \begin{bmatrix} sI_{n_0} - A_0^T & -A_{f0}^T & C_0^T \\ -A_{0f}^T & -A_f^T & C_f^T \end{bmatrix} = \bar{n}, \tag{16b}
$$

with $\forall s \in \mathbf{C}$, $\text{Re}[s] \geq 0$.

Before investigating the optimal control problem, we investigate the asymptotic structure of the MARE (8).

The MARE (8) can be partitioned into

$$
f_1 = P_{00}^T A_0 + A_0^T P_{00} + P_{f0}^T A_{f0} + A_{f0}^T P_{f0} - P_{00}^T S_{00} P_{00} -
$$
$$
- P_{f0}^T S_f P_{f0} - P_{00}^T S_{0f} P_{f0} - P_{f0}^T S_{0f}^T P_{00} + Q_{00} = 0, \tag{17a}
$$
$$
f_2 = A_0^T P_{f0}^T \Pi_\varepsilon + A_{f0}^T P_f + P_{00}^T A_{0f} + P_{f0}^T A_f - P_{00}^T S_{00} P_{f0}^T \Pi_\varepsilon -
$$
$$
- P_{f0}^T S_{0f}^T P_{f0}^T \Pi_\varepsilon - P_{00}^T S_{0f} P_f - P_{f0}^T S_f P_f + Q_{0f} = 0, \tag{17b}
$$
$$
f_3 = P_f^T A_f + A_f^T P_f + \Pi_\varepsilon P_{f0} A_{0f} + A_{0f}^T P_{f0}^T \Pi_\varepsilon - P_f^T S_f P_f
$$
$$
- P_f^T S_{0f}^T P_{f0}^T \Pi_\varepsilon - \Pi_\varepsilon P_{f0} S_{0f} P_f - \Pi_\varepsilon P_{f0} S_{00} P_{f0}^T \Pi_\varepsilon + Q_f = 0. \tag{17c}
$$

It is assumed that the limit of α_{ij} exists as ε_i and ε_j tend to zero (see e.g., [1, 2]), that is

$$\bar{\alpha}_{ij} = \lim_{\substack{\varepsilon_j \to +0 \\ \varepsilon_i \to +0}} \alpha_{ij} = \lim_{\substack{\varepsilon_j \to +0 \\ \varepsilon_i \to +0}} \frac{\varepsilon_j}{\varepsilon_i}. \tag{18}$$

Assumption 1 ensures that $A_{ii} - S_{ii}\bar{P}_{ii}^*$, $i = 1, \ldots, N$ are nonsingular. Substituting the solution of (17c) into (17b) and substituting $\bar{P}_{f0}^*$ into (17a) and making some lengthy calculations (the detail is omitted for brevity), we obtain the following zeroth-order equations (19)

$$\bar{P}_{00}^* \boldsymbol{A} + \boldsymbol{A}^T \bar{P}_{00}^* - \bar{P}_{00}^* \boldsymbol{S} \bar{P}_{00}^* + \boldsymbol{Q} = 0, \tag{19a}$$

$$\bar{P}_{f0}^* = -N_2^T + N_1^T \bar{P}_{00}^*, \tag{19b}$$

$$\bar{P}_f^* A_f + A_f^T \bar{P}_f^* - \bar{P}_f^* S_f \bar{P}_f^* + Q_f = 0, \tag{19c}$$

where

$$
\begin{aligned}
\boldsymbol{A} &:= A_0 + N_1 A_{f0} + S_{0f} N_2^T + N_1 S_f N_2^T, \\
\boldsymbol{S} &:= S_{00} + N_1 S_{0f}^T + S_{0f} N_1^T + N_1 S_f N_1^T, \\
\boldsymbol{Q} &:= Q_{00} - N_2 A_{f0} - A_{f0}^T N_2^T - N_2 S_f N_2^T, \\
\bar{P}_{f0}^* &:= \begin{bmatrix} \bar{P}_{10}^{*T} & \cdots & \bar{P}_{N0}^{*T} \end{bmatrix}^T, \ \bar{P}_f^* := \textbf{block diag} \begin{pmatrix} \bar{P}_{11}^* & \cdots & \bar{P}_{NN}^* \end{pmatrix}, \\
\bar{P}_{i0}^{*T} &:= -[\bar{P}_{00}^* D_{0i} + (A_{i0}^T \bar{P}_{ii}^* + Q_{0i})]D_{ii}^{-1}, \\
& \quad \bar{P}_{ii}^* A_{ii} + A_{ii}^T \bar{P}_{ii}^* - \bar{P}_{ii}^* S_{ii} \bar{P}_{ii}^* + Q_{ii} = 0, \ i = 1, \ldots, N, \\
N_1^T &:= -\bar{A}_f^{-T} \bar{A}_{0f}^T = \begin{bmatrix} -D_{01} D_{11}^{-1} \cdots -D_{0N} D_{NN}^{-1} \end{bmatrix}^T = \begin{bmatrix} N_{11} \cdots N_{1N} \end{bmatrix}^T, \\
N_2^T &:= \bar{A}_f^{-T} \bar{Q}_{0f}^T = \begin{bmatrix} \bar{Q}_{01} D_{11} & \cdots & \bar{Q}_{0N} D_{NN} \end{bmatrix}^T = \begin{bmatrix} N_{21} & \cdots & N_{2N} \end{bmatrix}^T, \\
\bar{A}_{0f} &:= A_{0f} - S_{0f} \bar{P}_f^* = \begin{bmatrix} D_{01} & \cdots & D_{0N} \end{bmatrix}, \\
\bar{A}_f &:= A_f - S_f \bar{P}_f^* = \textbf{block diag} \begin{pmatrix} D_{11} & \cdots & D_{NN} \end{pmatrix}, \\
\bar{Q}_{0f} &:= Q_{0f} + A_{f0}^T \bar{P}_f^* = \begin{bmatrix} \bar{Q}_{01} & \cdots & \bar{Q}_{0N} \end{bmatrix}, \\
D_{0i} &:= A_{0i} - S_{0i} \bar{P}_{ii}^*, \ D_{ii} := A_{ii} - S_{ii} \bar{P}_{ii}^*, \\
\bar{Q}_{0i} &:= Q_{0i} + A_{i0}^T \bar{P}_{ii}^*, \ i = 1, \ldots, N.
\end{aligned}
$$

In the following we established the relation between the MARE (8) and the zeroth-order equations (19). Before doing that, we give the results for the AREs (19).

Lemma 4. *[52] Under Assumptions 1 and 2, the following results hold.*

(i) *The matrices $\boldsymbol{A}$, $\boldsymbol{S}$ and $\boldsymbol{Q}$ do not depend on $\bar{P}^*_{ii}$, $i = 1, \ldots, N$. That is, following formulations are satisfied.*

$$\begin{bmatrix} \boldsymbol{A} & -\boldsymbol{S} \\ -\boldsymbol{Q} & -\boldsymbol{A}^T \end{bmatrix} = T_{00} - \sum_{j=1}^{N} T_{0j} T_{jj}^{-1} T_{j0}, \tag{20}$$

where

$$T_{00} := \begin{bmatrix} A_0 & -S_{00} \\ -Q_{00} & -A_0^T \end{bmatrix}, \; T_{0i} := \begin{bmatrix} A_{0i} & -S_{0i} \\ -Q_{0i} & -A_{i0}^T \end{bmatrix},$$

$$T_{i0} := \begin{bmatrix} A_{i0} & -S_{0i}^T \\ -Q_{0i}^T & -A_{0i}^T \end{bmatrix}, \; T_{ii} := \begin{bmatrix} A_{ii} & -S_{ii} \\ -Q_{ii} & -A_{ii}^T \end{bmatrix}, \; i = 1, \ldots, N.$$

(ii) *There exist a matrix $\boldsymbol{B} := \begin{bmatrix} B_{01} + N_{11}B_{11} & \cdots & B_{0N} + N_{1N}B_{NN} \end{bmatrix} \in \boldsymbol{R}^{n_0 \times \bar{m}}$, $\bar{m} := \sum_{j=1}^{N} m_j$ and a matrix $\boldsymbol{C}$ with the same dimension as C_0 such that $\boldsymbol{S} = \boldsymbol{B}R^{-1}\boldsymbol{B}^T$, $\boldsymbol{Q} = \boldsymbol{C}^T\boldsymbol{C}$. Moreover, the triple $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ is stabilizable and detectable.*

Remark 1. *Note the relation*

$$T_{ii} := \begin{bmatrix} A_{ii} & -S_{ii} \\ -Q_{ii} & -A_{ii}^T \end{bmatrix} = \begin{bmatrix} I_{n_i} & 0 \\ \bar{P}^*_{ii} & I_{n_i} \end{bmatrix} \begin{bmatrix} D_{ii} & -S_{ii} \\ 0 & -D_{ii}^T \end{bmatrix} \begin{bmatrix} I_{n_i} & 0 \\ -\bar{P}^*_{ii} & I_{n_i} \end{bmatrix}.$$

Since T_{ii} is nonsingular under Assumption 1 and the ARE (19c) has a stabilizing solution under Assumption 2, D_{ii} is also nonsingular. This means that T_{ii}^{-1} can be expressed explicitly in terms of D_{ii}^{-1}. Using the similar manner, we have the following relations.

$$T_{ii}^{-1} = \begin{bmatrix} I_{n_i} & 0 \\ \bar{P}^*_{ii} & I_{n_i} \end{bmatrix} \begin{bmatrix} D_{ii}^{-1} & -D_{ii}^{-1} S_{ii} D_{ii}^{-T} \\ 0 & -D_{ii}^{-T} \end{bmatrix} \begin{bmatrix} I_{n_i} & 0 \\ -\bar{P}^*_{ii} & I_{n_i} \end{bmatrix}.$$

Theorem 2. *[13, 52] Under Assumptions 1 and 2, there exists a positive scalar $\bar{\sigma}_2$ such that for all $\varepsilon \in H$ with $0 < \|\varepsilon\| \le \bar{\sigma}_2$ the MARE (8) admits a symmetric positive semidefinite stabilizing solution P_ε which can be written as*

$$P_\varepsilon = \Phi_\varepsilon \begin{bmatrix} \bar{P}^*_{00} + O(\|\varepsilon\|) & [\bar{P}^*_{f0} + O(\|\varepsilon\|)]^T \Pi_\varepsilon \\ \bar{P}^*_{f0} + O(\|\varepsilon\|) & \bar{P}^*_f + O(\|\varepsilon\|) \end{bmatrix} \tag{21}$$

$$= \begin{bmatrix} \bar{P}_{00}^* + O(\|\varepsilon\|) & [\bar{P}_{f0}^* + O(\|\varepsilon\|)]^T \Pi_\varepsilon \\ \Pi_\varepsilon[\bar{P}_{f0}^* + O(\|\varepsilon\|)] & \Pi_\varepsilon[\bar{P}_f^* + O(\|\varepsilon\|)] \end{bmatrix},$$

where $\Phi_\varepsilon = \mathbf{block\ diag}\begin{pmatrix} I_{n_0} & \varepsilon_1 I_{n_1} & \cdots & \varepsilon_N I_{n_N} \end{pmatrix}$.

This result can be easily extended to the other multimodeling-type ARE (see e.g., [48, 51, 53]). The cross-coupled MARE is discussed later.

3.3   Nonstandard MSPS

If one of the fast state matrices A_{ii}, $j = 1, \ldots, N$ is singular, the MSPS is called a nonstandard MSPS. In such a case, we cannot utilize the two-time-scale decomposition technique.

