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SINGULARLY PERTURBED CAUCHY PROBLEM FOR ABSTRACT LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER IN HILBERT SPACES*

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Abstract

We study the behavior of solutions to the problem

$$\begin{cases} \varepsilon\left(u_\varepsilon''(t)+A_1u_\varepsilon(t)\right)+u_\varepsilon'(t)+A_0u_\varepsilon(t)=f_\varepsilon(t), \quad t\in(0,T),\\ u_\varepsilon(0)=u_{0\varepsilon}, \quad u_\varepsilon'(0)=u_{1\varepsilon}, \end{cases}$$

as $\varepsilon\to 0$, where A_1 and A_0 are two linear self-adjoint operators in a Hilbert space H.

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keywords: singular perturbations; Cauchy problem; boundary layer function.

1 Introduction

Let H be a real Hilbert space endowed with the inner product $(\cdot,\cdot)$ and the norm $|\cdot|$. Let $A_i : D(A_i)\to H$, $i=0,1$, be two linear self-adjoint operators.

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Consider the following Cauchy problem:

$$\begin{cases} \varepsilon\left(u_\varepsilon''(t) + A_1 u_\varepsilon(t)\right) + u_\varepsilon'(t) + A_0 u_\varepsilon(t) = f_\varepsilon(t), \quad t \in (0,T), \\ u_\varepsilon(0) = u_{0\varepsilon}, \quad u_\varepsilon'(0) = u_{1\varepsilon}, \end{cases} \tag{P_ε}$$

where $\varepsilon > 0$ is a small parameter($\varepsilon \ll 1$), $u_\varepsilon, f_\varepsilon : [0,T) \to H$.

We will investigate the behavior of solutions $u_\varepsilon(t)$ to the perturbed system (P_ε) when $\varepsilon \to 0$, $u_{0\varepsilon} \to u_0$ and $f_\varepsilon \to f$. We will establish a relationship between solutions to the problem (P_ε) and the corresponding solutions to the following unperturbed system:

$$\begin{cases} v'(t) + A_0 v(t) = f(t), \quad t \in (0,T), \\ v(0) = u_0. \end{cases} \tag{P_0}$$

In our study we will use the following conditions:

(H1) *The operator $A_0 : D(A_0) \subseteq H \to H$ is self-adjoint and positive defined, i.e. there exists $\omega_0 > 0$ such that*

$$(A_0 u, u) \geq \omega_0 \, |u|^2, \quad \forall u \in D(A_0);$$

(H2) *The operator $A_1 : D(A_1) \subseteq H \to H$ is self-adjoint, $D(A_0) \subseteq D(A_1)$ and there exists $\omega_1 > 0$ such that*

$$|(A_1 u, u)| \leq \omega_1 \, (A_0 u, u), \quad \forall u \in D(A_0).$$

If, in some topology, $u_\varepsilon(t)$ tends to the corresponding solutions $v(t)$ of the unperturbed system (P_0) as $\varepsilon \to 0$, then the system (P_0) is called *regularly perturbed.* In the opposite case system (P_0) is called *singularly perturbed.* In the last case, a subset of $[0,\infty)$, in which the solution $u_\varepsilon(t)$ has a singular behavior relative to ε, arises. This subset is called *the boundary layer.* The function which defines the singular behavior of the solution $u_\varepsilon(t)$ within the boundary layer is called *the boundary layer function.*

Many physical processes are described by systems of type (P_ε). For example, the equation

$$\rho v_{tt} + \gamma v_t = \sigma \Delta v$$

(where ρ, γ, σ are the mass density per unit area of the membrane, the coefficient of viscosity of the medium, and the tension of the membrane, respectively), which characterizes the vibration of a membrane in a viscous medium, can be rewritten as

$$\varepsilon^2 u_{tt} + u_t = \Delta u,$$

with $\varepsilon = (\rho\sigma)^{1/2}/\gamma$.

In the case when the medium is highly viscous ($\gamma \gg 1$), or the density ρ is very small, we have $\varepsilon \to 0$ and the formal "limit" of this equation will be the following first order equation

$$u_t = \Delta u.$$

Let us mention some works dedicated to the study of singularly perturbed Cauchy problems for differential equations of second order in Hilbert spaces. In [2], [3], [4], [5], [7], [8], [9], the behaviour of the solutions u_ε to the abstract linear Cauchy problem (P_ε) has been studied as $\varepsilon \mapsto 0$ in the case when A_0 and A are positive operators, $B = 0$ or B is an linear integrodifferential operator. All results from these papers were obtained using the theory of semigroups of linear operators.

Our approach is based on two key points. The first one is the relationship between the solutions of the problems (P_ε) and (P_0). The second key point consists in obtaining a priori estimates for the solutions of the problems (P_ε), estimates which are uniform with respect to small parameter ε.

2 Preliminaries

The goal of this section is to remind the notations and main assertions which will be used in that follows.

