

A VIABILITY RESULT FOR EVOLUTION EQUATIONS ON LOCALLY CLOSED GRAPHS*

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Abstract

Using a tangency condition expressed with a set of integrals, we establish several necessary and sufficient conditions for viability referring to evolution equations on locally closed graphs.

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1 Introduction

Let X be a real Banach space, let $I \subseteq \mathbb{R}$ be a nonempty and bounded interval and let $K : I \rightsquigarrow X$ and $F : \mathcal{K} \rightsquigarrow X$ be two multi-functions with nonempty values, where $\mathcal{K} := \text{graph}(K)$. Let $A : D(A) \subseteq X \rightarrow X$ be the infinitesimal generator of a C_0 -semigroup $\{S(t); t \geq 0\}$.

Our aim here is to prove some new necessary and sufficient conditions in order that \mathcal{K} be viable with respect to $A + F$. This paper is an extension of the results established by Necula-Popescu-Vrabie [7].

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To be more precise, let us consider the Cauchy Problem

$$\begin{cases} u'(t) \in Au(t) + F(t, u(t)) \\ u(\tau) = \xi. \end{cases} \quad (1.1)$$

Definition 1.1. By a *mild solution* of (1.1) on $[\tau, T] \subseteq I$, we mean a function $u \in C([\tau, T]; X)$ satisfying $(t, u(t)) \in \mathcal{K}$, $u(\tau) = \xi$ and for which there exists $f \in L^1(\tau, T; X)$ with $f(t) \in F(t, u(t))$ a.e. for $t \in [\tau, T]$ and

$$u(t) = S(t - \tau)\xi + \int_{\tau}^t S(t - s)f(s) ds \quad (1.2)$$

for each $t \in [\tau, T]$.

Definition 1.2. We say that the graph, \mathcal{K} , of $K : I \rightsquigarrow X$, is *mild viable* with respect to $A + F$, where $F : \mathcal{K} \rightsquigarrow X$, if for each $(\tau, \xi) \in \mathcal{K}$, there exists $T > \tau$, such that $[\tau, T] \subseteq I$ and (1.1) has at least one mild solution $u : [\tau, T] \rightarrow X$. If $T \in (\tau, \sup I)$ can be taken arbitrary, we say that \mathcal{K} is *globally mild viable* with respect to $A + F$.

The first two sections of the paper are concerned with some prerequisites and basic concepts and results needed in the sequel. In Section 3 we prove the main necessary condition of viability, in Section 4 we give a relationship between two tangency conditions, Section 5 contains the statement of the two sufficient conditions for viability and the statement and proof of a technical approximation lemma, while in Section 6, we give the proofs of Theorems 5.1 and 5.2.

2 Preliminaries

If (Y, d) is a metric space, $y \in Y$ and $r > 0$, $D(y, r)$ denotes the closed ball with center y and radius $r > 0$, i.e. $D(y, r) = \{x \in Y; d(y, x) \leq r\}$, while $S(y, r)$ denotes the open ball with center y and radius $r > 0$, i.e. $S(y, r) = \{x \in Y; d(y, x) < r\}$. If $B \subseteq Y$ and $C \subseteq Y$, we denote by

$$\text{dist}(y, C) := \inf\{d(y, z); z \in C\}$$

and by

$$\text{dist}(B, C) := \inf\{d(x, y); x \in B, y \in C\}.$$

Also $\mathcal{B}(Y)$ denotes the family of all bounded subsets of Y .

Definition 2.1. Let $Y \subseteq X$ be nonempty. The function $\beta_Y : \mathcal{B}(X) \rightarrow \mathbb{R}_+$, defined by

$$\beta_Y(B) := \inf \left\{ \varepsilon > 0; \exists x_1, x_2, \dots, x_{n(\varepsilon)} \in Y, B \subseteq \bigcup_{i=1}^{n(\varepsilon)} D(x_i, \varepsilon) \right\},$$

is called the *Hausdorff measure of noncompactness on X subordinated to Y* . If $Y = X$, we simply denote β_X by β , and we simply call it the *Hausdorff measure of noncompactness on X* .

Remark 2.1. We have the following properties:

- (i) for each $B \in \mathcal{B}(X)$ and $r > 0$ with $B \subseteq D(0, r)$, we have $\beta(B) \leq r$;
- (ii) $\beta(B) = 0$ if and only if B is relatively compact;
- (iii) the restriction of β_Y to $\mathcal{B}(Y)$ coincides with the Hausdorff measure of noncompactness on Y ;
- (iv) for each $B \in \mathcal{B}(Y)$ we have $\beta(B) \leq \beta_Y(B) \leq 2\beta(B)$.

The next lemma is due to Mönch [4].

Lemma 2.1. *Let X be a separable Banach space and $\{f_m; m \in \mathbb{N}\}$ a subset in $L^1(\tau, T; X)$ for which there exists $\ell \in L^1(\tau, T; \mathbb{R}_+)$ such that*

$$\|f_m(s)\| \leq \ell(s)$$

for each $m \in \mathbb{N}$ and a.e. for $s \in [\tau, T]$. Then the mapping

$$s \mapsto \beta(\{f_m(s); m \in \mathbb{N}\})$$

is integrable on $[\tau, T]$ and, for each $t \in [\tau, T]$, we have

$$\beta \left(\left\{ \int_{\tau}^t f_m(s) ds; m \in \mathbb{N} \right\} \right) \leq \int_{\tau}^t \beta(\{f_m(s); m \in \mathbb{N}\}) ds. \quad (2.1)$$

For further details on the Hausdorff measure of noncompactness see Cârjă, Necula, Vrabie [3], Section 2.7, pp. 48~53.

Let X be a real Banach space, $I \subseteq \mathbb{R}$ a nonempty and bounded interval, $K : I \rightsquigarrow X$ a multi-function with nonempty values and let $\mathcal{K} := \text{graph}(K)$. Here and thereafter, \mathcal{K} is conceived as a metric space, whose metric, d , is

defined by $d((\tau, \xi), (\theta, \mu)) = \max\{|\tau - \theta|, \|\xi - \mu\|\}$, for all $(\tau, \xi), (\theta, \mu) \in \mathcal{K}$. Also, λ denotes the Lebesgue measure on \mathbb{R} . Furthermore, whenever we will use the term *strongly-weakly* we will mean that the domain of the multi-function in question is equipped with the strong topology, while the range is equipped with the weak topology. Otherwise, both domain and range are endowed with the strong, i.e. norm, topology.

Definition 2.2. The multi-function $F : \mathcal{K} \rightsquigarrow X$ is called (*strongly-weakly almost u.s.c.*) if for each $\varepsilon > 0$ there exists an open set $\mathcal{O}_\varepsilon \subseteq I$ such that $\lambda(\mathcal{O}_\varepsilon) \leq \varepsilon$ and $F|_{((I \setminus \mathcal{O}_\varepsilon) \times X) \cap \mathcal{K}}$ is (strongly-weakly) u.s.c.

Definition 2.3. The multi-function $F : \mathcal{K} \rightsquigarrow X$ is called *integrally-bounded* if for each $(\tau, \xi) \in \mathcal{K}$ there exist $\rho > 0$, $\delta > 0$, $\ell_1 \in L^1(I; \mathbb{R})$ and a negligible set $N_1 \subseteq I$ satisfying: for each $(t, u) \in (([\tau - \delta, \tau + \delta] \setminus N_1) \times S(\xi, \rho)) \cap \mathcal{K}$, we have

$$\|F(t, u)\| \leq \ell_1(t).$$

Remark 2.2. (i) If X is separable we can choose N_1 in Definition 2.3 the same for all $(\tau, \xi) \in \mathcal{K}$ and in this case for each $(\tau, \xi) \in ((I \setminus N_1) \times X) \cap \mathcal{K}$, $F(\tau, \xi)$ is bounded.

(ii) Moreover, if, in addition, F is closed valued and almost u.s.c., then, for each continuous function $u : I \rightarrow X$ with $(t, u(t)) \in \mathcal{K}$ for each $t \in I$, the multi-function $t \mapsto F(t, u(t))$ has at least one locally integrable selection on I . The same conclusion holds true if F is closed valued, strongly-weakly almost u.s.c. and has separable range. The latter assertion follows from Pettis' Measurability Theorem 1.1.3, p. 3, in Vrabie [10].