Recent theoretical advances in the descriptor system approach allow a revisiting of the various control problems [28]. Since the feedback controller in such problems can be expressed by solutions of the reduced-order and parameter independent AREs, the resulting feedback is derived without invertibility assumptions.

We focus on a specific linear state feedback controller which does not depend on the values of the small parameters. Our methodology is different from the methodology of [1]. This design method is based on the descriptor system approach. If $\|\varepsilon\|$ is very small, it is obvious that the optimal linear state feedback controller (7) can be approximated as

$$u_{\mathrm{app}}(t) = K_{\mathrm{app}}\xi(t) = -R^{-1}B^T \begin{bmatrix} \bar{P}_{00}^* & 0 \\ \bar{P}_{f0}^* & \bar{P}_f^* \end{bmatrix} \xi(t), \tag{22}$$

where

$$\bar{P}_{i0}^* = \begin{bmatrix} \bar{P}_{ii}^* & -I_{n_i} \end{bmatrix} T_{ii}^{-1} T_{i0} \begin{bmatrix} I_{n_0} \\ \bar{P}_{00}^* \end{bmatrix}.$$

Theorem 3. *[52] Under Assumptions 1 and 2, the use of the approximation controller* (22) *results in* J_{app} *satisfying*

$$J_{\mathrm{app}} = J_{\mathrm{opt}} + O(\|\varepsilon\|^2), \tag{23}$$

where $J_{\mathrm{app}} = \xi^T(0)U_\varepsilon\xi(0)$ *with*

$$U_\varepsilon(A_\varepsilon + B_\varepsilon K_{\mathrm{app}}) + (A_\varepsilon + B_\varepsilon K_{\mathrm{app}})^T U_\varepsilon + K_{\mathrm{app}}^T R K_{\mathrm{app}} + Q = 0.$$

The following theorem gives a relation between the composite controller (14) and the approximate controller (22).

Theorem 4. *[52] Under Assumptions 1 and 2, the following identities*

$$X_{ii} = \bar{P}_{ii}^*, \ X_{i0} = \bar{P}_{i0}^*, \ X_{00} = \bar{P}_{00}^*, \ i = 1, \ \dots \ , N \qquad (24)$$

hold. Hence the resulting composite controller (14) *is the same as the composite optimal controller* (22).

It can be observed that the new near-optimal controller (22) is equivalent to the existing one [1] in the case of the standard and the nonstandard MSPSs. We claim that the proposed controller (22) includes the composite near-optimal controller [1] as a special case since the proposed controller can be constructed even if the fast state matrices are singular.

3.4   Numerical Algorithms

In order to obtain the optimal solution to the multimodeling problems, we must solve the MARE, which are parameterized by the small, positive parameters ε_i, $i = 1, \ \dots \ , N$, which have the same order of magnitude. Various reliable approaches to the theory of ARE have been well documented in many literatures (see e.g. [32, 33]). One of the approaches is the invariant subspace approach, which is based on the Hamiltonian matrix [32]. However, such an approach is not adequate for the MSPS since the workspace dimensions required to carry out the calculations for the Hamiltonian matrix are twice those of the original full-system. Another disadvantage is that there is no guarantee of symmetry for the solution of the ARE when the ARE is known to be ill-conditioned [32]. It should be noted that it is very difficult to solve the MARE due to the high dimension and numerical stiffness [18, 19]. To avoid this drawback, various reliable approaches for solving the MARE have been well documented. Three types of numerical algorithms are presented in this paper: the first one is the exact slow-fast decomposition method, the second is a recursive algorithm and the third one is Newton's method.

3.4.1   Exact Slow-fast Decomposition Method

The exact slow-fast decomposition method for solving the MARE has been tackled in [15]. In order to simplify the notation, $N = 2$ is summarized [15].

Let us consider the nonlinear matrix algebraic equations.

$$T_{11}L_1 - T_{10} - \varepsilon_1 L_1(T_{00} - T_{01}L_1 - T_{02}L_2 + T_{02}L_3L_1) = 0, \qquad (25a)$$

$$T_{22}L_2 - \alpha_{12}L_3T_{10} - T_{20} - \varepsilon_2 L_2(T_{00} - T_{02}L_2) = 0, \qquad (25b)$$

$$T_{22}L_3 - \alpha_{12}L_3T_{11} - \varepsilon_2 L_2(T_{01} - T_{02}L_3) = 0, \qquad (25c)$$

$$-H_1T_{11} - \varepsilon_1 H_1 L_1(T_{01} - T_{02}L_3) + (T_{01} - T_{02}L_3) + \varepsilon_1(T_{00} - \qquad (25d)$$
$$-T_{01}L_1 - T_{02}L_2 + T_{02}L_3L_1)H_1 = 0,$$

$$-H_2T_{22} + \alpha_{12}T_{11}H_2 + \varepsilon_2 L_1(T_{01} - T_{02}L_3)H_2 + \qquad (25e)$$
$$+(L_1 - \varepsilon_2 H_2 L_2)T_{02} = 0,$$

$$-H_3T_{22} - \varepsilon_2 H_3 L_2 T_{02} - \varepsilon_2(T_{01} - T_{02}L_3) - T_{02} + \varepsilon_2(T_{00} - \qquad (25f)$$
$$-T_{01}L_1 - T_{02}L_2 + T_{02}L_3L_1)H_3 = 0.$$

These equations can be solved by utilizing the fixed point iterations for L_i and H_i, $i = 1$, 2, 3 [15]. On the other hand, reduced-order pure-slow and pure-fast asymmetric algebraic Riccati equations are derived as follows.

$$P_s a_1 - a_4 P_s - a_3 + P_s a_2 P_s = 0, \qquad (26a)$$

$$P_{f1} b_1 - b_4 P_{f1} - b_3 + P_{f1} b_2 P_{f1} = 0, \qquad (26b)$$

$$P_{f2} c_1 - c_4 P_{f2} - c_3 + P_{f2} c_2 P_{f2} = 0, \qquad (26c)$$

where

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} := T_{00} - T_{01}L_1 - T_{02}L_2 + T_{02}L_3L_1,$$

$$\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} := T_{11}\varepsilon_1 L_1(T_{01} - T_{02}L_3), \quad \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} := T_{22} + \varepsilon_2 L_2 T_{02}.$$

It should be noted that unique positive semidefinite stabilizing solutions exist for the asymmetric AREs defined in (26) exist. These solutions can be obtained by using Newton's method. It is well known that Newton's method converges quadratically under appropriate initial conditions. In fact, this important feature has been proved in [15]. Using the above results, the following matrix is defined.

$$\Pi := \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} = E_2^T K E_1, \qquad (27)$$

where

$$K := \begin{bmatrix} I_{n_0} - \varepsilon_1 H_1 L_1 + \varepsilon_1 \varepsilon_2 H_1 H_2 L_2 + \varepsilon_2 H_3 L_2 \\ L_1 - \varepsilon_2 H_2 L_2 \\ L_2 \end{bmatrix}$$

$$\begin{matrix} -\varepsilon_1 H_1 + \varepsilon_1 \varepsilon_2 H_1 H_2 L_3 + \varepsilon_2 H_3 L_2 & \varepsilon_2 (H_3 + \varepsilon_1 H_1 H_2) \\ I_{n_1} - \varepsilon_2 H_2 L_3 & -\varepsilon_2 H_2 \\ L_3 & I_{n_2} \end{matrix} \end{bmatrix},$$

$$E_1 := \begin{bmatrix} I_{n_0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_0} & 0 & 0 \\ 0 & I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_1^{-1} I_{n_1} & 0 \\ 0 & 0 & I_{n_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \varepsilon_2^{-1} I_{n_2} \end{bmatrix},$$

$$E_2 := \begin{bmatrix} I_{n_0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_0} & 0 & 0 \\ 0 & I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_2} & 0 \\ 0 & 0 & I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n_2} \end{bmatrix}.$$

Finally, we can express P_ε in terms of P_s, P_{f1} and P_{f2}.

$$P_\varepsilon = \left[\Omega_3 + \Omega_4 \cdot \textbf{block diag} \left(\begin{matrix} P_s & P_{f1} & P_{f2} \end{matrix} \right) \right]$$

$$\cdot \left[\Omega_1 + \Omega_2 \cdot \textbf{block diag} \left(\begin{matrix} P_s & P_{f1} & P_{f2} \end{matrix} \right) \right]^{-1}, \tag{28}$$

where

$$\Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} = \Pi^{-1}.$$

However, these results are restricted to the MSPS such that the Hamiltonian matrices for the fast subsystems have no eigenvalues in common (see e.g., Assumption 5, [17]). Thus, we cannot apply the technique proposed in [15] to the practical system.

3.4.2 Recursive Computation

Now, let us define $\phi := \|\varepsilon\| = \sqrt{\varepsilon_1^2 + \varepsilon_2^2}$. The solution (21) of MARE (8) can be changed as follows.

$$
P_\varepsilon = \begin{bmatrix}
\bar{P}_{00} + \phi E_{00} & \varepsilon_1(\bar{P}_{10} + \phi E_{10})^T & \varepsilon_2(\bar{P}_{20} + \phi E_{20})^T \\
\varepsilon_1(\bar{P}_{10} + \phi E_{10}) & \varepsilon_1(\bar{P}_{11} + \phi E_{11}) & \phi^2 E_{21}^T \\
\varepsilon_2(\bar{P}_{20} + \phi E_{20}) & \phi^2 E_{21} & \varepsilon_2(\bar{P}_{22} + \phi E_{22})
\end{bmatrix}, \qquad (29)
$$

where $E_{00} = E_{00}^T$, $E_{11} = E_{11}^T$, $E_{22} = E_{22}^T$.

The $O(\|\varepsilon\|)$ approximation of the error terms E_{pq} will result in $O(\|\varepsilon\|^2)$ approximation of the required matrix P_{pq}. That is why we are interested in finding equations of the error terms and a convenient algorithm to find their solutions. Substituting (29) into (17), we arrive at the recursive algorithm.