Let $k \in N^*$, $1 \leq p \leq +\infty$, $(a, b) \subset (-\infty, +\infty)$ and let X be the Banach space. We denote by $W^{k,p}(a, b; X)$ the Banach space of all vectorial distributions $u \in D'(a, b; X)$, $u^{(j)} \in L^p(a, b; X)$, $j = 0, 1, \ldots, k$, endowed with the norm

$$\|u\|_{W^{k,p}(a,b;X)} = \left(\sum_{j=0}^{k} \|u^{(j)}\|^p_{L^p(a,b;X)} \right)^{1/p}$$

for $p \in [1, \infty)$ and

$$\|u\|_{W^{k,\infty}(a,b;X)} = \max_{0 \leq j \leq k} \|u^{(j)}\|_{L^\infty(a,b;X)}$$

for $p = \infty$.

In the particular case $p = 2$, we denote $W^{k,2}(a,b;X) = H^k(a,b;X)$. If X is a Hilbert space, then $H^k(a,b;X)$ is also a Hilbert space with the inner product

$$(u,v)_{H^k(a,b;X)} = \sum_{j=0}^{k} \int_a^b \Big(u^{(j)}(t), v^{(j)}(t)\Big)_X \, dt.$$

For each arbitrary but fixed $s \in \mathbb{R}$, $k \in \mathbb{N}$ and $p \in [1,\infty]$, we define the Banach space

$$W_s^{k,p}(a,b;H) = \{f : (a,b) \to H; f^{(l)}(\cdot)e^{-st} \in L^p(a,b;X),\, l = 0,\dots,k\},$$

with the norm

$$\|f\|_{W_s^{k,p}(a,b;X)} = \|fe^{-st}\|_{W^{k,p}(a,b;X)}.$$

Theorem 1. *Let $p \in [1,\infty]$ and X be a reflexive Banach space. Then the embedding $W^{1,p}(0,T;X) \hookrightarrow C([0,T];X)$ is continuous, i.e., there exists $C(T,p) > 0$ such that, for each $f \in W^{1,p}(0,T;X)$, we have*

$$\|f\|_{C([0,T];X)} \le C(T,p)\, \|f\|_{W^{1,p}(0,T;X)}.$$

Theorem 2. *Let $k \in \mathbb{N}$, $p \in [1,\infty]$ and let X be a Banach space. Then there exists $C(k,p,T) > 0$ such that, for every $f \in W^{k,p}(0,T;X)$, there exists an extension $\tilde{f} \in W^{k,p}(0,\infty;X)$ of f satisfying*

$$\|\tilde{f}\|_{W^{k,p}(0,\infty;X)} \le C(k,p,T)\, \|f\|_{W^{k,p}(0,T;X)}.$$

Theorem 3. *Let X be a reflexive Banach space. Let $f : (0,T) \to X$ and let $f_h(t) = h^{-1}\left(f(t+h) - f(t)\right)$, $t, t+h \in (0,T)$.*

(i) If $1 \le p \le +\infty$ and for each $(a.b) \subseteq (0,T)$ $f \in W^{1,p}(a,b;X)$, then

$$\|f_h\|_{L^p(a,b;X)} \le \|f\|_{W^{1,p}(a,b;X)},\ 0 < |h| < \min\{a/2, (T-b)/2\}.$$

(ii) If $1 < p < +\infty$, $f \in L^p(a,b;X)$ and there exists $C > 0$ such that

$$\|f_h\|_{L^p(a,b;X)} \le C, \quad 0 < |h| < \min\{a/2, (T-b)/2\},$$

then $f \in W^{1,p}(a,b;X)$ and

$$\|f\|_{W^{1,p}(a,b;X)} \le C.$$

Theorem 4. *Let H be a real Hilbert space, and let $A : D(A) \subset H \to H$ be a linear self-adjoint positive operator. If $u \in W^{1,2}(0,T;H)$ such that $u(t) \in D(A)$ a.e. for $t \in [a,b] \subseteq [0,T]$ and $Au \in L^2(0,T;H)$, then the function $t \to (Au(t), u(t))$ is absolutely continuous on $[a,b]$ and*

$$\frac{d}{dt}(Au(t), u(t)) = 2(Au(t), u'(t)), \quad a.e. \quad t \in [a,b].$$

Definition 1. The operator $A : D(A) \subset H \to H$ is called *monotone* if

$$(Au_1 - Au_2, u_1 - u_2) \geq 0, \quad \forall u_1, u_2 \in D(A).$$

The operator A is called *maximal monotone* if it is monotone and A does not have (possible multivalued) monotone extensions in H.

Theorem 5. [1] *Let $A : D(A) \subset H \to H$ be a monotone operator in H. A is maximal monotone if and only if for every $\lambda > 0$ (equivalently for some $\lambda > 0$), $R(I + \lambda A) = H$.*

Theorem 6. [1] *The linear monotone operator $A : D(A) \subset H \to H$ is maximal monotone if and only if A is closed and $(A^*u, u) \geq 0$, $\forall u \in D(A^*)$, where A^* is the adjoint operator to A.*

For a maximal monotone operator $A : D(A) \subset H \to H$ and $\lambda > 0$, we denote by J_λ its resolvent $J_\lambda = (I + \lambda A)^{-1}$, and by $A_\lambda = \lambda^{-1}(I - J_\lambda)$ the Yosida approximation.