The next special class of graphs was considered for the first time by Necula [5].

Definition 2.4. Let $K : I \rightsquigarrow X$ be a multi-function with graph, \mathcal{K} . By a *simple solution issuing from* $(\tau, \xi) \in \mathcal{K}$ we mean a pair of functions $(g, v) \in L^1(\tau, T; X) \times C([\tau, T]; X)$ such that for all $t \in [\tau, T]$ we have $(t, v(t)) \in \mathcal{K}$ and

$$v(t) = S(t - \tau)\xi + \int_{\tau}^t S(t - s)g(s) ds$$

Definition 2.5. The graph, \mathcal{K} , of K is said to be *A-mild viable by itself* if for each $(\tau, \xi) \in \mathcal{K}$, there exist $T > \tau$, $\rho > 0$ and $\ell_2 \in L^1(I; \mathbb{R})$, so that for

each $(\tilde{\tau}, \tilde{\xi}) \in ([\tau, T) \times S(\xi, \rho)) \cap \mathcal{K}$, there exist a simple solution (\tilde{g}, \tilde{v}) issuing from $(\tilde{\tau}, \tilde{\xi})$ defined on $[\tilde{\tau}, \tilde{T}]$ such that

$$\|\tilde{g}(s)\| \leq \ell_2(s) \text{ a.e. for } s \in [\tilde{\tau}, \tilde{T}]$$

Remark 2.3. In other words, the graph, \mathcal{K} , of $K : I \rightsquigarrow X$ is A -mild viable by itself if and only if, for each $(\tau, \xi) \in \mathcal{K}$, there exist $T > \tau$, $\rho > 0$ and $\ell_2 \in L^1(I; \mathbb{R})$, so that $([\tau, T) \times S(\xi, \rho)) \cap \mathcal{K}$ is mild viable with respect to $A + G$, where the multi-function $G : ([\tau, T) \times X) \rightsquigarrow X$ is defined by

$$G(t, \xi) := \{v \in X; \|v\| \leq \ell_2(t)\},$$

for each $(t, \xi) \in ([\tau, T) \times X) \cap \mathcal{K}$

Remark 2.4. (i) Clearly, if $K : I \rightsquigarrow X$ is constant and $S(t)K \subseteq K$ for each $t \geq 0$, then \mathcal{K} is A -mild viable by itself. Indeed, in this case, $\ell_2 \equiv 0$ and $G(t, \xi) \equiv \{0\}$ satisfy all the requirements in Definition 2.5.

(ii) If \mathcal{K} is A -mild viable with respect to some integrally-bounded multi-function $F : \mathcal{K} \rightsquigarrow X$ then, one may easily check out that, for each $(\tau, \xi) \in \mathcal{K}$, the function G , defined as in Remark 2.3, with $\rho > 0$ given by Definition 2.3, and $\ell_2 = \ell_1$, where ℓ_1 are given by Definition 2.3, satisfies the conditions in Remark 2.2, and thus \mathcal{K} is viable by itself.

Let $(\tau, \xi) \in \mathcal{K}$ and let $E \in \mathcal{B}(X)$.

Definition 2.6. We say that E is A -right-quasi-tangent to \mathcal{K} at $(\tau, \xi) \in \mathcal{K}$ if

$$\liminf_{h \downarrow 0} \frac{1}{h} \text{dist} \left(S(h)\xi + \int_0^h S(h-s)\mathcal{F}_E ds, K(\tau+h) \right) = 0, \quad (2.2)$$

where

$$\mathcal{F}_E = \{f \in L^1_{\text{loc}}(\mathbb{R}; X); f(s) \in E \text{ a.e. for } s \in \mathbb{R}\}.$$

Throughout, we denote by $\mathcal{QTS}_{\mathcal{K}}^A(\tau, \xi)$ the set of all A -right-quasi-tangent sets to \mathcal{K} at (τ, ξ) . If K is constant, E is A -right-quasi-tangent to \mathcal{K} at (τ, ξ) if and only if it is A -quasi-tangent to K at ξ in the sense of Cârjă, Necula, Vrabie [2], [3]. The set $\mathcal{QTS}_{\mathcal{K}}^A(\tau, \xi)$ is used in Necula, Popescu, Vrabie [7] to establish necessary and sufficient conditions for viability. Next we introduce a new tangency condition which shall be used in the sequel, similar to the one used in Popescu [8].

Let \mathcal{K} be A -mild viable by itself, $F : \mathcal{K} \rightsquigarrow X$ be integrally bounded and let $(\tau, \xi) \in \mathcal{K}$. Let $\ell \in L^1(I, \mathbb{R})$ such that $\ell(s) \geq \max\{\ell_1(s), \ell_2(s)\}$ a.e. for

$s \in I$ where ℓ_1 is the function from Definition 2.3 and ℓ_2 is the function from Definition 2.5.

Let us denote by $\mathcal{C}_{\tau,\xi,\ell,h}$ the set of all continuous functions $v : [\tau, \tau+h] \rightarrow X$ for which there exists $g \in L^1(\tau, \tau+h; X)$ such that (g, v) is a simple solution issuing from (τ, ξ) and $\|g(s)\| \leq \ell(s)$ a.e. for $s \in [\tau, \tau+h]$. Obviously $\mathcal{C}_{\tau,\xi,\ell,h}$ is nonempty for h small enough.

Next, let us define by $\mathcal{E}_{\tau,\xi,\ell,h}$ the set of all functions $f \in L^1(\tau, \tau+h; X)$ for which there exists $v \in \mathcal{C}_{\tau,\xi,\ell,h}$ such that $f(s) \in F(s, v(s))$ for all $s \in [\tau, \tau+h]$. If F satisfies the conditions in Remark 2.2 then $\mathcal{E}_{\tau,\xi,\ell,h}$ is nonempty for h small enough.

We consider the generalized tangency condition

$$\liminf_{h \downarrow 0} \frac{1}{h} \text{dist} \left(S(h)\xi + \int_{\tau}^{\tau+h} S(\tau+h-s) \mathcal{E}_{\tau,\xi,\ell,h} ds, K(\tau+h) \right) = 0 \quad (2.3)$$

At this point, let us observe that (2.3) makes sense whenever $\mathcal{E}_{\tau,\xi,\ell,h}$ is nonempty. As we already pointed out, in order for the above set to be nonempty it is sufficient that \mathcal{K} be viable by itself and $F : \mathcal{K} \rightsquigarrow X$ be integrally bounded, closed valued and almost u.s.c. Here and thereafter, when we say that (2.3) takes place, we understand that \mathcal{K} is viable by itself, F is integrally bounded and $\mathcal{E}_{\tau,\xi,\ell,h} \neq \emptyset$ for h small enough (sufficiently for a certain h). The fact that (2.3) can take place even in the absence of continuity or measurability conditions for F is illustrated by the first very simple necessary condition for viability in the next section.

3 Necessary conditions for viability

The hypotheses we will use in the sequel are listed below.

- (H₁) $A : D(A) \subseteq X \rightarrow X$ is the infinitesimal generator of a C_0 -semigroup $\{S(t); t \geq 0\}$ of type (M, ω) , i.e., $\|S(t)\| \leq Me^{\omega t}$ for each $t \geq 0$;
- (H₂) the graph \mathcal{K} is A -mild viable by itself;
- (H₃) F has nonempty and closed values and is integrally bounded;
- (H₄) $F : \mathcal{K} \rightsquigarrow X$ is almost u.s.c.;
- (H₅) $F : \mathcal{K} \rightsquigarrow X$ is strongly-weakly almost u.s.c.;

(H₆) there exists a set $N \subseteq I$, with $\lambda(N) = 0$, and such that for each $(\tau, \xi) \in ((I \setminus N) \times X) \cap \mathcal{K}$, we have $F(\tau, \xi) \in \mathcal{QTS}_{\mathcal{K}}^A(\tau, \xi)$.

(H₇) there exists a set $N \subseteq I$, with $\lambda(N) = 0$, and such that for each $(\tau, \xi) \in ((I \setminus N) \times X) \cap \mathcal{K}$, we have (2.3)

(H₈) for each $(\tau, \xi) \in \mathcal{K}$, we have (2.3).