$$
\begin{aligned}
& D_{11}^T E_{11}^{(n+1)} + E_{11}^{(n+1)} D_{11} \\
&= -\frac{\varepsilon_1}{\phi}(D_{01}^T \bar{P}_{10}^T + \bar{P}_{10} D_{01}) - \varepsilon_1(D_{01}^T E_{10}^{(n)T} + E_{10}^{(n)} D_{01}) + \frac{\varepsilon_1^2}{\phi} P_{10}^{(n)} S_{00} P_{10}^{(n)T} \\
&\quad + \varepsilon_1(E_{11}^{(n)} S_{01}^T P_{10}^{(n)T} + P_{10}^{(n)} S_{01} E_{11}^{(n)}) + \varepsilon_1\sqrt{\alpha_{12}}(E_{21}^{(n)T} S_{02}^T P_{10}^{(n)T} + \\
&\quad + P_{10}^{(n)} S_{02} E_{21}^{(n)}) + \phi(E_{11}^{(n)} S_{11} E_{11}^{(n)} + \alpha_{12} E_{21}^{(n)T} S_{22} E_{21}^{(n)}), \qquad (30a)
\end{aligned}
$$

$$
\begin{aligned}
& D_{22}^T E_{22}^{(n+1)} + E_{22}^{(n+1)} D_{22} \\
&= -\frac{\varepsilon_2}{\phi}(D_{02}^T \bar{P}_{20}^T + \bar{P}_{20} D_{02}) - \varepsilon_2(D_{02}^T E_{20}^{(n)T} + E_{20}^{(n)} D_{02}) + \frac{\varepsilon_2^2}{\phi} P_{20}^{(n)} S_{00} P_{20}^{(n)T} \\
&\quad + \varepsilon_2(E_{22}^{(n)} S_{02}^T P_{20}^{(n)T} + P_{20}^{(n)} S_{02} E_{22}^{(n)}) + \frac{\varepsilon_2}{\sqrt{\alpha_{12}}}(E_{21}^{(n)} S_{01}^T P_{20}^{(n)T} + \\
&\quad + P_{20}^{(n)} S_{01} E_{21}^{(n)T}) + \phi(E_{22}^{(n)} S_{22} E_{22}^{(n)} + \frac{1}{\alpha_{12}} E_{21}^{(n)} S_{11} E_{21}^{(n)T}), \qquad (30b)
\end{aligned}
$$

$$\sqrt{\alpha_{12}}E_{21}^{(n+1)T}D_{22} + \frac{1}{\sqrt{\alpha_{12}}}D_{11}^T E_{21}^{(n+1)T}$$

$$= -\frac{\varepsilon_1}{\phi}\bar{P}_{10}D_{02} - \frac{\varepsilon_2}{\phi}D_{01}^T\bar{P}_{20}^T - \varepsilon_1 E_{10}^{(n)}D_{02} - \varepsilon_2 D_{01}^T E_{20}^{(n)T} + \varepsilon_1(P_{10}^{(n)}S_{02}E_{22}^{(n)} +$$

$$+ \frac{1}{\sqrt{\alpha_{12}}}P_{10}^{(n)}S_{01}E_{21}^{(n)T}) + \varepsilon_2(E_{11}^{(n)}S_{01}^T P_{20}^{(n)T} + \sqrt{\alpha_{12}}E_{21}^{(n)T}S_{02}^T P_{20}^{(n)T})$$

$$+ \frac{\varepsilon_1\varepsilon_2}{\phi}P_{10}^{(n)}S_{00}P_{20}^{(n)T} + \phi(\sqrt{\alpha_{12}}E_{21}^{(n)T}S_{22}E_{22}^{(n)} \qquad (30c)$$

$$+ \frac{1}{\sqrt{\alpha_{12}}}E_{11}^{(n)T}S_{11}^T E_{21}^{(n)T}),$$

$$D_0^T E_{00}^{(n+1)} + E_{00}^{(n+1)}D_0$$

$$= -D_{10}^T D_{11}^{-T}H_{01}^{(n)T} - H_{01}^{(n)}D_{11}^{-1}D_{10} - D_{20}^T D_{22}^{-T}H_{02}^{(n)T} - H_{02}^{(n)}D_{22}^{-1}D_{20}$$

$$+\phi(E_{00}^{(n)}S_{00}E_{00}^{(n)} + E_{10}^{(n)T}S_{01}^T E_{00}^{(n)} + E_{00}^{(n)}S_{01}E_{10}^{(n)}$$

$$+E_{20}^{(n)T}S_{02}^T E_{00}^{(n)} + E_{00}^{(n)}S_{02}E_{20}^{(n)} + E_{10}^{(n)T}S_{11}^T E_{10}^{(n)} + E_{20}^{(n)T}S_{22}E_{20}^{(n)}), \quad (30d)$$

$$E_{i0}^{(n+1)T} = (H_{0i}^{(n)} - E_{00}^{(n+1)}D_{0i})D_{ii}^{-1}, \; i = 1, \; 2, \qquad (30e)$$

where

$$H_{01}^{(n)} = -D_{10}^T E_{11}^{(n+1)} - \sqrt{\alpha_{12}}D_{20}^T E_{21}^{(n+1)} - \frac{\varepsilon_1}{\phi}D_{00}^T\bar{P}_{10}^T - \varepsilon_1 D_{00}^T E_{10}^{(n)T} +$$

$$+\phi(E_{00}^{(n)}S_{01}E_{11}^{(n)} + E_{10}^{(n)T}S_{11}E_{11}^{(n)}) + \phi\sqrt{\alpha_{12}}(E_{00}^{(n)}S_{02}E_{21}^{(n)} +$$

$$+E_{20}^{(n)T}S_{22}^T E_{21}^{(n)}) + \varepsilon_1(E_{00}^{(n)}S_{00} + E_{10}^{(n)T}S_{01}^T + E_{20}^{(n)T}S_{02}^T)P_{10}^{(n)T},$$

$$H_{02}^{(n)} = -D_{20}^T E_{22}^{(n+1)} - \frac{1}{\sqrt{\alpha_{12}}}D_{10}^T E_{21}^{(n+1)T} - \frac{\varepsilon_2}{\phi}D_{00}^T\bar{P}_{20}^T - \varepsilon_2 D_{00}^T E_{20}^{(n)T} +$$

$$+\phi(E_{00}^{(n)}S_{02}E_{22}^{(n)} + E_{20}^{(n)T}S_{22}E_{22}^{(n)}) + \frac{\phi}{\sqrt{\alpha_{12}}}(E_{00}^{(n)}S_{01}E_{21}^{(n)T} +$$

$$+E_{10}^{(n)T}S_{11}^T E_{21}^{(n)T}) + \varepsilon_2(E_{00}^{(n)}S_{00} + +E_{10}^{(n)T}S_{01}^T + E_{20}^{(n)T}S_{02}^T)P_{20}^{(n)T},$$

$$P_{10}^{(n)} = \bar{P}_{10} + \phi E_{10}^{(n)}, \quad P_{20}^{(n)} = \bar{P}_{20} + \phi E_{20}^{(n)},$$

$$E_{00}^{(0)} = E_{10}^{(0)} = E_{20}^{(0)} = E_{11}^{(0)} = E_{21}^{(0)} = E_{22}^{(0)} = 0.$$

The following theorem indicates the convergence of the algorithm (30).

Theorem 5. *[49] Under Assumptions 1 and 2, there exist the unique and bounded solutions E_{pq} of the error equation in a neighborhood of $\|\varepsilon\| = 0$.*

Moreover, the algorithm (30) *converges to the exact solution* E_{pq} *with the rate of convergence of* $O(\|\varepsilon\|^n)$, *that is*

$$\|E_{pq} - E_{pq}^{(n)}\| = O(\|\varepsilon\|^n), \ n = 1, \ 2, \ ... \ , \ pq = 00, \ 10, \ 20, \ 11, \ 21, \ 22. \ (31)$$

However, there exists the drawback that the recursive algorithm converges only to the approximation solution [49] since the convergence of the recursive algorithm depends on the zeroth-order solutions.

3.4.3 Newton's Method

In this section, we develop an elegant and simple algorithm which converges globally to the positive semidefinite solution of the MARE (8). The algorithm uses the Kleinman algorithm [33], which is equivalent to Newton's method. Thus, this paper presents important improvements upon some of the results of [15, 49] in the sense that one need not assume that the Hamiltonian matrices for the fast subsystems have no eigenvalues in common. Moreover, the convergence solution does not depend on the initial guess, and quadratic convergence is attained.

We propose the following algorithm for solving the MARE (8)

$$(A - SP^{(n)})^T P^{(n+1)} + P^{(n+1)T}(A - SP^{(n)}) + P^{(n)T}SP^{(n)} + Q = 0, \ (32)$$

$$i = 0, \ 1, \ 2, \ ... \ , \ P_\varepsilon^{(n)} = \Phi_\varepsilon P^{(n)} = P^{(n)T}\Phi_\varepsilon,$$

$$P^{(n)} = \begin{bmatrix} P_{00}^{(n)} & \varepsilon_1 P_{10}^{(n)T} & \varepsilon_2 \ P_{20}^{(n)T} \\ P_{10}^{(n)} & P_{11}^{(n)} & \dfrac{1}{\sqrt{\alpha_{21}}} P_{21}^{(n)T} \\ P_{20}^{(n)} & \sqrt{\alpha_{21}} P_{21}^{(n)} & P_{22}^{(n)} \end{bmatrix}, \ A = \Phi_\varepsilon A_\varepsilon, \ S = \Phi_\varepsilon S_\varepsilon \Phi_\varepsilon$$

with the initial condition

$$P^{(0)} = \begin{bmatrix} \bar{P}_{00} & \varepsilon_1 \bar{P}_{10}T & \varepsilon_2 \bar{P}_{20}^T \\ \bar{P}_{10} & \bar{P}_{11} & 0 \\ \bar{P}_{20} & 0 & \bar{P}_{22} \end{bmatrix}, \tag{33}$$

where $\bar{P}_{pq}$, $pq = 00, \ 10, \ 20, \ 11, \ 22$ are defined by (19).

The algorithm (32) has the feature given in the following theorem.

Theorem 6. *[50] Under Assumptions 1 and 2, there exists a positive scalar* $\tilde{\sigma}_1$ *such that for all* $\varepsilon \in H$ *with* $0 < \|\varepsilon\| \le \tilde{\sigma}_1$ *the iterative algorithm* (32) *converges to the exact solution* $P_\varepsilon^* = \Phi_\varepsilon P^* = P^{*T}\Phi_\varepsilon$ *with the rate of quadratic*

convergence, where $P_\varepsilon^{(n)} = \Phi_\varepsilon P^{(n)} = P^{(n)T} \Phi_\varepsilon$ is positive semidefinite. Moreover, zero-order solution $P^{(0)}$ is in the neighborhood of the exact solution P_ε^*. That is, the following conditions are satisfied.

$$\|P^{(n)} - P^*\| \le \frac{(2\theta)^{2^n}}{2^n \beta \boldsymbol{L}} = O(\|\varepsilon\|^{2^n}), \ n = 0, \ 1, \ 2, \ \cdots, \tag{34a}$$

$$\|P^{(0)} - P^*\| \le \frac{1}{\beta \boldsymbol{L}} [1 - \sqrt{1 - 2\theta}], \tag{34b}$$

where

$$\boldsymbol{L} := 2\|S\| < \infty, \ \beta := \|[\nabla \boldsymbol{F}(\boldsymbol{P}_0)]^{-1}\|, \ \theta := \beta \eta \boldsymbol{L}$$

with

$$\eta := \beta \cdot \|\boldsymbol{F}(\boldsymbol{P}_0)\|, \ \boldsymbol{F}(\boldsymbol{P}) := \begin{bmatrix} \mathrm{vec}F_{00} \\ \mathrm{vec}F_{10} \\ \mathrm{vec}F_{20} \\ \mathrm{vec}F_{11} \\ \mathrm{vec}F_{21} \\ \mathrm{vec}F_{22} \end{bmatrix},$$

$$A^T P + P^T A - P^T S P + Q = \begin{bmatrix} F_{00} & F_{10}^T & F_{20}^T \\ F_{10} & F_{11} & F_{21}^T \\ F_{20} & F_{21} & F_{22} \end{bmatrix},$$

and

$$\nabla \boldsymbol{F}(\boldsymbol{P}) := \frac{\partial \boldsymbol{F}(\boldsymbol{P})}{\partial \boldsymbol{P}^T}, \ \boldsymbol{P} = \begin{bmatrix} \mathrm{vec}P_{00} \\ \mathrm{vec}P_{10} \\ \mathrm{vec}P_{20} \\ \mathrm{vec}P_{11} \\ \mathrm{vec}P_{21} \\ \mathrm{vec}P_{22} \end{bmatrix}, \ \boldsymbol{P}_0 = \begin{bmatrix} \mathrm{vec}\bar{P}_{00} \\ \mathrm{vec}\bar{P}_{10} \\ \mathrm{vec}\bar{P}_{20} \\ \mathrm{vec}\bar{P}_{11} \\ 0 \\ \mathrm{vec}\bar{P}_{22} \end{bmatrix}.$$

These proofs can be derived by applying the Newton-Kantorovich theorem [34, 35].