Theorem 7. *Let $A : D(A) \subset H \to H$ be maximal monotone operator. Then for every $\lambda > 0$:*

(i) *J_λ is lipschitzian on H with the constant 1;*

(ii) $A_\lambda x = AJ_\lambda x, \quad \forall x \in H \quad$ *and* $A_\lambda x = J_\lambda Ax, \quad \forall x \in D(A)$;

(iii) *A_λ is a monotone and lipschitzian operator on H with the constant λ^{-1} ;*

(iv) $|A_\lambda x| \leq |Ax|, \quad \forall x \in D(A)$;

(v) $lim_{\lambda \to 0} A_\lambda x = Ax, \quad \forall x \in D(A)$;

(vi) $|A_\lambda x|^2 \leq (Ax, A_\lambda x), \quad \forall x \in D(A)$.

Definition 2. The function $u : [a, b] \to H$ is called *strong solution* to the Cauchy problem

$$\begin{cases} u'(t) + Au(t) = f(t), \quad t \in (a, b), \\ u(a) = u_0 \end{cases} \tag{2.1}$$

if u is absolutely continuous on $[a, b]$, $u' \in L^1(a, b; H)$, $u(t) \in D(A)$ a.e. for $t \in (a, b)$, $u(t)$ satisfies the first equality in (2.1) a.e. for $t \in (0, T)$ and $u(a) = u_0$.

Theorem 8. [1] *Let $A : D(A) \subset H \to H$ such that $A + \omega I$ is maximal monotone. If $u_0 \in D(A)$ and $f \in W^{1,1}(0, T; H)$ then there exists a unique strong solution $u \in W^{1,\infty}(0, T; H)$ to the problem*

$$\begin{cases} u'(t) + Au(t) = f(t), \quad t \in (0, T), \\ u(0) = u_0 \end{cases}$$

and

$$|u(t)| + \left(\int_0^t e^{\gamma\,(t-s)} \left((A + \omega I)u(s), u(s) \right) \, ds \right)^{1/2}$$

$$\leq e^{\omega\, t/2} \left(|u_0| + \int_0^t e^{-\omega\, s/2} \, |f(s)| \, ds \right), \quad \forall t \in [0, T],$$

$$\left| \frac{d^+ u}{dt}(t) \right| \leq e^{\omega t} \, |f(0) - Au_0| + \int_0^t e^{\omega(t-s)} \left| \frac{df}{ds}(s) \right| ds, \quad \forall t \in [0, T).$$

Lemma 1. [10] *Let $\psi \in L^1(a, b)$ $(-\infty < a < b < \infty)$ with $\psi \geq 0$ a.e. on (a,b) and c be a fixed real constant. If $h \in C[a, b]$ verifies*

$$\frac{1}{2} h^2(t) \leq \frac{1}{2} c^2 + \int_a^t \psi(s) h(s) ds, \quad \forall t \in [a, b],$$

then

$$h(t) \leq |c| + \int_a^t \psi(s) ds, \quad \forall t \in [a, b].$$

3 Existence of strong solutions to both (P_ε) and (P_0)

In this section we will study the solvability of problems (P_ε) and (P_0) and also the regularity of their solutions.

The following two theorems were inspired by [1].

Theorem 9. *Let $T > 0$ and let us assume that A_0 satisfies the condition* (**H1**). *If $u_0 \in D(A_0)$ and $f \in W^{1,1}(0,T;H)$, then there exists a unique strong solution $v \in W^{1,\infty}(0,T;H)$ to the problem (P_0). Moreover, v satisfies*

$$|v(t)| + \left(\int_0^t \left| A_0^{1/2} u(s) \right| ds \right)^{1/2} \leq |u_0| + \int_0^t |f(s)|\, ds, \quad \forall t \in [0,T],$$

$$\left| v'(t) \right| \leq |A_0 u_0 - f(0)| + \int_0^t \left| f'(s) \right| ds, \quad \forall t \in [0,T].$$

Theorem 10. *Let $T > 0$. Let us assume that $A : D(A) \subset H \to H$ is linear self-adjoint and positive. If $u_0 \in D(A)$, $u_1 \in H$ and $f \in W^{1,1}(0,T;H)$, then there exists a unique function $u : [0,T] \to H$ such that:*

$u \in W^{2,\infty}(0,T;H)$, $\quad A^{1/2}u \in W^{1,\infty}(0,T;H)$, $\quad Au \in L^{\infty}(0,T;H)$,

$A^{1/2}u$ *and* u' *are differentiable from to the right in H for every $t \in [0,T)$ and*

$$\frac{d^+}{dt}\frac{du}{dt}(t) + \frac{du}{dt}(t) + Au(t) = f(t), \quad t \in [0,T), \tag{3.1}$$

$$u(0) = u_0, \quad u'(0) = u_1. \tag{3.2}$$

In what follows this function will be called *the strong solution* to the problem (3.1), (3.2).