Theorem 3.1. *If \mathcal{K} is mild viable with respect to $A + F$ where F is an integrally bounded multi-function, then (H₂) and (H₈) hold true.*

Proof. First let us observe that even if F is not closed valued and almost u.s.c. the sets $\mathcal{C}_{\tau, \xi, \ell, h}$ and $\mathcal{E}_{\tau, \xi, \ell, h}$ are nonempty for h small enough. Indeed, let ρ and δ from the Definition 2.3 and $u : [\tau, T] \rightarrow S(\xi, \rho)$ be any solution of (1.1) with $T < \tau + \delta$. Then there exists $f \in L^1(\tau, T; X)$ with $f(t) \in F(t, u(t))$ a.e. for $t \in [\tau, T]$ and

$$u(t) = S(t - \tau)\xi + \int_{\tau}^t S(t - s)f(s) ds$$

for all $t \in [\tau, T]$. Hence, for each $h \in (0, T - \tau]$ we have $u \in \mathcal{C}_{\tau, \xi, \ell, h}$, $f \in \mathcal{E}_{\tau, \xi, \ell, h}$ and

$$\begin{aligned} & \text{dist} \left(S(h)\xi + \int_{\tau}^{\tau+h} S(\tau + h - s)\mathcal{E}_{\tau, \xi, \ell, h} ds, K(\tau + h) \right) \\ & \leq \text{dist} \left(S(h)\xi + \int_{\tau}^{\tau+h} S(\tau + h - s)f(s) ds, u(\tau + h) \right) = 0 \end{aligned}$$

and this completes the proof.

Let us remark that we have proved that for h sufficiently small

$$\{S(h)\xi + \int_{\tau}^{\tau+h} S(\tau + h - s)\mathcal{E}_{\tau, \xi, \ell, h} ds\} \cap K(\tau + h) \neq \emptyset$$

□

So, under more general hypotheses on F , (H₇) is necessary in order for \mathcal{K} be viable with respect to F . In that follows, we shall see that, under some additional natural assumptions on F , the converse statement is also true.

4 The relationship between (H_6) and (H_7)

Definition 4.1. We say that the multi-function $F : \mathcal{K} \rightsquigarrow X$ is almost ε - δ l.s.c. if for each $\gamma > 0$, there exists an open set $\mathcal{O}_\gamma \subset I$, with $\lambda(\mathcal{O}_\gamma) \leq \gamma$, and such that the mapping $(t, \xi) \mapsto F(t, \xi)$ is ε - δ l.s.c. on $((I \setminus \mathcal{O}_\gamma) \times X) \cap \mathcal{K}$.

Theorem 4.1. *Let X be separable and let \mathcal{K} and F satisfy (H_2) and (H_3) .*

- (i) *If F is almost ε - δ l.s.c., then (H_6) implies (H_7) .*
- (ii) *If F is almost u.s.c., then (H_7) implies (H_6) .*

Proof. From (H_3) and the fact that X is separable, it follows that there exist a finite or at most countable set Γ , $(\tau_i, \xi_i)_{i \in \Gamma} \subset \mathcal{K}$, $(\rho_i)_{i \in \Gamma} \subset (0, \infty)$, $(\delta_i)_{i \in \Gamma} \subset (0, \infty)$, $(\ell_i)_{i \in \Gamma} \subset L^1(I; \mathbb{R})$ and a negligible set $N_1 \subset I$ such that $\mathcal{K} \subseteq \cup_{i \in \Gamma} (\tau_i - \delta_i, \tau_i + \delta_i) \times S(\xi_i, \rho_i)$ and, for all $i \in \Gamma$, and all $(t, u) \in (((\tau_i - \delta_i, \tau_i + \delta_i) \setminus N_1) \times S(\xi_i, \rho_i)) \cap \mathcal{K}$, we have $\|F(t, u)\| \leq \ell_i(t)$.

We begin with the proof of (i). Since F is ε - δ l.s.c., it follows that, for each $n \in \mathbb{N}$, $n \geq 1$ there exists $I_n \subset I$, with $\lambda(I \setminus I_n) < \frac{1}{n}$, and such that the mapping $(t, \xi) \mapsto F(t, \xi)$ is ε - δ l.s.c. on $(I_n \times X) \cap \mathcal{K}$.

Let $A_n \subset I_n$ the set of all density points of I_n which are also Lebesgue points for ℓ_i for all $i \in \Gamma$. Let $A = (\cup_{n \geq 1} A_n) \cap (I \setminus (N_1 \cup N))$, where N is the negligible set in (H_6) . Obviously, $\lambda(I \setminus A) = 0$.

Let $(\tau, \xi) \in (A \times X) \cap \mathcal{K}$. We will show that

$$\liminf_{h \downarrow 0} \frac{1}{h} \text{dist} \left(S(h)\xi + \int_{\tau}^{\tau+h} S(\tau+h-s) \mathcal{E}_{\tau, \xi, \ell, h} ds, K(\tau+h) \right) = 0$$

Let $i_0 \in \Gamma$ and $n_0 \in \mathbb{N}$ such that $\tau \in A_{n_0} \cap (\tau_{i_0} - \delta_{i_0}, \tau_{i_0} + \delta_{i_0})$ and $\xi \in S(\xi_{i_0}, \rho_{i_0})$. From (H_6) , it follows that there exists $h_n \downarrow 0$, $f_n \in \mathcal{F}_{F(\tau, \xi)}$ and $p_n \in X$, with $\|p_n\| \rightarrow 0$, and such that

$$S(h_n)\xi + \int_{\tau}^{\tau+h_n} S(\tau+h_n-s) f_n(s) ds + h_n p_n \in K(\tau+h_n) \quad (4.1)$$

for all $n \in \mathbb{N}$, $n \geq 1$.

Let $\varepsilon > 0$ be arbitrary but fixed. Since \mathcal{K} is viable by itself there exists $\delta > 0$ and $v \in \mathcal{C}_{\tau, \xi, \ell, \delta}$. Diminishing δ if necessary we may assume that $\tau + \delta < \tau_{i_0} + \delta_{i_0}$ and $v(t) \in S(\xi_{i_0}, \rho_{i_0})$ for all $t \in [\tau, \tau + \delta]$ and

$$F(\tau, \xi) \subset F(t, v(t)) + D(0, \varepsilon) \text{ for all } t \in [\tau, \tau + \delta] \cap A_{n_0}.$$

At this point, let us observe that, for each $n \in \mathbb{N}$, $n \geq 1$, the multi-function $t \mapsto F(t, v(t)) \cap (f_n(t) + D(0, \varepsilon))$ is measurable, nonempty and closed valued from $[\tau, \tau + \delta] \cap A_{n_0}$ to X . Since X is separable, from Kuratowski and Ryll-Nardzewski Theorem 3.1.1, p. 86 in Vrabie [9], it follows that the multi-function above has at least one measurable selection. Let us denote by $g_n : [\tau, T] \cap A_{n_0} \rightarrow X$ such a selection. Next, let us extend g_n to a measurable selection of $F(\cdot, v(\cdot))$ on $[\tau, \tau + \delta]$, extension denoted, for simplicity, again by g_n . So, for each $n \in \mathbb{N}$, $n \geq 1$, and $t \in [\tau, \tau + \delta]$, we have

$$g_n(t) \in F(t, v(t)).$$

Also, for each $n \in \mathbb{N}$, $n \geq 1$, and $t \in [\tau, \tau + \delta] \cap A_{n_0}$, we have

$$\|f_n(t) - g_n(t)\| \leq \varepsilon.$$

From (4.1) and the fact that $g_n \in \mathcal{E}_{\tau, \xi, \ell, \delta}$ we deduce that for each $h_n \in (0, \delta)$