It should be noted that the proposed algorithm, which is based on the Kleinman algorithm, has quadratic convergence. It may also be noted that to solve the multiparameter algebraic Lyapunov equation (MALE), a fixed-point algorithm can be combined. See [50] for details. In addition, it has

been proved that the resulting $O(\|\varepsilon\|^{2^n})$ accuracy controller achieves the cost $J_{\text{opt}} + O(\|\varepsilon\|^{2^{n+1}})$.

Remark 2. *Using the Newton-Kantorovich theorem [34, 35], which will be presented later in this paper, it is clear that there exists a positive scalar $\tilde{\sigma}_2$ such that for all $\varepsilon \in H$ with $0 < \|\varepsilon\| \leq \tilde{\sigma}_2$, the MARE (8) has positive semidefinite solutions within the limits of the sufficiency condition. Moreover, it should be noted that the asymptotic structure of (21) can also be obtained by applying the Newton-Kantorovich theorem.*

4 Extension to Other Problem

The above-mentioned techniques can be demonstrated for the filtering and the various control.

4.1 Filtering Problem

Filtering problems for MSPS have been investigated extensively. In [51], a new design method for the near-optimal Kalman filters has been proposed. As a result, the high-dimensional ill-conditioned MARE is replaced by the low-order singular perturbation parameter-independent ARE. Furthermore, the proposed filters can be implemented even if the fast state matrices are singular and the perturbation parameters are unknown. In [12], the well-posedness of multimodel strategies for a LQ-Gaussian (LQG) optimal control problem has been studied. In addition, numerical stiffness is avoided by using the exact slow-fast decomposition method for solving the filtered MARE in [17]. The local control problem of a control agent of the above paper is obtained by neglecting the fast dynamics of the other agent's subsystem, and each agent uses the optimal solution of its local control problem. However, the nonsingularity assumptions for the fast state matrices A_{ii}, $i = 1, \ldots, N$ are also needed. To avoid this drawback, a new recursive algorithm for solving the MARE has been proposed [54]. It has been proved that the solution of the MARE converges to a positive semi-definite stabilizing solution with the rate of convergence of $O(\|\varepsilon\|^{n+1})$, where i denotes the number of required iterations. Moreover, it has been recently proved that the resulting Kalman filter achieves a performance level, i.e. $O(\|\varepsilon\|^{2n+1})$, close to the optimal mean square error.

4.2 H_∞ Control Problem

The asymptotic expansions for MARE with a sign-indefinite quadratic term that arises in the H_∞ control problem and an iterative technique for solving such MARE are described in [48]. In [59], a new iterative algorithm for solving MARE with a sign-indefinite quadratic term has been proposed for the general case. The proposed algorithm consists of Newton's method and two fixed-point algorithms. As a result, it has been proven that the solution of the MARE converges to a positive semi-definite stabilizing solution with a rate of convergence of $O(\|\varepsilon\|^{2^n})$. Moreover, compared with the existing results [48], a reduction in the size of the computational work space can be achieved even if the MSPS has many fast subsystems. This algorithm for solving the MARE and MALE is applied to a wide class of control law synthesis methods involving a solution to the MARE, such as in the robust stabilizing control problem. On the other hand, a reliable H_∞ control for linear time-invariant MSPS against sensor failures has been investigated [30]. The main contribution of this paper was an extension of the previous study of the reliable H_∞ control.

4.3 Guaranteed Cost Control Problem

The multi-parameter singularly perturbed guaranteed cost control problem has been demonstrated [56]. By solving the reduced-order slow and fast AREs, the new ε-independent guaranteed cost controller can be obtained. The new technique has the following advantages: It does not need information on the small parameters ε_i. The required work space is the same as that of the reduced-order slow and fast subsystems. The present new results can be applied to the MSPS without the need for the various assumptions that have been made for the fast subsystems in the existing results, although the fast subsystems have the uncertainty. Therefore, the new design approach has been successfully applied to a more practical uncertain MSPS. Furthermore, if the parameters are known, we can obtain the exact GCC by using the above-mentioned numerical technique. As another important approach to the uncertain MSPS except for the guaranteed cost control problem, the fault diagnosis of two-time-scale MSPSs has been considered in [31].

5 Nash Games

The LQ Nash games for the MSPS have been studied by using composite controller design [5, 57, 58]. Furthermore, a decentralized stochastic Nash game has been presented for two decision makers controlling MSPS [8]. According to this result, in order to obtain near-equilibrium Nash strategies, the decision makers need only to solve two coupled low-order stochastic control problems. Furthermore, decentralized team strategies for decision makers using MSPS have been developed [10]. The well-posedness of the multimodel solution was demonstrated. Recently, computational approaches for Nash games have been studied [53, 55, 62]. For obtaining the strategies, Newton's method [55] seems to be very powerful tool. In this section, existing and recent progress on the use of the two-time-scale decomposition method and numerical analysis related to Nash games for MSPSs will be reviewed.

5.1 Parameter Independent Strategies

Consider a linear time-invariant MSPS

$$\dot{x}(t) = A_0 x(t) + \sum_{j=1}^{N} A_{0j} z_j(t) + \sum_{j=1}^{N} B_{0j} u_j(t), \ x(0) = x^0, \tag{35a}$$

$$\varepsilon_i \dot{z}_i(t) = A_{i0} x(t) + A_{ii} z_i(t) + B_{ii} u_i(t), \ z_i(0) = z_i^0, \ i = 1, \ ... \ , N, \tag{35b}$$

with the quadratic cost functions

$$J_i(u_1, \ ... \ , u_N) = \frac{1}{2} \int_0^\infty [y_i^T y_i + u_i^T R_{ii} u_i] dt, \tag{36a}$$

$$y_i = C_{i0} x + C_{ii} z_i = C_i \xi. \tag{36b}$$

These conditions are quite natural since at least one control agent has to be able to control and observe unstable modes. Our purpose is to find a linear feedback strategy set $(u_1^*, \ ... \ , u_N^*)$ such that

$$J_i(u_1^*, \ ... \ , \ u_N^*) \le J_i(u_1^*, \ ... \ , \ u_{i-1}^*, \ u_i, \ u_{i+1}^*, \ ... \ , \ u_N^*), \ i = 1, \ ... \ , N \tag{37}$$

The decision makers are required to select the closed loop strategy u_i^*, if they exist, such that (37) holds. Moreover, each player uses the strategy u_i^* such that the closed-loop system is asymptotically stable for sufficiently small ε_i. The following lemma is already known [36].

Lemma 5. *There exists an admissible strategy such that the inequality* (37) *holds iff the cross-coupled multiparameter algebraic Riccati equations* (CMAREs)

$$P_{i\varepsilon}\left(A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon}P_{j\varepsilon}\right) + \left(A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon}P_{j\varepsilon}\right)^T P_{i\varepsilon} + P_{i\varepsilon}S_{i\varepsilon}P_{i\varepsilon} + Q_i = 0, \quad (38)$$

$i = 1, \ldots, N,$ *have solutions* $P_{i\varepsilon} \geq 0,$ *where*

$$P_{i\varepsilon} := \begin{bmatrix} P_{i00} & P_{if0}^T\Pi_\varepsilon \\ \Pi_\varepsilon P_{if0} & \Pi_\varepsilon P_{if} \end{bmatrix}, \ P_{i00} = P_{i00}^T, \ P_{if0} \begin{bmatrix} P_{i10} \\ \vdots \\ P_{iN0} \end{bmatrix},$$

$$P_{if} := \begin{bmatrix} P_{i11} & \alpha_{12}P_{i21}^T & \alpha_{13}P_{i31}^T & \cdots & \alpha_{1N}P_{iN1}^T \\ P_{i21} & P_{i22} & \alpha_{23}P_{i32}^T & \cdots & \alpha_{2N}P_{iN2}^T \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{i(N-1)1} & P_{i(N-1)2} & P_{i(N-1)3} & \cdots & \alpha_{(N-1)N}P_{iN(N-1)}^T \\ P_{iN1} & P_{iN2} & P_{iN3} & \cdots & P_{iNN} \end{bmatrix},$$

$$B_1 := \begin{bmatrix} B_{10} \\ B_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, B_i := \begin{bmatrix} B_{i0} \\ \vdots \\ B_{ii} \\ \vdots \\ 0 \end{bmatrix}, \cdots, B_N := \begin{bmatrix} B_{0N} \\ 0 \\ 0 \\ \vdots \\ B_{NN} \end{bmatrix},$$

$$S_{i\varepsilon} := \Phi_\varepsilon^{-1} B_i R_{ii}^{-1} B_i^T \Phi_\varepsilon^{-1},$$

$$S_i := B_i R_{ii}^{-1} B_i^T = \begin{bmatrix} S_{i00} & O & S_{i0i} & O \\ O & O & O & O \\ S_{i0i}^T & O & S_{iii} & O \\ O & O & O & O \end{bmatrix},$$

$$Q_i := C_i C_i^T = \begin{bmatrix} Q_{i00} & O & Q_{i0i} & O \\ O & O & O & O \\ Q_{i0i}^T & O & Q_{iii} & O \\ O & O & O & O \end{bmatrix},$$

$$\Phi_\varepsilon := \textbf{\textit{block diag}} \begin{pmatrix} I_{n_0} & \varepsilon_1 I_{n_1} & \cdots & \varepsilon_N I_{n_N} \end{pmatrix}.$$

Then the closed-loop linear Nash equilibrium solutions to the full-order prob-

lem are given by

$$u_i^*(t) = -R_{ii}^{-1} B_{i\varepsilon}^T P_{i\varepsilon} \xi(t). \tag{39}$$

It should be noted that it is impossible to solve the CMARE (38) if the small perturbed parameter ε_i are unknown. Thus, the purpose of this section is to find the parameter-independent Nash strategies.

The parameter-independent Nash strategies for the MSPS will be studied under the following basic assumption.