Proof. Let us denote by $\mathcal{H} = D(A^{1/2}) \times H$ which, endowed with the inner product

$$(U_1, U_2)_{\mathcal{H}} = (A^{1/2}u_1, A^{1/2}u_2) + (v_1, v_2), \quad U_i = (u_i; v_i) \in \mathcal{H}, \quad i = 1, 2,$$

is the real Hilbert space. Let us further denote by $\mathcal{L} : \mathcal{D}(\mathcal{L}) \subseteq \mathcal{H} \to \mathcal{H}$, the operator defined by

$$D(\mathcal{L}) = D(A) \times H, \quad \mathcal{L}U = (-v, Au + v), \quad \forall U = (u; v) \in D(\mathcal{L}).$$

As

$$(\mathcal{L}U, U)_{\mathcal{H}} = -(Av, u) + (Au + v, v) = |v|^2 \geq 0, \quad \forall U \in D(\mathcal{L}),$$

it follows that $\mathcal{L}$ is monotone. Now we are going to show that it is maximal monotone. To this aim, let us consider the equation $(\lambda I + \mathcal{L})\, U = F$, $\lambda > 0$, where $F = (f, g) \in \mathcal{H}$ and $U = (u, v) \in D(\mathcal{L})$, which is equivalent to the system

$$\begin{cases} \lambda u - v = f \\ \lambda v + Au + v = g, \end{cases}$$

i.e.

$$\begin{cases} \lambda u - v = f \\ \lambda(\lambda+1)u + Au = g_1, \end{cases} \tag{3.3}$$

where $g_1 = g + (\lambda+1)f$.

As A is a positive self-adjoint operator, therefore using Theorem 6, we can infer that A is a maximal monotone operator. Due to Theorem 5, we have that

$$\forall \beta > 0 \quad D((\beta I + A)^{-1}) = H, \quad R((\beta I + A)^{-1}) \subseteq D(A).$$

Therefore (3.3) is equivalent to the system

$$\begin{cases} \lambda u - v = f \\ u = (\beta I + A)^{-1} g_1, \end{cases} \tag{3.4}$$

with $\beta = \lambda(\lambda+1)$. Hence, if $f \in D(A^{1/2})$ and $g \in H$, it follows that $u = (\beta I + A)^{-1} g_1 \in D(A)$. From the first equation in (3.4), we deduce that $v = \lambda u - f \in D(A^{1/2})$. So, for every $F \in \mathcal{H}$ there exists a unique solution $U \in D(\mathcal{L})$ to the equation $(\lambda I + \mathcal{L})\, U = F$. So, $R(\lambda I + \mathcal{L}) = \mathcal{H}$ and, by Theorem 5, the operator $\mathcal{L}$ is maximal monotone. By Theorem 8, the problem

$$\begin{cases} U'(t) + \mathcal{L}U(t) = F(t), \quad t \in (0,T), \\ U(0) = U_0, \end{cases} \tag{P.U}$$

where $U(t) = (u(t); v(t))$, $U_0 = (u_0, u_1)$, $F(t) = (0, f(t))$ has a unique strong solution $U = (u, v) \in W^{1,\infty}(0,T;\mathcal{H})$ which implies that $A^{1/2}u, v \in W^{1,\infty}(0,T;H)$. As the equation in $(P.U)$ is equivalent to the system

$$\begin{cases} u'(t) - v(t) = 0 \\ v'(t) + Au(t) + v(t) = f(t), \end{cases}$$

it follows that u satisfies (3.1) and (3.2). Thus, (3.1), (3.2) has a unique strong solution $u \in W^{2,\infty}(0,T;H)$.

Finally, we have $A^{1/2}u \in W^{1,\infty}(0,T;H)$ and $Au \in L^\infty(0,T;H)$ and this completes the proof. □

4 A priori estimates for solutions to the problem (P_ε)

The goal of this section is to establish some *a priori* estimations for solutions to (P_ε) which are uniform relative to the small parameter ε.

Consider the following problem:

$$\begin{cases} \varepsilon\left(u_\varepsilon''(t) + A_1 u_\varepsilon(t)\right) + u_\varepsilon'(t) + A_0 u_\varepsilon(t) = f(t), \quad t \in (0,T), \\ u_\varepsilon(0) = u_0, \quad u_\varepsilon'(0) = u_1. \end{cases} \tag{4.1}$$

Lemma 2. *Let $T > 0$. Suppose that, for each $\varepsilon \in (0,1)$, the operator $A(\varepsilon) = (\varepsilon A_1 + A_0) : D\left(A(\varepsilon)\right) \subseteq H \to H$ is self-adjoint and satisfies*

$$(A(\varepsilon)u, u) \geq \omega\, |u|^2, \quad \forall u \in D\left(A(\varepsilon)\right), \quad \omega > 0, \quad \varepsilon \in (0,1]. \tag{4.2}$$