$$\begin{aligned} & \frac{1}{h_n} \text{dist} \left(S(h_n)\xi + \int_{\tau}^{\tau+h_n} S(\tau+h_n-s)\mathcal{E}_{\tau, \xi, \ell, h_n} ds, K(\tau+h_n) \right) \\ & \leq \left\| \frac{1}{h_n} \int_{\tau}^{\tau+h_n} S(\tau+h_n-s)(g_n(s) - f_n(s)) ds \right\| + \|p_n\| \\ & \leq Me^{\omega\delta} \frac{1}{h_n} \int_{[\tau, \tau+h_n] \cap A_{n_0}} \|f_n(s) - g_n(s)\| ds \\ & \quad + \frac{1}{h_n} \int_{[\tau, \tau+h_n] \setminus A_{n_0}} \|f_n(s) - g_n(s)\| ds + \|p_n\| \\ & \leq Me^{\omega\delta} \varepsilon + \frac{1}{h_n} \int_{[\tau, \tau+h_n] \setminus A_{n_0}} (\|f_n(s)\| + \|g_n(s)\|) ds + \|p_n\| \\ & \leq Me^{\omega\delta} \varepsilon + \frac{1}{h_n} \int_{[\tau, \tau+h_n] \setminus A_{n_0}} (\ell_{i_0}(\tau) + \ell_{i_0}(s)) ds + \|p_n\| \\ & \leq Me^{\omega\delta} \varepsilon + \frac{1}{h_n} \int_{[\tau, \tau+h_n] \setminus A_{n_0}} |\ell_{i_0}(s) - \ell_{i_0}(\tau)| ds + \frac{2}{h_n} \int_{[\tau, \tau+h_n] \setminus A_{n_0}} \ell_{i_0}(\tau) ds + \|p_n\| \\ & \leq Me^{\omega\delta} \varepsilon + \frac{1}{h_n} \int_{\tau}^{\tau+h_n} |\ell_{i_0}(s) - \ell_{i_0}(\tau)| ds + 2\ell_{i_0}(\tau) \frac{\lambda([\tau, \tau+h_n] \setminus A_{n_0})}{h_n} + \|p_n\| \end{aligned}$$

Passing to limsup in the inequality above and taking into account that τ is a density point and a Lebesgue point, we get

$$\limsup_{n \rightarrow \infty} \frac{1}{h_n} \text{dist} \left(S(h_n)\xi + \int_{\tau}^{\tau+h_n} S(\tau+h_n-s) \mathcal{E}_{\tau, \xi, \ell, h_n} ds, K(\tau+h_n) \right) \leq M e^{\omega \delta} \varepsilon$$

and therefore (H_7) holds true and this completes the proof of the first part of Theorem 4.1.

Now let us prove (ii) . Since F is almost u.s.c., it follows that for each $n \in \mathbb{N}$, $n \geq 1$ there exists $I_n \subset I$, with $\lambda(I \setminus I_n) < \frac{1}{n}$, such that the mapping $(t, \xi) \mapsto F(t, \xi)$ is u.s.c. on $(I_n \times X) \cap \mathcal{K}$.

Let $A_n \subset I_n$ the set of all density points of I_n which are Lebesgue points too for ℓ_i , for all $i \in \Gamma$. Let $A = (\cup_{n \geq 1} A_n) \cap (I \setminus (N_1 \cup N))$, where N is the negligible set in (H_7) . Obviously, $\lambda(I \setminus A) = 0$.

Let $(\tau, \xi) \in \mathcal{K}$. We will show that

$$\liminf_{h \downarrow 0} \frac{1}{h} \text{dist} \left(S(h)\xi + \int_{\tau}^{\tau+h} S(\tau+h-s) \mathcal{F}_{F(\tau, \xi)} ds, K(\tau+h) \right) = 0.$$

Let $i_0 \in \Gamma$ and $n_0 \in \mathbb{N}$ such that $\tau \in A_{n_0} \cap (\tau_{i_0} - \delta_{i_0}, \tau_{i_0} + \delta_{i_0})$ and $\xi \in S(\xi_{i_0}, \rho_{i_0})$. From (H_7) , it follows that there exists $h_n \downarrow 0$, $v_n \in \mathcal{C}_{\tau, \xi, \ell, h_n}$, $f_n \in \mathcal{E}_{\tau, \xi, \ell, h_n}$ and $p_n \in X$, with $\|p_n\| \rightarrow 0$, such that for all $n \in \mathbb{N}$, $n \geq 1$ and all $t \in [\tau, \tau + h_n]$ we have $f_n(t) \in F(t, v_n(t))$ and

$$S(h_n)\xi + \int_{\tau}^{\tau+h_n} S(\tau+h_n-s) f_n(s) ds + h_n p_n \in K(\tau+h_n) \quad (4.2)$$

Let $\varepsilon > 0$ be arbitrary but fixed and let $\delta > 0$ be such that

$$F(s, \mu) \subset F(\tau, \xi) + D(0, \varepsilon), \text{ for all } (s, \mu) \in ([\tau, \tau + \delta] \cap A_{n_0} \times S(\xi, \delta)) \cap \mathcal{K}$$

Since for all $n \in \mathbb{N}$, $n \geq 1$ and all $t \in [\tau, \tau + h_n]$ we have

$$\|v_n(t) - \xi\| \leq \|S(t-\tau)\xi - \xi\| + M e^{\omega h_n} \int_{\tau}^{\tau+h_n} \ell(s) ds$$

and diminishing δ , if necessary, we may suppose that $\tau + \delta < \tau_{i_0} + \delta_{i_0}$ and $v_n(t) \in S(\xi_{i_0}, \rho_{i_0}) \cap S(\xi, \delta)$ for all $n \geq 1$ with $h_n < \delta$ and all $t \in [\tau, \tau + h_n]$. Then, for all $n \geq 1$ with $h_n < \delta$, we get

$$f_n(t) \in F(t, v_n(t)) \subset F(\tau, \xi) + D(0, \varepsilon) \text{ for all } t \in [\tau, \tau + h_n] \cap A_{n_0}$$

Using the same arguments as in the first part of the proof we deduce that there exists a measurable selection $g_n : [\tau, \tau + h_n] \cap A_{n_0} \rightarrow F(\tau, \xi)$ of the multi-function $t \mapsto F(\tau, \xi) \cap (f_n(t) + D(0, \varepsilon))$ on $[\tau, \tau + h_n] \cap A_{n_0}$. Next, let us extend g_n to \mathbb{R} by using a fixed element in $F(\tau, \xi)$, extension denoted, for simplicity, again by g_n .

From (4.2) and the fact that $g_n \in \mathcal{F}_{F(\tau, \xi)}$ we deduce that for each $h_n \in (0, \delta)$

$$\begin{aligned} & \frac{1}{h_n} \text{dist} \left(S(h_n)\xi + \int_{\tau}^{\tau+h_n} S(\tau + h_n - s) \mathcal{F}_{F(\tau, \xi)} ds, K(\tau + h_n) \right) \\ & \leq \left\| \frac{1}{h_n} \int_{\tau}^{\tau+h_n} S(\tau + h_n - s) (g_n(s) - f_n(s)) ds \right\| + \|p_n\| \end{aligned}$$

From now on the proof is identical to the one used in the first part of the Theorem. □

5 Sufficient conditions for viability

Definition 5.1. We say that the graph \mathcal{K} is:

- (i) *locally closed from the left* if for each $(\tau, \xi) \in \mathcal{K}$ there exist $T > \tau$ and $\rho > 0$ such that, for each $(\tau_n, \xi_n) \in ([\tau, T] \times D(\xi, \rho)) \cap \mathcal{K}$, with $(\tau_n)_n$ nondecreasing, $\lim_n \tau_n = \tilde{\tau}$ and $\lim_n \xi_n = \xi$, we have $(\tilde{\tau}, \xi) \in \mathcal{K}$;
- (ii) *closed from the left* if for each $(\tau_n, \xi_n) \in \mathcal{K}$, with $(\tau_n)_n$ nondecreasing, $\lim_n \tau_n = \tilde{\tau}$ and $\lim_n \xi_n = \xi$, we have $(\tilde{\tau}, \xi) \in \mathcal{K}$;
- (iii) *locally compact from the left* if, it is locally closed from the left and, for each $(\tau, \xi) \in \mathcal{K}$ there exist $T > \tau$ and $\rho > 0$ such that, for each $(\tau_n, \xi_n) \in ([\tau, T] \times D(\xi, \rho)) \cap \mathcal{K}$, with $(\tau_n)_n$ nondecreasing, and $\lim_n \tau_n = \tilde{\tau}$, there exists a convergent subsequence $(\xi_{n_k})_k$ of $(\xi_n)_n$;
- (iv) *compact from the left* if, it is closed from the left and, for each $(\tau_n, \xi_n) \in \mathcal{K}$ with $(\tau_n)_n$ nondecreasing, $\lim_n \tau_n = \tilde{\tau}$, and $(\xi_n)_n$ bounded, there exists a convergent subsequence $(\xi_{n_k})_k$ of $(\xi_n)_n$.