Assumption 3. *The Hamiltonian matrices T_{iii}, $i = 1$, ... N are nonsingular, where*

$$T_{iii} := \begin{bmatrix} A_{ii} & -S_{iii} \\ -Q_{iii} & -A_{ii}^T \end{bmatrix}. \tag{40}$$

Under Assumptions 1-3, the following zeroth-order equations of the CMAREs (38) are given as $\|\varepsilon\| \to +0$.

$$\bar{P}_{i00} \left(A_s - \sum_{j=1}^N S_{s_j} \bar{P}_{j00} \right) + \left(A_s - \sum_{j=1}^N S_{s_j} \bar{P}_{j00} \right)^T \bar{P}_{i00} + \tag{41a}$$

$$+ \bar{P}_{i00} S_{s_i} \bar{P}_{i00} + Q_{s_i} = 0,$$

$$A_{ii}^T \bar{P}_{iii} + \bar{P}_{iii} A_{ii} - \bar{P}_{iii} S_{iii} \bar{P}_{iii} + Q_{iii} = 0, \tag{41b}$$

$$\bar{P}_{ikl} = 0, \ k > l, \ \bar{P}_{ijj} = 0, \ i \neq j \tag{41c}$$

$$\begin{bmatrix} \bar{P}_{110} & \bar{P}_{210} & \cdots & \bar{P}_{N10} \end{bmatrix} = \begin{bmatrix} \bar{P}_{111} \\ -I_{n_1} \end{bmatrix}^T T_{111}^{-1} T_{110} \begin{bmatrix} I_{n_0} & 0 & \cdots & 0 \\ \bar{P}_{100} & \bar{P}_{200} & \cdots & \bar{P}_{N00} \end{bmatrix},$$

$$\begin{bmatrix} \bar{P}_{120} & \bar{P}_{220} & \cdots & \bar{P}_{N20} \end{bmatrix} = \begin{bmatrix} \bar{P}_{222} \\ -I_{n_2} \end{bmatrix}^T T_{222}^{-1} T_{220} \begin{bmatrix} 0 & I_{n_0} & \cdots & 0 \\ \bar{P}_{100} & \bar{P}_{200} & \cdots & \bar{P}_{N00} \end{bmatrix},$$

$$\vdots$$

$$\begin{bmatrix} \bar{P}_{1N0} & \bar{P}_{2N0} \cdots \bar{P}_{NN0} \end{bmatrix} = \begin{bmatrix} \bar{P}_{NNN} \\ -I_{n_N} \end{bmatrix}^T T_{NNN}^{-1} T_{NN0} \begin{bmatrix} 0 & 0 & \cdots & I_{n_0} \\ \bar{P}_{100} & \bar{P}_{200} \cdots \bar{P}_{N00} \end{bmatrix}, \tag{41d}$$

where

$$\begin{bmatrix} A_s & * \\ * & -A_s^T \end{bmatrix} = \begin{bmatrix} A_0 & * \\ * & -A_0^T \end{bmatrix} - \sum_{i=1}^{N} T_{i0i} T_{iii}^{-1} T_{ii0},$$

$$\begin{bmatrix} * & -S_{s_i} \\ -Q_{s_i} & * \end{bmatrix} = T_{i00} - T_{i0i} T_{iii}^{-1} T_{ii0},$$

$$T_{i00} = \begin{bmatrix} A_0 & -S_{i00} \\ -Q_{i00} & -A_0^T \end{bmatrix}, T_{i0i} = \begin{bmatrix} A_{0i} & -S_{i0i} \\ -Q_{i0i} & -A_{i0}^T \end{bmatrix}, T_{ii0} = \begin{bmatrix} A_{i0} & -S_{i0i}^T \\ -Q_{i0i}^T & -A_{0i}^T \end{bmatrix},$$

$$i = 1, \dots, N.$$

The following theorem shows the relation between the solutions P_i and the zeroth-order solutions $\bar{P}_{ikl}$ $i = 1, \dots, N$, $k \geq l$, $0 \leq k$, $l \leq N$.

$$\det \begin{bmatrix} \hat{A}_s^T \otimes I_{n_0} + I_{n_0} \otimes \hat{A}_s^T & -(S_{s_2}\bar{P}_{100}) \otimes I_{n_0} - I_{n_0} \otimes (S_{s_2}\bar{P}_{100}) & \cdots \\ -(S_{s_1}\bar{P}_{200}) \otimes I_{n_0} - I_{n_0} \otimes (S_{s_1}\bar{P}_{200}) & \hat{A}_s^T \otimes I_{n_0} + I_{n_0} \otimes \hat{A}_s^T & \cdots \\ \vdots & \vdots & \ddots \\ -(S_{s_1}\bar{P}_{N00}) \otimes I_{n_0} - I_{n_0} \otimes (S_{s_1}\bar{P}_{N00}) & -(S_{s_2}\bar{P}_{N00}) \otimes I_{n_0} - I_{n_0} \otimes (S_{s_2}\bar{P}_{N00}) & \cdots \end{bmatrix}$$

$$\begin{matrix} \cdots & -(S_{s_N}\bar{P}_{100}) \otimes I_{n_0} - I_{n_0} \otimes (S_{s_N}\bar{P}_{100}) \\ \cdots & -(S_{s_N}\bar{P}_{200}) \otimes I_{n_0} - I_{n_0} \otimes (S_{s_N}\bar{P}_{200}) \\ \ddots & \vdots \\ \cdots & \hat{A}_s^T \otimes I_{n_0} + I_{n_0} \otimes \hat{A}_s^T \end{matrix} \Bigg] \neq 0, \qquad (42)$$

where $\hat{A}_s := A_s - \sum_{j=1}^{N} S_{s_j} \bar{P}_{j00}$ and $\hat{A}_s$ are stable matrix.

Theorem 7. *Suppose that the condition* (42) *holds. Under Assumptions 1 and 2, there is a neighborhood* $\boldsymbol{V}(0)$ *of* $\|\varepsilon\| = 0$ *such that for all* $\|\varepsilon\| \in \boldsymbol{V}(0)$ *there exists a solution* $P_i = P_i(\varepsilon_1, \dots, \varepsilon_N)$. *These solutions are unique in a neighborhood of* $\bar{P}_i = P_i(0, \dots, 0)$. *Then, the MARE* (38) *possess the power series expansion at* $\|\varepsilon\| = 0$. *That is, the following form is satisfied.*

$$P_{i\varepsilon} := \Phi_\varepsilon P_i, \, P_i = \bar{P}_i + O(\|\varepsilon\|) = \begin{bmatrix} \bar{P}_{i00} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \bar{P}_{i10} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{P}_{ii0} & 0 & \cdots & 0 & \bar{P}_{iii} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{P}_{iN0} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} + O(\|\varepsilon\|). \quad (43)$$

5.2   Numerical Algorithms

When the parameters represent small unknown perturbations whose values are not known exactly, the previously introduced composite design is very useful. However, the composite Nash equilibrium solution achieves only a performance level of $O(\|\varepsilon\|)$, close to the full-order performance. Another important drawback is that since the closed-loop solution of the reduced Nash problem depends on the path along $\varepsilon_1/\varepsilon_2$ as $\|\varepsilon\| \to +0$, we cannot conclude that the closed-loop solution of the full problem converges to the closed-loop solution of the reduced problem [2]. Therefore, as long as the small perturbation parameters ε_i are known, much effort should be made towards finding the exact strategies which guarantees Nash equilibrium without ill-conditioning. In this subsection, the iterative algorithms for solving the CMAREs are summarized.

5.2.1   Recursive Computation

A recursive algorithm for solving singularly perturbed Nash games has been attempted [53]. It has been shown that the recursive algorithm is very effective in solving the CMAREs when the system matrices are functions of a small perturbation parameter ε_i. However, the recursive algorithm converges only to the approximation solution because the convergence solutions depend on the zeroth-order solutions. In addition, the recursive algorithm has the property of linear convergence. Thus, the convergence speed is very slow.

5.2.2   Newton's Method

In order to improve the convergence rate of the recursive algorithm, we propose the following algorithm which is based on the Newton's method.

$$\Phi^{(n)T}\boldsymbol{P}^{(n+1)} + \boldsymbol{P}^{(n+1)T}\Phi^{(n)} - \Theta^{(n)T}\boldsymbol{P}^{(n+1)}\boldsymbol{J} - \boldsymbol{J}\boldsymbol{P}^{(n+1)T}\Theta^{(n)} + \Xi^{(n)} = 0,$$

$$n = 0,\ 1,\ \dots\ , \tag{44}$$

$$\Leftrightarrow \begin{cases} \Phi_1^{(n)T}P_1^{(n+1)} + P_1^{(n+1)T}\Phi_1^{(n)} - \Theta_2^{(n)T}P_2^{(n+1)} - P_2^{(n+1)T}\Theta_2^{(n)} + \Xi_1^{(n)} = 0, \\ \Phi_2^{(n)T}P_2^{(n+1)} + P_2^{(n+1)T}\Phi_2^{(n)} - \Theta_1^{(n)T}P_1^{(n+1)} - P_1^{(n+1)T}\Theta_1^{(n)} + \Xi_2^{(n)} = 0, \end{cases}$$

where

$$\Phi^{(n)} := \tilde{A} - \tilde{S}\boldsymbol{P}^{(n)} - \boldsymbol{J}\tilde{S}\boldsymbol{P}^{(n)}\boldsymbol{J} = \begin{bmatrix} \Phi_1^{(n)} & 0 \\ 0 & \Phi_2^{(n)} \end{bmatrix},$$

$$\Theta^{(n)} := \tilde{S} \boldsymbol{J} \boldsymbol{P}^{(n)} = \begin{bmatrix} 0 & \Theta_1^{(n)} \\ \Theta_2^{(n)} & 0 \end{bmatrix},$$

$$\Xi^{(n)} := \tilde{Q} + \boldsymbol{P}^{(n)T} \tilde{S} \boldsymbol{P}^{(n)} + \boldsymbol{J} \boldsymbol{P}^{(n)T} \tilde{S} \boldsymbol{J} \boldsymbol{P}^{(n)} + \boldsymbol{P}^{(n)T} \boldsymbol{J} \tilde{S} \boldsymbol{P}^{(n)} \boldsymbol{J}$$

$$= \begin{bmatrix} \Xi_1^{(n)} & 0 \\ 0 & \Xi_2^{(n)} \end{bmatrix},$$

$$\Phi_i^{(n)} := \begin{bmatrix} \Phi_{00i}^{(n)} & \Phi_{01i}^{(n)} & \Phi_{02i}^{(n)} \\ \Phi_{10i}^{(n)} & \Phi_{11i}^{(n)} & \Phi_{12i}^{(n)} \\ \Phi_{20i}^{(n)} & \Phi_{21i}^{(n)} & \Phi_{22i}^{(n)} \end{bmatrix}, \quad \Theta_i^{(n)} := \begin{bmatrix} \Theta_{00i}^{(n)} & \Theta_{01i}^{(n)} & \Theta_{02i}^{(n)} \\ \Theta_{10i}^{(n)} & \Theta_{11i}^{(n)} & \Theta_{12i}^{(n)} \\ \Theta_{20i}^{(n)} & \Theta_{21i}^{(n)} & \Theta_{22i}^{(n)} \end{bmatrix},$$

$$\Xi_i^{(n)} := \begin{bmatrix} \Xi_{00i}^{(n)} & \Xi_{01i}^{(n)} & \Xi_{02i}^{(n)} \\ \Xi_{01i}^{(n)T} & \Xi_{11i}^{(n)} & \Xi_{12i}^{(n)} \\ \Xi_{02i}^{(n)T} & \Xi_{12i}^{(n)T} & \Xi_{22i}^{(n)} \end{bmatrix}, \ i = 1, \ 2,$$

$$\boldsymbol{P}^{(n)} := \begin{bmatrix} P_1^{(n)} & 0 \\ 0 & P_2^{(n)} \end{bmatrix},$$

$$P_1^{(n)} := \begin{bmatrix} P_{100}^{(n)} & \varepsilon_1 P_{110}^{(n)T} & \varepsilon_2 P_{120}^{(n)T} \\ P_{110}^{(n)} & P_{111}^{(n)} & \sqrt{\alpha_{21}}^{-1} P_{121}^{(n)T} \\ P_{120}^{(n)} & \sqrt{\alpha_{21}} P_{121}^{(n)} & P_{122}^{(n)} \end{bmatrix},$$

$$P_2^{(n)} := \begin{bmatrix} P_{200}^{(n)} & \varepsilon_1 P_{210}^{(n)T} & \varepsilon_2 P_{220}^{(n)T} \\ P_{210}^{(n)} & P_{211}^{(n)} & \sqrt{\alpha_{21}}^{-1} P_{221}^{(n)T} \\ P_{220}^{(n)} & \sqrt{\alpha_{21}} P_{221}^{(n)} & P_{222}^{(n)} \end{bmatrix},$$

$$\tilde{A} := \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \ \tilde{Q} := \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, \ \tilde{S} := \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix},$$

$$\boldsymbol{J} := \begin{bmatrix} 0 & I_{\bar{n}} \\ I_{\bar{n}} & 0 \end{bmatrix}, \ A := \Phi_\varepsilon A_\varepsilon.$$

and the initial condition $\boldsymbol{P}^{(0)}$ has the following form

$$\boldsymbol{P}^{(0)} = \begin{bmatrix} P_1^{(0)} & 0 \\ 0 & P_2^{(0)} \end{bmatrix} = \begin{bmatrix} \bar{P}_{100} & \varepsilon_1 \bar{P}_{110}^T & \varepsilon_2 \bar{P}_{120}^T & 0 & 0 & 0 \\ \bar{P}_{110} & \bar{P}_{111} & 0 & 0 & 0 & 0 \\ \bar{P}_{120} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{P}_{200} & \varepsilon_1 \bar{P}_{210}^T & \varepsilon_2 \bar{P}_{220}^T \\ 0 & 0 & 0 & \bar{P}_{210} & 0 & 0 \\ 0 & 0 & 0 & \bar{P}_{220} & 0 & \bar{P}_{222} \end{bmatrix}. \quad (45)$$