If $f \in W^{1,1}(0,T;H), u_0 \in D\left(A(\varepsilon)\right)$, $u_1 \in H$, then the unique strong solution, u_ε, of the problem (4.1) *satisfies*

$$\|A^{1/2}(\varepsilon)u_\varepsilon\|_{C([0,t];\,H)} + \|u_\varepsilon'\|_{L^2(0,\,t;\,H)} \leq C(\omega)\, M(t), \tag{4.3}$$

for each $t \in [0,T]$ and each $\varepsilon \in (0,1/2]$. If, in addition, $u_1 \in D\left(A^{1/2}(\varepsilon)\right)$, then

$$\|u_\varepsilon'\|_{C([0,t];\,H)} + \|A^{1/2}(\varepsilon)\, u_\varepsilon'\|_{L^2(0,\,t;\,H)} \leq C(\omega)\, M_1(t), \tag{4.4}$$

for each $t \in [0,T]$, and each $\varepsilon \in (0,1]$, and

$$\|A(\varepsilon)u_\varepsilon\|_{L^\infty(0,\,t;\,H)} \leq C(\omega) M_1(t), \quad \forall t \in [0,T], \quad \forall \varepsilon \in (0,1], \tag{4.5}$$

where

$$M(t) = M(t, u_0, u_1, f) = \left|A^{1/2}(\varepsilon)u_0\right| + |u_1| + \|f\|_{W^{1,1}(0,t;\,H)} + |f(0)|,$$

$$M_1(t) = M_1(t, u_0, u_1, f) = \left|A^{1/2}(\varepsilon)u_1\right| + |A(\varepsilon)u_0| + \|f\|_{W^{1,1}(0,t;\,H)} + |f(0)|.$$

Proof. We begin with the proof of (4.3). Let us denote by

$$E(u,t) = \varepsilon\, \left(u'(t), u(t)\right) + \int_0^t \left(A(\varepsilon)u(\tau), u(\tau)\right)\, d\tau + \frac{1}{2}\, |u(t)|^2$$

$$+\varepsilon \int_0^t \left|u'(\tau)\right|^2 d\tau + \varepsilon^2 \left|u'(t)\right|^2 + \varepsilon \left(A(\varepsilon)u(t), u(t)\right).$$

For every solution, u_ε, of (4.1), by direct computation, we obtain

$$\frac{d}{dt}E(u_\varepsilon, t) = \big(f(t), u_\varepsilon(t) + 2\varepsilon u_\varepsilon'(t)\big), \quad \text{a.e.} \quad t \in (0, T).$$

As

$$E(u_\varepsilon, t) \geq 0, \quad \left|u_\varepsilon(t) + 2\varepsilon u_\varepsilon'(t)\right| \leq 2\left(E(u_\varepsilon, t)\right)^{1/2},$$

for each $t \in [0, T]$, and each $\varepsilon \in (0, 1]$, it follows that

$$\frac{d}{dt}E(u_\varepsilon, t) \leq 2\left|f(t)\right| \left(E(u_\varepsilon, t)\right)^{1/2}, \quad \text{a.e.} \quad \forall t \in (0, T).$$

Integrating the last inequality, we obtain

$$\frac{1}{2}E(u_\varepsilon, t) \leq \frac{1}{2}E(u_\varepsilon, 0) + \int_0^t \left|f(\tau)\right| \left(E(u_\varepsilon, \tau)\right)^{1/2} d\tau, \quad \forall t \in [0, T].$$

Applying Lemma 1 to the last inequality, we get

$$\left(E(u_\varepsilon, t)\right)^{1/2} \leq \left(E(u_\varepsilon, 0)\right)^{1/2} + \int_0^t \left|f(\tau)\right| d\tau, \quad \forall t \in [0, T],$$

from which we deduce

$$\|u_\varepsilon\|_{C([0,t];\,H)} + \|A^{1/2}(\varepsilon)\, u_\varepsilon\|_{L^2(0,t;\,H)} \leq C(\omega)M(t), \tag{4.6}$$

for each $t \in [0, T]$ and each $\forall \varepsilon \in (0, 1]$. Let now

$$\mathcal{E}(u, t) = \varepsilon|u'(t)|^2 + |u(t)|^2 + (A(\varepsilon)u(t), u(t)) + 2(1-\varepsilon)\int_0^t |u'(s)|^2 ds$$

$$+2\varepsilon\left(u(t), u'(t)\right) + 2\int_0^t \left(A(\varepsilon)u(s), u(s)\right) ds.$$

Then, for every strong solution u_ε to the problem (4.1), we have

$$\frac{d}{dt}\mathcal{E}(u_\varepsilon, t) = 2\left(f(t), u_\varepsilon(t) + u_\varepsilon'(t)\right), \quad \text{a.e. } t \in (0, T),$$

and thus

$$\mathcal{E}(u_\varepsilon,t)=\mathcal{E}(u_\varepsilon,0)+2\,(u_\varepsilon,f(t))-2\,(u_0,f(0))$$
$$+2\int_0^t \big(f(s)-f'(s),u_\varepsilon(s)\big)\,ds,\quad \forall t\in[0,T]. \tag{4.7}$$

Since

$$\mathcal{E}(u_\varepsilon,0)\le C(\omega)\,M^2(t),\quad \forall t\in[0,T],\quad \forall\varepsilon\in(0,1]$$

and, in view of (4.6), we have

$$2\,|(u_\varepsilon,f(t))-(u_0,f(0))|\le C(\omega)\,M^2(t),\quad \forall t\in[0,T],\quad \forall\varepsilon\in(0,1],$$

from (4.7), we get

$$\mathcal{E}(u_\varepsilon,t)\le C(\omega)\,M^2(t),\quad t\in[0,t],\quad \forall\varepsilon\in(0,1],$$

which implies (4.3).