Remark 5.1. Let $(\xi_{n_k})_k$ be the subsequence of $(\xi_n)_n$ whose existence is ensured by (ii) in Definition 5.1 and let $\xi = \lim_k \xi_{n_k}$. Then $(\tau, \xi) \in \mathcal{K}$.

Definition 5.2. By a *Carathéodory uniqueness function* we mean a function $\alpha : I \times R_+ \rightarrow R_+$ such that:

- (i) for each $x \in \mathbb{R}_+$, $t \mapsto \alpha(t, x)$ is locally integrable;
- (ii) for a.e. $t \in I$, $x \mapsto \alpha(t, x)$ is continuous, nondecreasing;
- (iii) for each $\tau \in I$, the only absolutely continuous solution of the Cauchy problem

$$\begin{cases} x'(t) = \alpha(t, x(t)) \\ x(\tau) = 0 \end{cases}$$

is $x \equiv 0$.

Definition 5.3. We say that $A + F$ is β -compact if for all $(\tau, \xi) \in \mathcal{K}$ there exists $\delta > 0$, $\rho > 0$, a Carathéodory uniqueness function, $\alpha : I \times R_+ \rightarrow R_+$, a negligible set $N \subset I$ and a continuous function $m : [0, \infty) \rightarrow [0, \infty)$, such that, for all $B \subseteq D(\xi, \rho)$, all $t \in (0, \infty)$ and all $s \in [\tau - \delta, \tau + \delta] \setminus N$ we have

$$\beta(S(t)F(\{s\} \times B) \cap \mathcal{K}) \leq m(t)\alpha(s, \beta(B)). \quad (5.1)$$

Remark 5.2.

- (i) If the C_0 -semigroup $\{S(t); t \geq 0\}$ is compact and F is integrally bounded then $A + F$ is β -compact.
- (ii) If F is β -compact (see definition 5.3 in Popescu [8]), then $A + F$ is β -compact.

Theorem 5.1. *Let \mathcal{K} be locally closed from the left and let $F : \mathcal{K} \rightsquigarrow X$ be nonempty, convex and weakly compact valued. If (H_2) , (H_3) and (H_5) are satisfied and $A + F$ is β -compact then a necessary and sufficient condition in order that \mathcal{K} be mild viable with respect to $A + F$ is (H_7) .*

Theorem 5.2. *Let \mathcal{K} be locally compact from the left and let $F : \mathcal{K} \rightsquigarrow X$ be nonempty, convex and weakly compact valued. If (H_2) , (H_3) and (H_5) are satisfied, then a necessary and sufficient condition in order that \mathcal{K} be viable with respect to $A + F$ is (H_7) .*

From Theorems 5.1, 5.2 and Brezis-Browder Ordering Principle, i.e. Theorem 2.1.1, p. 30 in Cârjă, Necula, Vrabie [3], we easily deduce the two global viability results stated below.

Theorem 5.3. *Let \mathcal{K} be closed from the left and let $F : \mathcal{K} \rightsquigarrow X$ be nonempty, convex and weakly compact valued. If (H_2) , (H_3) and (H_5) are satisfied and $A + F$ is β -compact then a necessary and sufficient condition in order that \mathcal{K} be globally mild viable with respect to $A + F$ is (H_7) .*

Theorem 5.4. *Let \mathcal{K} be compact from the left and let $F : \mathcal{K} \rightsquigarrow X$ be nonempty, convex and weakly compact valued. If (H_2) , (H_3) and (H_5) are satisfied, then a necessary and sufficient condition in order that \mathcal{K} be globally viable with respect to $A + F$ is (H_7) .*

The next lemma, essentially inspired from Cârjă, Monteiro-Marques [1], is the main step through the proof of both Theorems 5.1 and 5.2.

Lemma 5.1. *Let I be a nonempty and bounded interval and $K : I \rightsquigarrow X$ a multi-function with locally closed from the left graph, \mathcal{K} , let $(\tau, \xi) \in \mathcal{K}$ and let $F : \mathcal{K} \rightsquigarrow X$ be a nonempty valued multi-function. Suppose (H_1) , (H_2) , (H_3) and (H_7) are satisfied. Let $Z \subseteq I$ be a negligible set including the negligible set in (H_7) and $\ell \in L^1(I, \mathbb{R})$ be the function from the definition of $\mathcal{E}_{\tau, \xi, \ell, h}$.*

Let $\rho > 0$ and $T > \tau$ be such that:

- (1) $([\tau, T] \times D(\xi, \rho)) \cap \mathcal{K}$ is closed from the left;
- (2) $\|F(t, u)\| \leq \ell(t)$ a.e. for $t \in [\tau, T]$ and for all $u \in K(t) \cap D(\xi, \rho)$;
- (3) T and ρ satisfy Definition 2.5;
- (4) $\sup_{t \in [\tau, T]} \|S(t - \tau)\xi - \xi\| + Me^{\omega(T - \tau)} \int_{\tau}^T l(s) ds + Me^{\omega(T - \tau)}(T - \tau) < \rho$.

Then, for each $\varepsilon \in (0, 1)$ and each open set $\mathcal{O} \subseteq I$, with $Z \subseteq \mathcal{O}$, there exist a family $\mathcal{P}_T = \{[t_m, s_m]; m \in \Gamma\}$, of disjoint intervals, with Γ finite or at most countable, and five functions $f, r, v \in L^1(\tau, T; X)$, $\theta : \{(t, s); \tau \leq s \leq t \leq T\} \rightarrow [0, T - \tau]$ measurable, and $u \in C([\tau, T]; X)$ such that:

- (i) $\cup [t_m, s_m] = [\tau, T]$ and $s_m - t_m \leq \varepsilon$, for all $m \in \Gamma$;
- (ii) if $t_m \in \mathcal{O}$, then $[t_m, s_m] \subseteq \mathcal{O}$;
- (iii) $u(t_m) \in D(\xi, \rho) \cap K(t_m)$, for all $m \in \Gamma$, $u(T) \in D(\xi, \rho) \cap K(T)$;
- (iv) $\theta(t, s) \leq t - s$; $t \mapsto \theta(t, s)$ nonexpansive on $(s, T]$ and, for each $t \in (\tau, T]$, $s \mapsto \theta(t, s)$ measurable on $[\tau, t]$;

- (v) $v \in C([t_m, s_m]; X)$; $(t, v(t)) \in ([\tau, T] \times S(\xi, \rho)) \cap \mathcal{K}$ for all $t \in [\tau, T]$ and $\|v(t) - u(t_m)\| \leq \varepsilon$ for all $t \in [t_m, s_m)$;
- (vi) $f(s) \in F(s, v(s))$ a.e. for $s \in [t_m, s_m)$ if $t_m \notin \mathcal{O}$ and $\|f(s)\| \leq \ell(s)$ a.e. for $s \in [\tau, T]$;
- (vii) $\|r(s)\| \leq \varepsilon$ a.e. for $s \in [\tau, T]$;
- (viii) $u(t) = S(t-\tau)\xi + \int_{\tau}^t S(t-s)f(s)ds + \int_{\tau}^t S(\theta(t, s))r(s)ds$ for all $t \in [\tau, T]$;
- (ix) $\|u(t) - u(t_m)\| \leq \varepsilon$ for all $t \in [t_m, s_m)$ and $m \in \Gamma$.

Proof. Let ε be arbitrary but fixed in $(0, 1)$ and let $\mathcal{O} \subseteq R$ be an open subset with $Z \subseteq \mathcal{O}$. We will show that there exist $\delta = \delta(\varepsilon, \mathcal{O}) \in (\tau, T)$ and $\mathcal{P}_\delta, f, r, v, \theta, u$ such that (i)~(ix) hold true with δ instead of T . We distinguish between the following different cases.