Note that the considered algorithm (44) is original. The new algorithm (44) can be constructed by setting $\boldsymbol{P}^{(n+1)} = \boldsymbol{P}^{(n)} + \Delta\boldsymbol{P}^{(n)}$ and neglecting $O(\Delta\boldsymbol{P}^{(n)T}\Delta\boldsymbol{P}^{(n)})$ term. Newton's method is well-known and is widely used to find a solution of the algebraic equations, and its local convergence properties are well understood.

Theorem 8. *Under Assumptions 1-3, the new iterative algorithm (44) converges to the exact solution $\boldsymbol{P}^*$ of the CMAREs (38) with the rate of quadratic convergence. Furthermore, the unique bounded solution $\boldsymbol{P}^{(n)}$ of the CMAREs (38) is in the neighborhood of the exact solution $\boldsymbol{P}^*$. That is, the following conditions are satisfied.*

$$\|\boldsymbol{P}^{(n)} - \boldsymbol{P}^*\| \leq O(\|\varepsilon\|^{2^n}), \ n = 0, \ 1, \ \dots \ , \tag{46a}$$

$$\|\boldsymbol{P}^{(n)} - \boldsymbol{P}^*\| \leq \frac{1}{\tilde{\beta}\tilde{\boldsymbol{L}}}[1 - \sqrt{1 - 2\tilde{\theta}}], \ n = 0, \ 1, \ \dots \ , \tag{46b}$$

where

$$\boldsymbol{P} = \boldsymbol{P}^* = \begin{bmatrix} P_1^* & 0 \\ 0 & P_2^* \end{bmatrix}, \ \tilde{\boldsymbol{L}} := 6\|\tilde{S}\|, \ \tilde{\beta} := \|[\nabla\boldsymbol{F}(\boldsymbol{P}^{(0)})]^{-1}\|, \ \tilde{\theta} := \tilde{\beta}\tilde{\eta}\tilde{\boldsymbol{L}},$$

$$\tilde{\eta} := \|[\nabla\boldsymbol{F}(\boldsymbol{P}^{(0)})]^{-1}\| \cdot \|\boldsymbol{F}(\boldsymbol{P}^{(0)})\|.$$

6 Stochastic MSPS Governed by Itô Equations

The various control problems for stochastic systems governed by Itô's differential equation have attracted considerable research interest. The stabilization, LQ optimal control and H_∞ control problems for singularly perturbed stochastic systems (SPSS) with state-dependent noise were investigated [37, 43, 44]. Although these results are very elegant and despite it being easy to obtain a controller, the multiparameter singularly perturbed stochastic systems (MSPSS) remain to be considered. The problem of exponential stability of the zero state equilibrium of a linear stochastic system modeled by a system of singularly perturbed Itô differential equations is investigated in [20, 37, 42],

The LQ optimal stochastic control problem for MSPSS in which N lower-level fast subsystems are interconnected through a higher-level slow subsystem has been investigated [60]. The stochastic H_∞ control problem for the MSPSS has been discussed [61]. In particular, a new iterative algorithm for

solving the stochastic multimodeling algebraic Riccati equation (SMARE) that has sign-indefinite quadratic form has been proposed. Stochastic Nash games have been studied for stochastic multimodeling systems [62]. The main contribution of this paper is the new strategy set that is independent of the small parameters. In [63], the guaranteed cost control problem for MSPSS has been re-formulated as an extension of [56].

In this section, the numerical solution to the SMARE with a sign-indefinite quadratic term related to the stochastic H_∞ control problem with state-dependent noise is investigated. It may be noted that a similar technique can be used for several stochastic control problems [60, 62, 63].

We consider the following MSPSS that consist of N-fast subsystems with specific structure of lower level interconnected through the dynamics of a higher level slow subsystem.

$$d\xi(t) = [A_\varepsilon \xi(t) + B_\varepsilon u(t) + D_\varepsilon v(t)]dt + \sum_{p=1}^{M} A_{p\varepsilon} \xi(t) dw_p(t), \qquad (47\text{a})$$

$$z(t) = \left[\begin{array}{c} C\xi(t) \\ Hu(t) \end{array} \right], \qquad (47\text{b})$$

where

$$\xi(t) := \left[\begin{array}{c} x(t) \\ z_1(t) \\ \vdots \\ z_N(t) \end{array} \right] \in \Re^{\bar{n}}, \ u(t) := \left[\begin{array}{c} u_1(t) \\ \vdots \\ u_N(t) \end{array} \right] \in \Re^{\bar{m}},$$

$$v(t) := \left[\begin{array}{c} v_1(t) \\ \vdots \\ v_N(t) \end{array} \right] \in \Re^{\bar{l}},$$

$$\bar{n} := \sum_{j=0}^{N} n_j, \ \bar{m} := \sum_{j=1}^{N} m_j, \ \bar{l} := \sum_{j=1}^{N} l_j,$$

$$A_{p\varepsilon} := \left[\begin{array}{cc} A_{p0} & \mu A_{p0f} \\ \Pi_\varepsilon^{-1} \bar{\varepsilon}^\delta A_{pf0} & \Pi_\varepsilon^{-1} \bar{\varepsilon}^\delta A_{pf} \end{array} \right], \ A_{p0f} := \left[\begin{array}{ccc} A_{p01} & \cdots & A_{p0N} \end{array} \right],$$

$$A_{pf0} := \left[\begin{array}{ccc} A_{p10}^T & \cdots & A_{pN0}^T \end{array} \right]^T,$$

$$A_{pf} := \textbf{block diag} \left(\begin{array}{ccc} A_{p11} & \cdots & A_{pNN} \end{array} \right),$$

$$D_\varepsilon := \begin{bmatrix} D_0 \\ \Pi_\varepsilon^{-1} D_f \end{bmatrix}, \ D_0 := \begin{bmatrix} D_{01} & \cdots & D_{0N} \end{bmatrix},$$

$$D_f := \textbf{block diag} \begin{pmatrix} D_{11} & \cdots & D_{NN} \end{pmatrix},$$

$$H := \textbf{block diag} \begin{pmatrix} H_{11} & \cdots & H_{NN} \end{pmatrix}.$$

$v_i(t) \in L_2^{l_i}(0, \infty)$, $i = 1, \ldots, N$ is considered to be an unknown finite-energy deterministic disturbance [45, 46]. $z(t) \in \Re^p$ is the controlled output. $\varepsilon_i > 0$, $i = 1, \ldots, N$ and $\mu > 0$ are small parameters and $\delta > 1/2$ is independent of $\bar\varepsilon := \min\{\varepsilon_1, \ldots, \varepsilon_N\}$. It should be noted that the parameters μ and δ have been introduced in [43, 44] for the first time. Moreover, the considered MSPSS consists of N-fast subsystems as compared to [43]. $w_p(t) \in \Re$, $p = 1, \ldots, M$ is a one-dimensional standard Wiener process defined in the filtered probability space. Note that one of the fast state matrices A_{ii}, $i = 1, \ldots, N$ may be singular.

Remark 3. *In stochastic problems, careful treatment is required to establish the validity of the multimodel problem [11]. In addition to the usual difficulties encountered in modeling a fast stochastic variable, the problem is rether involved due to the presence of information patterns. To simplify this aspect, the scaling parameter μ is considered.*

Without loss of generality, the stochastic H_∞ control problem for the MSPSS is investigated under the following basic assumption [45, 46].

Assumption 4. $H^T H = I_{\bar m}$.

It should be noted that the matrix pair (E, G) is deemed stable, if $d\xi(t) = E\xi(t)dt + G\xi(t)dw$ is asymptotically mean square stable [46].

The stochastic H_∞ control problem for MSPSS is given below [45, 46].

Given a constant $\gamma > 0$, find a matrix K satisfying the following conditions:

i) *The system*

$$d\xi(t) = [A_\varepsilon + B_\varepsilon K]\xi(t)dt + \sum_{p=1}^{M} A_{p\varepsilon}\xi(t)dw_p(t) \tag{48}$$

is exponentially mean-square stable (EMSS) *internally, i.e. it satisfies the following equation.*

$$E\|\xi(t)\|^2 \le \rho e^{-\psi(t-s)} E\|\xi(s)\|^2, \ \exists\rho, \ \psi > 0. \tag{49}$$

ii) *The closed-loop system*

$$d\xi(t) = [(A_\varepsilon + B_\varepsilon K)\xi(t) + D_\varepsilon v(t)]dt + \sum_{p=1}^{M} A_{p\varepsilon}\xi(t)dw_p(t), \quad (50a)$$

$$z(t) = \begin{bmatrix} C \\ HK \end{bmatrix} \xi(t), \quad (50b)$$

corresponding to the system in equation (50) *with feedback control* $u(t) = K\xi(t)$, *satisfies following condition.*

$$\sup_{\substack{v \in L_2^{\bar{l}}(0,\ \infty), \\ v \neq 0,\ \xi(0) = 0}} \frac{\|z\|_2^2}{\|v\|_2^2} := \sup_{\substack{v \in L_2^{\bar{l}}(0,\ \infty), \\ v \neq 0,\ x(0) = 0}} \frac{E \int_0^{+\infty} [\xi^T(t)C^T C\xi(t) + u^T(t)u(t)]dt}{E \int_0^{+\infty} v^T(t)v(t)dt} < \gamma^2. \quad (51)$$

The following result is well known [45, 46].

Lemma 6. *Suppose that Assumption 4 is satisfied. The stochastic* H_∞ *state-feedback control problem has a solution if and only if there exists a symmetric non-negative definite solution* Z_ε *to the following SMARE*

$$\boldsymbol{G}(Z_\varepsilon) := A_\varepsilon^T Z_\varepsilon + Z_\varepsilon A_\varepsilon + \sum_{p=1}^{M} A_{p\varepsilon}^T Z_\varepsilon A_{p\varepsilon}$$

$$-Z_\varepsilon(B_\varepsilon B_\varepsilon^T - \gamma^{-2} D_\varepsilon D_\varepsilon^T)Z_\varepsilon + C^T C = 0 \quad (52)$$

such that the stochastic system

$$d\xi(t) = [A_\varepsilon - B_\varepsilon B_\varepsilon^T Z_\varepsilon + \gamma^{-2} D_\varepsilon D_\varepsilon^T Z_\varepsilon]\xi(t)dt + \sum_{p=1}^{M} A_{p\varepsilon}\xi(t)dw_p(t) \quad (53)$$

is EMSS.