Proof of (4.4). Let $h>0$ such that $t,t+h\in[0,T]$. Denote by $u_{\varepsilon h}(t)=u_\varepsilon(t+h)-u_\varepsilon(t)$, where u_ε is the strong solution to problem (4.1). Then for $u_{\varepsilon h}$ we have the equality

$$\frac{d}{dt}E(u_{\varepsilon h},t)=\big(f_h(t),u_{\varepsilon h}(t)+2\varepsilon\,u'_{\varepsilon h}(t)\big)\quad \text{a.e.}\quad \in(0,T-h).$$

Integrating this equality and applying Lemma 1 and Theorem 3, we obtain

$$(E(u_{\varepsilon h},t))^{1/2}\le(E(u_{\varepsilon h},0))^{1/2}+\int_0^t\big|f'(\tau)\big|\,d\tau,\quad \forall t\in[0,T-h].$$

As $u'(0)=u_1$ and

$$\lim_{h\downarrow 0}\varepsilon\left|h^{-1}\,u'_h(0)\right|=|f(0)-u_1-A(\varepsilon)u_0|\,,$$

$$\lim_{h\downarrow 0}h^{-1}\,\left|A^{1/2}(\varepsilon)u_{\varepsilon h}(0)\right|=\left|A^{1/2}(\varepsilon)u_1\right|,$$

dividing the last equality by h and passing to the limit as $h\to 0$, we get (4.4).

Proof of (4.5). Let $A_\lambda(\varepsilon)$ be the Yosida approximation of the operator $A(\varepsilon)$. Let

$$E_1(u,t)=\varepsilon\,\big(A_\lambda(\varepsilon)u'(t),u'(t)\big)+(A_\lambda(\varepsilon)u(t),u(t))$$

$$+\left(A_\lambda(\varepsilon)u(t),A(\varepsilon)u(t)\right)+2\varepsilon\left(A_\lambda(\varepsilon)u(t),u'(t)\right)$$

$$+2(1-\varepsilon)\int_0^t\left(A_\lambda(\varepsilon)u'(s),u'(s)\right)\,ds+2\int_0^t\left(A_\lambda(\varepsilon)u(\tau),A(\varepsilon)u(s)\right)ds.$$

Then every strong solution, u_ε, of the problem (4.1) satisfies

$$\frac{d}{dt}E_1(u_\varepsilon,t)=2\left(f(t),\mathcal{A}_\lambda u_\varepsilon(t)+\mathcal{A}_\lambda u_\varepsilon'(t)\right),\quad \text{a.e.}\quad t\in(0,T).$$

Integrating this equality, we obtain

$$E_1(u_\varepsilon,t)=E_1(u_\varepsilon,0)+I_1(t,\varepsilon)+I_2(t,\varepsilon),\quad \forall t\in[0,T], \tag{4.8}$$

where

$$I_1(t,\varepsilon)=2\left(f(t),A_\lambda(\varepsilon)u_\varepsilon(t)\right)-2\left(f(0),A_\lambda(\varepsilon)u_0\right),$$

$$I_2(t,\varepsilon)=2\int_0^t\left(f(s)-f'(s),A_\lambda(\varepsilon)u_\varepsilon(s)\right)\,ds.$$

Let us evaluate $I_1(t,\varepsilon),I_2(t,\varepsilon)$. Using (iv), (vi) in Theorem 7, we get

$$|I_1(t,\varepsilon)|\le\frac{1}{2}\left|A_\lambda(\varepsilon)u_\varepsilon(t)\right|^2+2|f(t)|^2+|f(0)|^2+|A_\lambda(\varepsilon)u_0|^2$$

$$\le\frac{1}{2}\left(A_\lambda(\varepsilon)u_\varepsilon(t),A(\varepsilon)u_\varepsilon(t)\right)+C(\omega)\,M_1^2(t),\quad \forall t\in[0,T]. \tag{4.9}$$

As $(A_\lambda(\varepsilon)u,u)\ge0$, $\forall u\in H$, it follows that

$$\left(A_\lambda(\varepsilon)u,v\right)^2\le\left(A_\lambda(\varepsilon)u,u\right)\,\left(A_\lambda(\varepsilon)v,v\right),\quad \forall u,v\in H.$$

Therefore, due to (vi) in Theorem 7, we get

$$\varepsilon\left(A_\lambda(\varepsilon)u_\varepsilon'(t),u_\varepsilon'(t)\right)+\left(A_\lambda(\varepsilon)u_\varepsilon(t),u_\varepsilon(t)\right)+\left(A_\lambda(\varepsilon)u_\varepsilon(t),A(\varepsilon)u_\varepsilon(t)\right)$$

$$+2\varepsilon\left(A_\lambda(\varepsilon)u_\varepsilon(t),u_\varepsilon'(t)\right)=(1-\varepsilon)\,\left(A_\lambda(\varepsilon)u_\varepsilon(t),u_\varepsilon(t)\right)$$

$$+\varepsilon\left(A_\lambda(u_\varepsilon(t)+u_\varepsilon'(t)),(u_\varepsilon(t)+u_\varepsilon'(t)\right)+\left(A_\lambda(\varepsilon)u_\varepsilon(t),A(\varepsilon)u_\varepsilon(t)\right)$$

$$\ge\left(A_\lambda(\varepsilon)u_\varepsilon(t),A(\varepsilon)u_\varepsilon(t)\right)\ge\left|A_\lambda(\varepsilon)u_\varepsilon(t)\right|^2,\quad \forall\varepsilon\in(0,1].$$