Case 1. If $\tau \in \mathcal{O}$, we take $\Gamma = \{1\}$, $t_1 = \tau$, $s_1 = \delta$ with $\delta \in (\tau, T)$ small enough in order to $[\tau, \delta] \subseteq \mathcal{O}$, $\tau - \delta \leq \varepsilon$ and there exists a simple solution (f, v) issuing from (τ, ξ) , defined on $[\tau, \delta]$ with $\|f(s)\| \leq \ell(s)$ a.e. for $s \in [\tau, \delta]$. Further, let us diminish δ such that $\|v(t) - \xi\| < \min\{\varepsilon, \rho\}$ for all $t \in [\tau, \delta]$ and let us define $\mathcal{P}_\delta = \{[\tau, \delta]\}$, $\theta = 0$, $r = 0$ and $u(t) = v(t)$ for all $t \in [\tau, \delta]$.

Case 2. If $\tau \notin \mathcal{O}$ then $\tau \notin Z$ which implies that there exist $h_n \downarrow 0$, $v_n \in \mathcal{C}_{\tau, \xi, \ell, h_n}$, $f_n \in \mathcal{E}_{\tau, \xi, \ell, h_n}$ such that $f_n(s) \in F(s, v_n(s))$ a.e. for $s \in [\tau, \tau + h_n]$ and $p_n \in X$, with $\|p_n\| \rightarrow 0$, such that

$$S(h_n)\xi + \int_{\tau}^{\tau+h_n} S(\tau+h_n-s)f_n(s)ds + p_nh_n \in K(\tau+h_n)$$

for all $n \in \mathbb{N}$, $n \geq 1$. Let $n_0 \in \mathbb{N}$ and $\delta = \tau + h_{n_0}$ be such that $\delta \in (\tau, T)$, $h_{n_0} < \varepsilon$, $\|p_{n_0}\| < \varepsilon$ and

$$\sup_{t \in [\tau, \tau+h_{n_0}]} \|S(t-\tau)\xi - \xi\| + Me^{\omega h_{n_0}} \int_{\tau}^{\tau+h_{n_0}} \ell(s)ds + h_{n_0} \leq \min\{\varepsilon, \rho\}$$

We define $\mathcal{P}_\delta = \{[\tau, \delta]\}$, $f(s) = f_{n_0}(s)$, $\theta(t, s) = 0$ for $\tau \leq s \leq t \leq \delta$, $r(s) = p_{n_0}$, $v(s) = v_{n_0}(s)$ for $s \in [\tau, \delta]$, and let $u : [\tau, \delta] \rightarrow X$ be given by (viii). We may easily see that (i)~(ix) are satisfied.

Let

$$\mathcal{U} = \{(\mathcal{P}_\delta, f, r, v, \theta, u); \delta \in (\tau, T], \text{ (i)~(ix) hold true with } \delta \text{ instead of } T\}.$$

As we already have shown, $\mathcal{U} \neq \emptyset$. On \mathcal{U} we define a partial order by:

$$(\mathcal{P}_{\delta_1}, f_1, r_1, v_1, \theta_1, u_1) \preceq (\mathcal{P}_{\delta_2}, f_2, r_2, v_2, \theta_2, u_2),$$

if

$$\begin{cases} \delta_1 \leq \delta_2, & \mathcal{P}_{\delta_1} \subseteq \mathcal{P}_{\delta_2}, \\ f_1(s) = f_2(s), r_1(s) = r_2(s), v_1(s) = v_2(s) & \text{a.e. for } s \in [\tau, \delta_1] \\ \theta_1(t, s) = \theta_2(t, s) & \text{for } \tau \leq s \leq t \leq \delta_1 \\ u_1(s) = u_2(s), & \text{for all } s \in [\tau, \delta_1]. \end{cases}$$

We will prove that each nondecreasing sequence in \mathcal{U} is bounded from above. Let $(\mathcal{P}_{\delta_j}, f_j, r_j, v_j, \theta_j, u_j)_{j \geq 1}$ be a nondecreasing sequence in \mathcal{U} and let $\delta = \sup_{j \geq 1} \delta_j$. If there exists $j_0 \in \mathbb{N}$ such that $\delta_{j_0} = \delta$, then $(\mathcal{P}_{\delta_{j_0}}, f_{j_0}, r_{j_0}, v_{j_0}, \theta_{j_0}, u_{j_0})$ is an upper bound for the sequence. So, let us assume that $\delta_j < \delta$, for all $j \geq 1$. Obviously, $\delta \in (\tau, T]$. We define $\mathcal{P}_\delta = \cup_{j \geq 1} \mathcal{P}_{\delta_j}$, $f(s) = f_j(s)$, $\theta(t, s) = \theta_j(t, s)$ for $\tau \leq s \leq t \leq \delta_j$, $v(s) = v_j(s)$ and $r(s) = r_j(s)$ for all j and all $s \in [\tau, \delta_j)$. Clearly, $f, r, v \in L^1(\tau, \delta; X)$. Since $|\theta_j(\delta_j, s) - \theta_i(\delta_i, s)| \leq |\delta_j - \delta_i|$ for all $i, j \geq 1$ and $\tau \leq s < \min\{\delta_i, \delta_j\}$, we may define $\theta(\delta, s) = \lim_{j \rightarrow \infty} \theta_j(\delta_j, s)$ for all $\tau \leq s < \delta$. One may easily see that θ satisfies (iv). Next, we define $u : [\tau, \delta] \rightarrow X$ by

$$u(t) = S(t - \tau)\xi + \int_{\tau}^t S(t - s)f(s) ds + \int_{\tau}^t S(\theta(t, s))r(s) ds,$$

for all $t \in [\tau, \delta]$. We have $u \in C([\tau, \delta]; X)$ and $u(s) = u_j(s)$, for all $j \geq 1$ and all $s \in [\tau, \delta_j]$. Since $u(\delta) = \lim_{t \uparrow \delta} u(t) = \lim_{j \rightarrow \infty} u(\delta_j) = \lim_{j \rightarrow \infty} u_j(\delta_j)$, and $u_j(\delta_j) \in D(\xi, \rho) \cap K(\delta_j)$ and the latter is closed from the left, we deduce that $u(\delta) \in D(\xi, \rho) \cap K(\delta)$. The rest of conditions in lemma being obviously satisfied, it follows that $(\mathcal{P}_\delta, f, r, v, \theta, u)$ is an upper bound for the sequence. Thus, the partially ordered set (\mathcal{U}, \preceq) and the function $\mathcal{N} : (\mathcal{U}, \preceq) \rightarrow R$, defined by $\mathcal{N}(\mathcal{P}_\delta, f, r, v, \theta, u) = \delta$, for each $(\mathcal{P}_\delta, f, r, v, \theta, u) \in \mathcal{U}$, satisfy the hypotheses of the Brezis-Browder Ordering Principle, i.e. Theorem 2.1.1, p. 30 in Cârjă, Necula, Vrabie [3]. Accordingly, there exists an \mathcal{N} -maximal element in \mathcal{U} . This means that there exists $(\mathcal{P}_{\delta^*}, f^*, r^*, v^*, \theta^*, u^*) \in \mathcal{U}$ such that, whenever

$$(\mathcal{P}_{\delta^*}, f^*, r^*, v^*, \theta^*, u^*) \preceq (\mathcal{P}_{\bar{\delta}}, \bar{f}, \bar{r}, \bar{v}, \bar{\theta}, \bar{u}),$$

we necessarily have

$$\mathcal{N}(\mathcal{P}_{\delta^*}, f^*, r^*, v^*, \theta^*, u^*) = \mathcal{N}(\mathcal{P}_{\bar{\delta}}, \bar{f}, \bar{r}, \bar{v}, \bar{\theta}, \bar{u}).$$

We will show that $\delta^* = T$. To this aim, let us assume by contradiction that $\delta^* < T$. We distinguish between two cases.