The controller solving this H_∞ *problem is given by equation* (54).

$$u(t) = K\xi(t) = -B_\varepsilon^T Z_\varepsilon \xi(t). \quad (54)$$

6.1 Asymptotic Structure of SMARE

In this section, we need to first analyze the asymptotic structure of SMARE (52) to obtain the controller. In order to simplify the presentation, the following matrices are defined.

$$\hat{S}_\varepsilon := B_\varepsilon B_\varepsilon^T - \gamma^{-2} D_\varepsilon D_\varepsilon^T = \begin{bmatrix} \hat{S}_{00} & \hat{S}_{0f} \Pi_\varepsilon^{-1} \\ \Pi_\varepsilon^{-1} \hat{S}_{0f}^T & \Pi_\varepsilon^{-1} \hat{S}_f \Pi_\varepsilon^{-1} \end{bmatrix},$$

$$\hat{S}_{0f} := \begin{bmatrix} \hat{S}_{01} & \cdots & \hat{S}_{0N} \end{bmatrix}, \ \hat{S}_f := \textbf{block diag} \begin{pmatrix} \hat{S}_{11} & \cdots & \hat{S}_{NN} \end{pmatrix}.$$

Let $\bar{Z}_{00}$, $\bar{Z}_{f0}$ and $\bar{Z}_f$ be the limiting solutions of the above SMARE (52) as $\mu \to +0$, $\varepsilon_i \to +0$, $i = 1, \, ... \, , \, N$, then we obtain the following reduced-order equations (55).

$$\bar{Z}_{00} A_0 + A_0^T \bar{Z}_{00} + \bar{Z}_{f0}^T A_{f0} + A_{f0}^T \bar{Z}_{f0} + \sum_{p=1}^M A_{p00}^T \bar{Z}_{00} A_{p00}$$

$$- \bar{Z}_{00} S_{00} \bar{Z}_{00} - \bar{Z}_{f0}^T S_f \bar{Z}_{f0} - \bar{Z}_{00} S_{0f} \bar{Z}_{f0} - \bar{Z}_{f0}^T S_{0f}^T \bar{Z}_{00} + Q_{00} = 0, \quad (55a)$$

$$A_{f0}^T \bar{Z}_f + \bar{Z}_{00} A_{0f} + \bar{Z}_{f0}^T A_f - \bar{Z}_{00} S_{0f} \bar{Z}_f - \bar{Z}_{f0}^T S_f \bar{Z}_f + Q_{0f} = 0, \quad (55b)$$

$$\bar{Z}_f^T A_f + A_f^T \bar{Z}_f - \bar{Z}_f^T S_f \bar{Z}_f + Q_f = 0, \quad (55c)$$

First, the following AREs are introduced.

$$\bar{Z}_{ii}^* A_{ii} + A_{ii}^T \bar{Z}_{ii}^* - \bar{Z}_{ii}^* \hat{S}_{ii} \bar{Z}_{ii}^* + Q_{ii} = 0, \ i = 1, \, ... \, , N. \quad (56)$$

Moreover, let us define the following sets.

$\Gamma_{f_i} = \{\gamma > 0|$ the ARE (56) with $\hat{S}_{ii} = B_{ii} B_{ii}^T - \gamma^{-2} D_{ii} D_{ii}^T$ has a positive semidefinite and stabilizing solution $\bar{Z}_{ii}^*\}$, $i = 1, \, ... \, , N$.

Assumption 5. *The sets Γ_{f_i} are not empty.*

Lemma 7. *Under Assumption 5, the asymmetric ARE (55c) admits a unique symmetric positive semidefinite stabilizing solution $\bar{Z}_f$ which can be written as*

$$\bar{Z}_f^* := \textbf{block diag} \begin{pmatrix} \bar{Z}_{11}^* & \cdots & \bar{Z}_{NN}^* \end{pmatrix}. \quad (57)$$

Assumption 5 ensures that $A_{ii} - \hat{S}_{ii} \bar{Z}_{ii}^*$, $i = 1, \, ... \, , \, N$ are nonsingular. Substituting the solution of (55c) into (55b) and substituting $\bar{Z}_{f0}^*$ into (55a)

and making some lengthy calculations, we obtain the following zeroth-order
equations (58).

$$\bar{Z}_{00}^* \hat{A} + \hat{A}^T \bar{Z}_{00}^* + \sum_{p=1}^{M} A_{p00}^T \bar{Z}_{00}^* A_{p00} - \bar{Z}_{00}^* \hat{S} \bar{Z}_{00}^* + \hat{Q} = 0, \qquad (58a)$$

$$\bar{Z}_{i0}^{*T} := \begin{bmatrix} \bar{Z}_{ii}^* & -I_{n_i} \end{bmatrix} \hat{T}_{ii}^{-1} \hat{T}_{i0} \begin{bmatrix} I_{n_0} \\ \bar{Z}_{00}^* \end{bmatrix}, \qquad (58b)$$

$$\bar{Z}_{ii}^* A_{ii} + A_{ii}^T \bar{Z}_{ii}^* - \bar{Z}_{ii}^* \hat{S}_{ii} \bar{Z}_{ii}^* + Q_{ii} = 0, \qquad (58c)$$

where $\bar{Z}_{f0}^* := \begin{bmatrix} \bar{Z}_{10}^{*T} & \cdots & \bar{Z}_{N0}^{*T} \end{bmatrix}^T$,

$$\begin{bmatrix} \hat{A} & -\hat{S} \\ -\hat{Q} & -\hat{A}^T \end{bmatrix} := \hat{T}_{00} - \sum_{j=1}^{N} \hat{T}_{0j} \hat{T}_{jj}^{-1} \hat{T}_{j0},$$

$$\hat{T}_{00} := \begin{bmatrix} A_0 & -\hat{S}_{00} \\ -Q_{00} & -A_0^T \end{bmatrix}, \quad \hat{T}_{0i} := \begin{bmatrix} A_{0i} & -\hat{S}_{0i} \\ -Q_{0i} & -A_{i0}^T \end{bmatrix},$$

$$\hat{T}_{i0} := \begin{bmatrix} A_{i0} & -\hat{S}_{0i}^T \\ -Q_{0i}^T & -A_{0i}^T \end{bmatrix}, \quad \hat{T}_{ii} := \begin{bmatrix} A_{ii} & -\hat{S}_{ii} \\ -Q_{ii} & -A_i^T \end{bmatrix}, \; i = 1, \; \dots, N.$$

Remark 4. *For each $i \in \{1, \dots, N\}$ equation (56) is a Riccati equation
arising in connection with the deterministic H_∞ problem. Hence, if Γ_{f_i} is
not empty then $\Gamma_{f_i} = (\gamma_{f_i}, \infty)$. On the other hand, if $\gamma \in \Gamma_{f_i}$ then the matrix
$A_{ii} - \hat{S}_{ii} \bar{Z}_{ii}^*$ is a stable matrix. Therefore the hamiltonian $\hat{T}_{ii}$ is invertible.*

The ARE (58c) produces a positive semidefinite solution if γ is sufficiently
large. Hence, let us define the set.
$\Gamma_s = \{\gamma > 0|$ the SARE (58a) has a positive semidefinite and stabilizing
solution $\bar{Z}_{00}^*\}$.
We introduce the assumption:

Assumption 6. *The set Γ_s is not empty and it has the form $\Gamma_s = (\gamma_s, \infty)$.*

Remark 5. a) *In the considered general case it is not clear how the co-
efficients $\hat{A}$, $\hat{S}$, $\hat{Q}$ are depending upon γ. That is why we have to
introduce as an assumption the fact that the set Γ_s takes the form of
a right unbounded interval. It is worth mentioning that this happens if
all matrices A_{ii} are invertible.*

b) *The fact that $\bar{Z}_{00}^*$ is the stabilizing solution of* (58a) *means that the trajectory $x(t) = 0$ of the Itô differential equation*

$$dx(t) = [\hat{\boldsymbol{A}} - \hat{\boldsymbol{S}}\bar{Z}_{00}^*]x(t)dt + \sum_{p=1}^{M} A_{p00}x(t)dw_p(t) \tag{59}$$

is EMSS. This is equivalent to the fact that the Lyapunov operator $X \to [\hat{\boldsymbol{A}} - \hat{\boldsymbol{S}}\bar{Z}_{00}^*]^T X + X[\hat{\boldsymbol{A}} - \hat{\boldsymbol{S}}\bar{Z}_{00}^*] + \sum_{p=1}^{M} A_{p00}^T X A_{p00}$ are located in the half plane $\mathbf{Re}\lambda < 0$. *This means that* (59) *is true.*

The limiting behavior of Z_ε is described by the following theorem.

Theorem 9. *Under Assumptions 5 and 6, if a parameter $\gamma > \bar{\gamma} := \max\{\gamma_s, \ \gamma_{f_1}, \ ... \ ,\gamma_{f_N}\}$ is selected, there exists a small σ^* such that for all $\|\nu\| \in (0, \ \sigma^*)$, the SMARE (52) admits the unique symmetric positive semidefinite stabilizing solution Z_ε for stochastic system (47) which can be written as*

$$\begin{aligned} Z_\varepsilon &= \Phi_\varepsilon \left[\begin{array}{cc} \bar{Z}_{00}^* + O(\|\nu\|) & [\bar{Z}_{f0}^* + O(\|\nu\|)]^T \Pi_\varepsilon \\ \bar{Z}_{f0}^* + O(\|\nu\|) & \bar{Z}_f^* + O(\|\nu\|) \end{array} \right] \\ &= \left[\begin{array}{cc} \bar{Z}_{00}^* + O(\|\nu\|) & [\bar{Z}_{f0}^* + O(\|\nu\|)]^T \Pi_\varepsilon \\ \Pi_\varepsilon[\bar{Z}_{f0}^* + O(\|\nu\|)] & \Pi_\varepsilon[\bar{Z}_f^* + O(\|\nu\|)] \end{array} \right], \end{aligned} \tag{60}$$

where $\nu := \left[\begin{array}{cccc} \varepsilon_1 & \cdots & \varepsilon_N & \mu \end{array} \right] \in \Re^{N+1}$.

It should be noted that there is no solution of to the SMARE (52) as long as there are no positive semi-definite solutions $\bar{Z}_{ii}$ to the SARE (58c). Conversely, the asymptotic structure of the solution to the SMARE (52) can be established by using the reduced-order solution $\bar{Z}_{ii}$ of the SARE (58c) via an implicit function theorem. Therefore, the existence of the reduced-order solution $\bar{Z}_{ii}$ of the SARE (58c) will play an important role in this study. In this case, it is easy to verify that the magnitude of the disturbance attenuation level γ_{f_i} influences the existence of the reduced-order solution $\bar{Z}_{ii}$. In fact, when γ_{f_i} tends to zero, it is hard to obtain the reduced-order solution $\bar{Z}_{ii}$ except for the special case. Finally, the problem considered in this study is restricted for the disturbance attenuation level γ_{f_i} such that the reduced-order SAREs (58c) have the solutions $\bar{Z}_{ii}$.