As

$$E_1(u_\varepsilon,t)\ge0,\quad |A_\lambda(\varepsilon)u_\varepsilon|\le E_1^{1/2}(u_\varepsilon,t),\quad \forall t\in[0,T],\quad \forall\varepsilon\in(0,1],$$

we have

$$|I_2(t,\varepsilon)| \le 2\int_0^t \big(|f(s)| + |f'(s)|\big)\, E_1^{1/2}(u_\varepsilon, s)\, ds, \quad \forall t \in [0,T]. \tag{4.10}$$

Due to (vi) in Theorem 7, we get

$$E_1(u_\varepsilon, 0) \le C(\omega)\left(|A(\varepsilon)u_0|^2 + |A^{1/2}(\varepsilon)u_1|^2\right), \quad \forall \varepsilon \in (0,1]. \tag{4.11}$$

Using (4.9), (4.10) and (4.11), from (4.8), we obtain

$$\begin{aligned} E_1(u_\varepsilon, t) &\le C(\omega)\, M_1^2(t) \\ &+2\int_0^t \big(|f(s)| + |f'(s)|\big)\, E_1^{1/2}(u_\varepsilon, s)\, ds, \end{aligned} \tag{4.12}$$

for all $t \in [0,T]$ and all $\varepsilon \in (0,1]$.

Applying Lemma 1 to (4.12), we deduce

$$E_1^{1/2}(u_\varepsilon, t) \le C(\omega)\, M_1(t), \quad \forall t \in [0,T], \quad \forall \varepsilon \in (0,1],$$

from which it follows that

$$(A_\lambda(\varepsilon)u_\varepsilon(t), A(\varepsilon)u_\varepsilon(t)) \le C(\omega)\, M_1^2(t) \forall \quad t \in [0,T], \quad \forall \varepsilon \in (0,1].$$

Finally, passing to the limit in the last inequality as $\lambda \to 0$ and using (v) in Theorem 7, we get (4.5) and this completes the proof. □

Let u_ε be a strong solution of the problem (4.1) and let us denote by

$$z_\varepsilon(t) = u_\varepsilon'(t) + \alpha e^{-t/\varepsilon}, \quad \alpha = f(0) - u_1 - A(\varepsilon)u_0. \tag{4.13}$$

Lemma 3. *Let $T > 0$ and let us assume that, for each $\varepsilon \in (0,1)$, the operator $A(\varepsilon) = \varepsilon A_1 + A_0$ is self-adjoint and satisfies (4.2). If u_1, $f(0) - A(\varepsilon)u_0 \in D\left(A(\varepsilon)\right)$ and $f \in W^{2,1}(0,T;H)$, then there exist $C(\omega) > 0$, such that the function z_ε, defined by (4.13), satisfies*

$$\begin{aligned} \|A^{1/2}(\varepsilon)z_\varepsilon\|_{C([0,\,t];\,H)} + \|z_\varepsilon'\|_{C([0,\,t];\,H)} + \left\|A^{1/2}(\varepsilon)z_\varepsilon'\right\|_{L^2(0,\,t;\,H)} \\ \le C(\omega)\, M_2(t), \quad \forall t \in [0,T], \quad \forall \varepsilon \in (0,1], \end{aligned} \tag{4.14}$$

where

$$M_2(t) = |A(\varepsilon)f(0) - A^2(\varepsilon)u_0| + \|f\|_{W^{2,1}(0,t;H)} + |A(\varepsilon)u_1| + |f'(0)|.$$

Proof. If u_1, $f(0) - A(\varepsilon)u_0 \in D\left(A(\varepsilon)\right)$ and $f \in W^{2,1}(0,T;H)$, then, due to Theorem 10, z_ε is the strong solution of the problem

$$\begin{cases} \varepsilon z_\varepsilon''(t) + z_\varepsilon'(t) + A(\varepsilon) z_\varepsilon(t) = \mathcal{F}(t,\varepsilon), \quad \text{a.e.} \quad t \in (0,T), \\ z_\varepsilon(0) = f(0) - A(\varepsilon)u_0, \quad z_\varepsilon'(0) = 0, \end{cases}$$

where

$$\mathcal{F}(t,\varepsilon) = f\,'(t) + e^{-t/\varepsilon} A(\varepsilon)\alpha.$$

Finally, let us observe that z_ε satisfies $A^{1/2}(\varepsilon) z_\varepsilon \in W^{1,\infty}(0,T;H)$, $z_\varepsilon \in W^{2,\infty}(0,T;H)$ and $A(\varepsilon)z_\varepsilon \in L^\infty(0,T;H)$. Therefore, (4.14) follows from Lemma 2 and the proof is complete. □

5 The relationship between the solution of (P_ε) and (P_0)

Now we are going to establish the relationship between the solution to the problem (P_ε) and the corresponding solution to the problem (P_0). This relationship was inspired by [6]. To this end, we begin by defining the transformation kernel which realizes this relationship.