Case 1. If $\delta^* \in \mathcal{O}$, we take $\bar{\delta} \in (\delta^*, T)$ such that $[\delta^*, \bar{\delta}] \subseteq \mathcal{O}$ and $\bar{\delta} - \delta^* < \varepsilon$ and there exists a simple solution (g, v) issuing from $(\delta^*, u^*(\delta^*))$ defined on $[\delta^*, \bar{\delta}]$ with $\|g(s)\| \leq \ell(s)$ a.e. for $s \in [\delta^*, \bar{\delta}]$. We may diminish $\bar{\delta}$ such that $\|v(t) - u^*(\delta^*)\| \leq \varepsilon$ for all $t \in [\delta^*, \bar{\delta}]$. Let us define

$$\bar{f}(s) = \begin{cases} f^*(s) & \text{for } s \in [\tau, \delta^*] \\ g(s) \text{ a.e.} & \text{for } s \in (\delta^*, \bar{\delta}] \end{cases}, \quad \bar{r}(s) = \begin{cases} r^*(s) & \text{for } s \in [\tau, \delta^*] \\ 0 & \text{for } s \in (\delta^*, \bar{\delta}] \end{cases},$$

$$\bar{\theta}(t, s) = \begin{cases} \theta^*(t, s) & \text{for } \tau \leq s \leq t \leq \delta^* \\ t - \delta^* + \theta^*(\delta^*, s) & \text{for } \tau \leq s < \delta^* < t < \bar{\delta} \\ 0 & \text{for } \delta^* \leq s < t \leq \bar{\delta} \end{cases},$$

$$\bar{v}(s) = \begin{cases} v^*(s), & \text{for } s \in [\tau, \delta^*] \\ v(s), & \text{for } s \in (\delta^*, \bar{\delta}] \end{cases}, \quad \bar{u}(s) = \begin{cases} u^*(s), & \text{for } s \in [\tau, \delta^*] \\ v(s), & \text{for } s \in (\delta^*, \bar{\delta}] \end{cases}$$

and $\mathcal{P}_{\bar{\delta}} = \mathcal{P}_{\delta^*} \cup \{[\delta^*, \bar{\delta}]\}$.

It follows that $(\mathcal{P}_{\bar{\delta}}, \bar{f}, \bar{r}, \bar{v}, \bar{\theta}, \bar{u}) \in \mathcal{U}$, $(\mathcal{P}_{\delta^*}, f^*, r^*, v^*, \theta^*, u^*) \preceq (\mathcal{P}_{\bar{\delta}}, \bar{f}, \bar{r}, \bar{v}, \bar{\theta}, \bar{u})$, but $\delta^* < \bar{\delta}$ which contradicts the maximality of $(\mathcal{P}_{\delta^*}, f^*, r^*, v^*, \theta^*, u^*)$.

Case 2. If $\delta^* \notin \mathcal{O}$ then $\delta^* \notin Z$ which implies that there exist $h_n \downarrow 0$, $v_n \in \mathcal{C}_{\delta^*, u^*(\delta^*), \ell^*, h_n}$, $f_n \in \mathcal{E}_{\delta^*, u^*(\delta^*), \ell^*, h_n}$ such that $f_n(s) \in F(s, v_n(s))$ a.e. for $s \in [\delta^*, \delta^* + h_n]$ and $p_n \in X$, with $\|p_n\| \rightarrow 0$, such that

$$S(h_n)u^*(\delta^*) + \int_{\delta^*}^{\delta^* + h_n} S(\delta^* + h_n - s)f_n(s) ds + p_n h_n \in K(\delta^* + h_n)$$

for all $n \in \mathbb{N}$, $n \geq 1$. Since by (4) in Lemma 5.1 $u^*(\delta^*) \in S(\xi, \rho)$ we may choose $n_0 \in \mathbb{N}$ and $\bar{\delta} = \delta^* + h_{n_0}$ be such that $\bar{\delta} \in (\tau, T)$, $h_{n_0} < \varepsilon$, $\|p_{n_0}\| < \varepsilon$ and

$$\sup_{t \in [\delta^*, \delta^* + h_{n_0}]} \|S(t - \delta^*)u^*(\delta^*) - u^*(\delta^*)\| + M e^{\omega h_{n_0}} \int_{\delta^*}^{\delta^* + h_{n_0}} \ell^*(s) ds + M e^{\omega(T - \tau)} h_{n_0} \leq \nu$$

where $\nu = \min\{\varepsilon, \rho - \|u^*(\delta^*) - \xi\|\}$.

Let us define $\mathcal{P}_{\bar{\delta}} = \mathcal{P}_{\delta^*} \cup \{[\delta^*, \bar{\delta}]\}$, $\bar{\theta}$ as in **Case 1** and

$$\bar{f}(s) = \begin{cases} f^*(s), & s \in [\tau, \delta^*] \\ f_{n_0}(s), & s \in (\delta^*, \bar{\delta}] \end{cases}, \quad \bar{r}(s) = \begin{cases} r^*(s), & s \in [\tau, \delta^*] \\ p_{n_0}, & s \in (\delta^*, \bar{\delta}] \end{cases}, \quad \bar{v}(s) = \begin{cases} v^*(s), & s \in [\tau, \delta^*] \\ v_{n_0}, & s \in (\delta^*, \bar{\delta}] \end{cases},$$

$$\bar{u}(t) = \begin{cases} u^*(t), t \in [\tau, \delta^*] \\ S(t - \delta^*)u^*(\delta^*) + \int_{\delta^*}^t S(t-s)f_{n_0}(s) ds + (t - \delta^*)p_{n_0}, \text{ for } t \in (\delta^*, \bar{\delta}]. \end{cases}$$

We can easily see that (i)~(ix) are satisfied. So, $(\mathcal{P}_{\bar{\delta}}, \bar{f}, \bar{r}, \bar{v}, \bar{\theta}, \bar{u}) \in \mathcal{U}$ and, in addition, $(\mathcal{P}_{\delta^*}, f^*, r^*, v^*, \theta^*, u^*) \preceq (\mathcal{P}_{\bar{\delta}}, \bar{f}, \bar{r}, \bar{v}, \bar{\theta}, \bar{u})$. But $\delta^* < \bar{\delta}$ which contradicts the maximality of $(\mathcal{P}_{\delta^*}, f^*, r^*, v^*, \theta^*, u^*)$. Hence $\delta^* = T$, and $\mathcal{P}_{\delta^*}, f^*, r^*, v^*, \theta^*$ and u^* satisfy all the conditions (i)~(ix). The proof is complete. \square

Definition 5.4. Let $\varepsilon > 0$, Z and \mathcal{O} be as in Lemma 5.1. An element $(\mathcal{P}_T, f, r, v, \theta, u)$ satisfying (i)~(ix) in Lemma 5.1, is called an $(\varepsilon, \mathcal{O})$ -*approximate solution of (1.1)*.

6 Proof of Theorems 5.1 and 5.2

Proof. Since the necessity follows from Theorem 3.1, we will confine ourselves only to the proof of the sufficiency.

Let $Z \subseteq \mathbb{R}$ be a negligible set including the negligible sets appearing in (H_7) and Definition 5.3. Let $\varepsilon_n \in (0, 1)$, with $\varepsilon_n \downarrow 0$, let $(\mathcal{O}_n)_{n \geq 1} \subseteq \mathbb{R}$ be a sequence of open sets, and let ℓ the function in Lemma 5.1. We notice that we may assume with no loss of generality that the sequence $(\mathcal{O}_n)_{n \geq 1}$ is so chosen to satisfy :

- (a) $Z \subseteq \mathcal{O}_n$ for each $n \in \mathbb{N}$, $n \geq 1$;
- (b) $\mathcal{O}_{n+1} \subseteq \mathcal{O}_n$ and $\lambda([\tau, T] \cap \mathcal{O}_n) \leq \varepsilon_n$ for each $n \in \mathbb{N}$, $n \geq 1$;
- (c) $F|_{[(T \setminus \mathcal{O}_n) \times D(\xi, \rho)] \cap \mathcal{X}}$ is strongly-weakly u.s.c., for each $n \in \mathbb{N}$, $n \geq 1$;

Let $\rho > 0$ and $T > \tau$ be as in Lemma 5.1, and such that ρ satisfies Definition 5.3 and let $n \in \mathbb{N}$, $n \geq 1$ be arbitrary but fixed. Let $((\mathcal{P}_T^n, f_n, r_n, v_n, \theta_n, u_n))_n$ be a sequence of $(\varepsilon_n, \mathcal{O}_n)$ -approximate solutions of (1.1), sequence whose existence is ensured, again by Lemma 5.1. If $\mathcal{P}_T^n = \{t_m^n, s_m^n\}$; $m \in \Gamma_n$ with Γ_n finite or at most countable, we denote by $a_n : [\tau, T) \rightarrow [\tau, T)$ the step function, defined by $a_n(s) = t_m^n$ for each $s \in [t_m^n, s_m^n)$. Clearly

$$\lim_n a_n(s) = s \tag{6.1}$$

uniformly for $s \in [\tau, T)$.