6.2   Newton's Method

Let us consider Newton's method (61).

$$
Z_\varepsilon^{(n+1)}(A_\varepsilon - \hat{S}_\varepsilon Z_\varepsilon^{(n)}) + (A_\varepsilon - \hat{S}_\varepsilon Z_\varepsilon^{(n)})^T Z_\varepsilon^{(n+1)}
$$
$$
+ \sum_{p=1}^{M} A_{p\varepsilon}^T Z_\varepsilon^{(n+1)} A_{p\varepsilon} + Z_\varepsilon^{(n)} \hat{S}_\varepsilon Z_\varepsilon^{(n)} + Q = 0, \tag{61}
$$

where $n = 0, 1, \ldots$, and the initial conditions are chosen as follows.

$$
Z_\varepsilon^{(0)} := \Phi_\varepsilon \left[\begin{array}{cc} \bar{Z}_{00}^* & \bar{Z}_{f0}^{*T}\Pi_\varepsilon \\ \bar{Z}_{f0}^* & \bar{Z}_f^* \end{array} \right] = \Phi_\varepsilon \bar{Z}. \tag{62}
$$

Using the asymptotic structure of (60), it should be noted that the initial condition is chosen as (62).

The algorithm represented by equation (61) has the feature given in the following theorem for the MSPSS.

Theorem 10. *Suppose that Assumptions 5 and 6 are satisfied. If the parameter-independent reduced-order SARE (58c) has a positive semidefinite solution, there exists a positive scalar $\hat{\sigma}$ such that for all $\varepsilon \in H$ with $0 < \|\varepsilon\| \leq \hat{\sigma}$, the iterative algorithm represented by equation (61) converges to the exact solution of Z_ε with a rate equal to that of quadratic convergence; here, $Z_\varepsilon^{(n)}$ is positive semidefinite. Moreover, the convergence solutions equal those of Z_ε in the SMARE (52) in the neighborhood of the initial condition $Z_\varepsilon^{(0)} = \Phi_\varepsilon \bar{Z}$. In other words, the following condition is satisfied.*

$$
\|Z_\varepsilon^{(n)} - Z_\varepsilon\| = \frac{(2\hat{\theta})^{2^n}}{2^n \hat{\beta}\hat{\boldsymbol{L}}} = O(\|\nu\|^{2^n}), \ n = 0, 1, \ldots, \tag{63}
$$

where
$\hat{\boldsymbol{L}} = 2\|\hat{S}_\varepsilon\| < \infty$, $\hat{\beta} = \|[\nabla \boldsymbol{G}(Z_\varepsilon^{(0)})]^{-1}\|$, $\hat{\theta} = \hat{\beta}\hat{\eta}\hat{\boldsymbol{L}} < 2^{-1}$ $\hat{\eta} = \|[\nabla \boldsymbol{G}(Z_\varepsilon^{(0)})]^{-1}\| \cdot \|\boldsymbol{G}(Z_\varepsilon^{(0)})\|$.

7   Simulation Example

In order to demonstrate the efficiency of the stochastic H_∞ control for MSPSS, we present results for practical multiarea electric energy systems.

The state variable model of the megawatt-frequency control problem was developed in [47].

In developing the state space model, the following basis equations were used:

$$\Delta P_{\text{tie}i} = \sum_v T_{iv}^* \left(\int \Delta f_i dt - \int \Delta f_v dt \right),$$

$$\Delta P_{gi} - \Delta P_{di} = \frac{2H_i}{f^*} \frac{d}{dt} \Delta f_i + D_i \Delta f_i + \Delta P_{\text{tie}i},$$

$$\frac{d}{dt} \Delta P_{gi} = -\frac{1}{T_{ti}} \Delta P_{gi} + \frac{1}{T_{ti}} \Delta X_{gvi},$$

$$\frac{d}{dt} \Delta X_{gvi} = -\frac{1}{T_{gvi}} \Delta X_{gvi} - \frac{1}{T_{gvi} R_i} \Delta f_i + \frac{1}{T_{gvi}} \Delta P_{ci}.$$

Some system parameters used in our study are referred to [47] for details.

For a two-area MSPSS, the following state, control and disturbance variables can be defined.

$$\xi(t) := \left[\int \Delta P_{\text{tie}1} dt \ \int \Delta f_1 dt \ \Delta f_1 \ \int \Delta f_2 dt \ \Delta f_2 \ \Delta P_{g1} \ \Delta P_{g2} \big| \Delta X_{gv1} \big| \Delta X_{gv2} \ \right]^T$$

$$= \left[\ x(t) \ \big| \ z_1(t) \ \big| \ z_2(t) \ \right]^T,$$

$$u(t) := \left[\ \Delta P_{c1} \ \ \Delta P_{c2} \ \right]^T, \ v(t) := \left[\ \Delta P_{d1} \ \ \Delta P_{d2} \ \right]^T.$$

The following system data were used for the numerical calculation.

$$P_{r1} = P_{r2} = 2000 \ [\text{MW}], \ H_1 = H_2 = 5 \ [\text{sec}],$$
$$D_1 = D_2 = 8.33 \times 10^{-3} \ [\text{puMW/Hz}],$$
$$T_{t1} = T_{t2} = 0.3 \ [\text{sec}], \ T_{gv1} = 0.030,$$
$$T_{gv2} = 0.029 \ [\text{sec}], \ \delta_1^* - \delta_2^* = 60 \ [\text{degree}],$$
$$R_1 = R_2 = 2.4 \ [\text{Hz/puMW}], \ f^* = 60 \ [\text{Hz}],$$
$$T_{12}^* = 0.315 \ [\text{puMW}], \ \Delta P_{di} = 0.1 \ [\text{puMW}].$$

$$A_{00} = \begin{bmatrix} 0 & 0.315 & 0 & -0.315 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1.888 & -0.0498 & 1.888 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1.888 & 0 & -1.888 & -0.0498 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3.333 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3.333 \end{bmatrix},$$

$$A_{01} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3.333 \\ 0 \end{bmatrix}, \ A_{02} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 6 \\ 0 \\ 3.333 \end{bmatrix},$$

$$A_{10} = \begin{bmatrix} 0 & 0 & 0.41666 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.41666 & 0 & 0 \end{bmatrix},$$

$$A_{20} = \begin{bmatrix} 0 & 0 & 0.41666 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.41666 & 0 & 0 \end{bmatrix}, \ A_{11} = A_{22} = -1,$$

$A_{100} = \textbf{block diag}\begin{pmatrix} 0 & 0 & 0.00249 & 0 & 0.00249 & 0 & 0 \end{pmatrix},$

$A_{110} = A_{120} = 0 \in \Re^{1\times7}, \ A_{111} = A_{112} = A_{122} = 0, \ B_{01} = B_{02} = 0 \in \Re^{7\times1},$

$B_{11} = 1, \ B_{22} = 1, \ D_{01} = \begin{bmatrix} 0 & 0 & -0.6 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$

$D_{02} = \begin{bmatrix} 0 & 0 & 0 & 0 & -0.6 & 0 & 0 \end{bmatrix}^T, \ D_{11} = D_{22} = 0,$

$Q = \textbf{block diag}\begin{pmatrix} I_7 & 0.25I_2 \end{pmatrix}.$

The system matrices are given by the top of this page. It is assumed that time constant of the governors represents the small singular perturbations. Hence, small parameters are $T_{gv1} := \varepsilon_1 = 0.030$ and $T_{gv2} := \varepsilon_2 = 0.029$. Moreover, it should be noted that $\mu = 0$.

It should be noted that the deterministic disturbance distribution $v(t) := [\Delta P_{d1} \ \Delta P_{d2}]^T = [0.1 \ 0.1]^T$ and the state-dependent noise related to the load frequency constant [47] are both considered compared with the existing results [48, 49]. We suppose that the error in the load frequency constant is within 5% of the nominal value. Therefore, the proposed design method is very useful because the resulting strategy can be implemented on more practical MSPSS.

For every boundary value $\gamma > \bar{\gamma} := \max\{\gamma_s, \ \gamma_{f_1}, \ \gamma_{f_2}\} = 2.2608e - 1$, the SMARE (52) has a positive definite stabilizing solution because the AREs (55c) and the SARE (55a) have a positive definite solution, where $\gamma_s = 2.2608e - 1$, $\gamma_{f_1} = \gamma_{f_2} = \infty$.

Now, we choose $\gamma = 0.3 \ (> \bar{\gamma})$ to solve the MSARE (7). The efficiency of Newton's method (61) is demonstrated. It is easy to verify that algorithm (61) converges to the exact solution with an accuracy of $\|\boldsymbol{G}(Z_\varepsilon^{(n)})\| < 1.0e{-}11$

after five iterations.

<p align="center">Table 1. Errors per iterations.</p>

n	$\|\boldsymbol{G}(Z_\varepsilon^{(n)})\|$
0	1.5667
1	$4.2489e - 01$
2	$3.3631e - 03$
3	$2.0470e - 05$
4	$1.5710e - 11$
5	$9.1508e - 12$

In order to verify the accuracy of the solution, the remainder per iteration is substituted as $Z_\varepsilon^{(n)}$ into SMARE (52). In Table 1, the results of the error $\|\boldsymbol{G}(Z_\varepsilon^{(n)})\|$ per iteration are given. It can be seen that algorithm (61) yields quadratic convergence. Using the obtained iterative solution, the high-order approximate stochastic H_∞ controller is given as follows.

$$u^{(5)}(t) = \begin{bmatrix} 1.5893 & 9.4531e - 1 & 4.1393 & 1.6120 & 1.8547e - 1 \\ -7.8321e - 1 & 1.7522e - 3 & 2.3204e - 1 & 1.1581 & 9.5872e - 1 \end{bmatrix}$$
$$\begin{bmatrix} 4.2214 & -2.8374e - 2 & 4.6816e - 1 & 2.1536e - 2 \\ 2.6205e - 1 & 9.3331e - 2 & 2.2279e - 2 & 2.6668e - 1 \end{bmatrix} \xi(t).$$

In addition, when the small parameters ε_i, $i = 1,\ 2$ are unknown, we can obtain the parameter-independent control as follows by using the similar technique in section 3.3.

$$u_{\text{app}}(t) = \begin{bmatrix} 1.3707 & 8.7785e - 1 & 3.5978 & 1.3178 & 1.3358e - 1 \\ -7.8269e - 1 & -4.5742e - 2 & 1.8744e - 1 & 1.1557 & 9.1813e - 1 \end{bmatrix}$$
$$\begin{bmatrix} 3.5938 & -2.5123e - 2 & 1.1803e - 1 & 0 \\ 2.1534e - 1 & 1.0543e - 1 & 0 & 1.1803e - 1 \end{bmatrix} \xi(t).$$

This control will also be reliable because they seem to be close.

8 Conclusion

The existing results and recent research trends in the various multimodeling analysis and design methods have been briefly summarized. A thorough study of both the parameter-independent methodology and the numerical algorithms revealed the properties of the different methods have been given.

The following conclusion can be drawn: When the small perturbation parameters ε_i are not known, it is strongly recommended that the two-time-scale decomposition method or descriptor systems approach be used. On the other hand, as long as the small perturbation parameters ε_i are known, effort should be made towards finding the exact solutions by means of numerical algorithm. In particular, since the closed-loop solution of the reduced Nash problem depends on the path, the required solution has to be solved numerically.

This survey has mostly concentrated on some classical and recent developments in parameter-independent and computational methods for designing the strategy. Although the choice of topics was necessarily somewhat limited, there are some topics which deserve further attention. For example, the mathematical model described by Itô, i.e. differential equations with Markovian switching in the multimodel situation, is very interesting. This problem will be addressed in future investigations.

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