We will show that, on a subsequence at least, $(u_n)_n$ is uniformly convergent on $[\tau, T]$ to some function u .

We analyze first the case when X is separable. From (vii) in Lemma 5.1, it follows that, for each $t \in [\tau, T]$, we have

$$\beta\left(\left\{\int_{\tau}^t S(\theta_n(t, s))r_n(s) ds; n \geq 1\right\}\right) = 0. \quad (6.2)$$

Next, let us observe that

$$\|f_n(t)\| \leq \ell(t) \quad (6.3)$$

for each $n \geq 1$ and a.e for $t \in [\tau, T]$.

From (v) and (ix), we deduce that

$$\lim_n \|u_n(a_n(s)) - u_n(s)\| = 0 \text{ and } \lim_n \|u_n(a_n(s)) - v_n(s)\| = 0$$

uniformly for $s \in [\tau, T]$. So we have $\lim_n \|v_n(s) - u_n(s)\| = 0$ uniformly for $s \in [\tau, T]$. Then

$$\beta(\{v_n(s) - u_n(s); n \geq 1\}) = 0 \quad (6.4)$$

for each $s \in [\tau, T]$.

Next, by (viii) in Lemma 5.1, we obtain

$$u_n(t) = S(t - \tau)\xi + \int_{\tau}^t S(t - s)f_n(s) ds + \int_{\tau}^t S(\theta_n(t, s))r_n(s) ds \quad (6.5)$$

for all $n \geq 1$ and $t \in [\tau, T]$.

Let $k \in \mathbb{N}$, $k \geq 1$ and $t \in [\tau, T]$. In view of (6.2), (6.5) and Lemma 2.1, we have

$$\begin{aligned} & \beta(\{u_n(t); n \geq k\}) \\ & \leq \beta\left(\left\{\int_{\tau}^t S(t - s)f_n(s) ds; n \geq k\right\}\right) + \beta\left(\left\{\int_{\tau}^t S(\theta_n(t, s))r_n(s) ds; n \geq k\right\}\right) \\ & \leq \int_{[\tau, t] \setminus \mathcal{O}_k} \beta(\{S(t - s)f_n(s); n \geq k\}) ds + \int_{\mathcal{O}_k} \beta(\{S(t - s)f_n(s); n \geq k\}) ds \end{aligned} \quad (6.6)$$

Since $f_n(s) \in F(s, v_n(s))$ a.e. for $s \in [\tau, T] \setminus \mathcal{O}_k$ and $A + F$ is β -compact we deduce that

$$\beta(\{S(t - s)f_n(s); n \geq k\}) \leq m(t - s)\alpha(s, \beta(\{v_n(s); n \geq k\}))$$

for all $t \in [\tau, T]$ and a.e. for $s \in [\tau, T] \setminus \mathcal{O}_k$. Let $\alpha_0 = (\sup_{s \in [0, T-\tau]} m(s))\alpha$, then α_0 is a Carathéodory uniqueness function, too.

So, from (6.6) and (6.3) it follows that

$$\beta(\{u_n(t); n \geq k\}) \leq \int_{[\tau, t] \setminus \mathcal{O}_k} \alpha_0(s, \beta(\{v_n(s); n \geq k\})) ds + M e^{\omega(T-\tau)} \int_{\mathcal{O}_k} \ell(s) ds$$

Since by (6.4) we have $\beta(\{u_n(t); n \geq k\}) = \beta(\{v_n(t); n \geq k\})$ and $\beta(\{u_n(t); n \geq k\}) = \beta(\{u_n(t); n \geq 1\})$, passing to the limit for $k \rightarrow \infty$ in the inequality above and taking into account that α_0 is a Carathéodory uniqueness function, it follows that $\beta(\{u_n(t); n \geq 1\}) = 0$. Thus $\{u_n(t); n \geq 1\}$ is relatively compact for each $t \in [\tau, T]$. In view of (6.3) and using (6.2) and Theorem 8.4.1, p. 194 in Vrabie [10] we conclude that, on a subsequence at least, $(u_n)_n$ is uniformly convergent on $[\tau, T]$ to some function u . But $\lim_n v_n(t) = u(t)$, uniformly for $t \in [\tau, T]$, and hence, for each $k \geq 1$, the set

$$C_k = \overline{\{(t, v_n(t)); n \geq k, t \in [\tau, T] \setminus \mathcal{O}_k\}}$$

is compact. Since F is strongly-weakly u.s.c. and has weakly compact values, by Lemma 2.6.1, p. 47, in Cârjă, Necula, Vrabie [3], it follows that, for each $k \geq 1$, the set

$$B_k := \overline{\text{conv}} \left(\bigcup_{n \geq k} \bigcup_{t \in [\tau, T] \setminus \mathcal{O}_k} F(t, v_n(t)) \right)$$

is weakly compact. We notice that $\|f_n(s)\| \leq \ell(s)$ a.e. for $s \in [\tau, T]$ and $f_n(s) \in B_k$ for each $k \geq 1$ and $n \geq k$ and a.e. for $s \in [\tau, T] \setminus \mathcal{O}_k$. Since $\ell \in L^1(\tau, T; \mathbb{R})$, B_k is weakly compact and $\lim_k \lambda(\mathcal{O}_k) = 0$, by Diestel's Theorem 1.3.8, p. 10, in Cârjă, Necula, Vrabie [3], it follows that, on a subsequence at least, $\lim_n f_n = f$ weakly in $L^1(\tau, T; X)$. As $\lim_n v_n(t) = u(t)$ uniformly for $t \in [\tau, T]$, and, by Lemma 5.1, for each $k \geq 1$, each $n \geq k$, we have $f_n(s) \in F(s, v_n(s))$ a.e. for $s \in [\tau, T] \setminus \mathcal{O}_k$, from Theorem 3.1.2, p. 88, in Vrabie [9], we conclude that $f(s) \in F(s, u(s))$ for each $k \geq 1$ and a.e. for $s \in [\tau, T] \setminus \mathcal{O}_k$. Since $\lim_k \lambda(\mathcal{O}_k) = 0$, we get

$$f(s) \in F(s, u(s)) \text{ a.e. for } s \in [\tau, T] \quad (6.7)$$

Finally, passing to the limit both sides in (6.5), for $n \rightarrow \infty$, we get

$$u(t) = S(t - \tau)\xi + \int_{\tau}^t S(t - s)f(s) ds,$$

for each $t \in [\tau, T]$. Since $v_n(t) \in K(t)$ and $\lim_n v_n(t) = u(t)$ for all $t \in [\tau, T]$ and \mathcal{K} is locally closed from the left, it follows that $u(t) \in K(t)$ for each $t \in [\tau, T]$. By (6.7), we conclude that u is a mild solution of (1.1), and this completes the proof when X is separable.

If X is not separable, we have to observe that there exists a separable and closed subspace $Y \subseteq X$ such that the families: $\{S(\cdot)f_n(\cdot); n \geq 1\}$, $\{S(\cdot)u_n(\cdot); n \geq 1\}$, $\{S(\cdot)v_n(\cdot); n \geq 1\}$ and $\{S(\cdot)r_n(\cdot); n \geq 1\}$ are Y -valued. Then, to complete the proof, it suffices to follow the very same arguments as before and to make use of (iv) in Remark 2.1.

The proof of Theorem 5.2 is exactly the same with the exception of obtaining the fact that $\{u_n(t); n \geq 1\}$ is relatively compact. Indeed, since \mathcal{K} is locally compact from the left, it follows that the set $\{v_n(t); n \geq 1\}$ is relatively compact. Moreover, recalling that

$$\lim_n \|v_n(s) - u_n(s)\| = 0$$

for $s \in [\tau, T)$, it follows that $\{u_n(t); n \geq 1\}$ is relatively compact for all $t \in [\tau, T)$. The remaining of the proof is identical to the one of Theorem 5.1. \square

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