Namely, for $\varepsilon > 0$, let us denote

$$K(t,\tau,\varepsilon) = \frac{1}{2\,\varepsilon\,\sqrt{\pi}} \left(K_1(t,\tau,\varepsilon) + 3K_2(t,\tau,\varepsilon) - 2K_3(t,\tau,\varepsilon) \right),$$

where

$$K_1(t,\tau,\varepsilon) = \exp\left\{ \frac{3t - 2\tau}{4\varepsilon} \right\} \lambda\left(\frac{2t-\tau}{2\sqrt{\varepsilon t}} \right),$$

$$K_2(t,\tau,\varepsilon) = \exp\left\{ \frac{3t + 6\tau}{4\varepsilon} \right\} \lambda\left(\frac{2t+\tau}{2\sqrt{\varepsilon t}} \right),$$

$$K_3(t,\tau,\varepsilon) = \exp\left\{ \frac{\tau}{\varepsilon} \right\} \lambda\left(\frac{t+\tau}{2\sqrt{\varepsilon t}} \right), \quad \lambda(s) = \int_s^\infty e^{-\eta^2} d\eta.$$

The properties of the kernel $K(t,\tau,\varepsilon)$ are collected in the next lemma.

Lemma 4. [11]. *The function $K(t,\tau,\varepsilon)$ has the following properties*:

(*i*) $K \in C([0,\infty) \times [0,\infty)) \cap C^2((0,\infty) \times (0,\infty))$;

(*ii*) $K_t(t,\tau,\varepsilon) = \varepsilon K_{\tau\tau}(t,\tau,\varepsilon) - K_\tau(t,\tau,\varepsilon), \quad \forall t > 0, \forall \tau > 0$;

(*iii*) $\varepsilon K_\tau(t,0,\varepsilon) - K(t,0,\varepsilon) = 0, \quad \forall t \geq 0\,;$

(*iv*) $K(0,\tau,\varepsilon) = \dfrac{1}{2\varepsilon} \exp\left\{-\dfrac{\tau}{2\varepsilon}\right\}, \quad \forall \tau \geq 0\,;$

(*v*) *For every $t > 0$ and every $q, s \in \mathbb{N}$, there exist $C_1(q,s,t,\varepsilon) > 0$ and $C_2(q,s,t) > 0$ such that*

$$|\partial_t^s \partial_\tau^q K(t,\tau,\varepsilon)| \leq C_1(q,s,t,\varepsilon) \exp\{-C_2(q,s,t)\tau/\varepsilon\}, \quad \forall \tau > 0\,;$$

Moreover, for every $\gamma \in \mathbb{R}$, there exist $C_1 > 0$, $C_2 > 0$ and $\varepsilon_0 > 0$, depending on γ, such that:

$$\int_0^\infty e^{\gamma\tau}\, |K_t(t,\tau,\varepsilon)|\, d\tau \leq C_1\, \varepsilon^{-1}\, e^{C_2 t}, \quad \forall t \geq 0, \quad \forall \varepsilon \in (0,\varepsilon_0],$$

$$\int_0^\infty e^{\gamma\tau}\, |K_\tau(t,\tau,\varepsilon)|\, d\tau \leq C_1\, \varepsilon^{-1}\, e^{C_2 t} \quad \forall t \geq 0, \quad \forall \varepsilon \in (0,\varepsilon_0],$$

$$\int_0^\infty e^{\gamma\tau}\, |K_{\tau\tau}(t,\tau,\varepsilon)|\, d\tau \leq C_1\, \varepsilon^{-2}\, e^{C_2 t}, \quad \forall t \geq 0, \quad \forall \varepsilon \in (0,\varepsilon_0]\,;$$

(*vi*) $K(t,\tau,\varepsilon) > 0, \quad \forall t \geq 0, \quad \forall \tau \geq 0\,;$

(*vii*) *For every continuous $\varphi : [0,\infty) \to H$, with $|\varphi(t)| \leq M \exp\{\gamma\, t\}$, we have*:

$$\lim_{t\to 0} \left\| \int_0^\infty K(t,\tau,\varepsilon)\varphi(\tau)d\tau - \int_0^\infty e^{-\tau}\varphi(2\varepsilon\tau)d\tau \right|_H = 0,$$

for every $\varepsilon \in \left(0, (2\,\gamma)^{-1}\right)$;

(*viii*)

$$\int_0^\infty K(t,\tau,\varepsilon)d\tau = 1, \quad \forall t \geq 0.$$

(*ix*) *For every $\gamma > 0$ and $q \in [0,1]$, there exist $C_1 > 0$, $C_2 > 0$ and $\varepsilon_0 > 0$, depending on γ and on q, such that* :

$$\int_0^\infty K(t,\tau,\varepsilon)\, e^{\gamma\tau} |t-\tau|^q\, d\tau \leq C_1\, e^{C_2 t}\, \varepsilon^{q/2}, \quad \forall t > 0, \quad \forall \varepsilon \in (0,\varepsilon_0].$$

If $\gamma \leq 0$ and $q \in [0,1]$, then

$$\int_0^\infty K(t,\tau,\varepsilon)\, e^{\gamma\tau}\, |t-\tau|^q\, d\tau \leq C\, \varepsilon^{q/2} \left(1+\sqrt{t}\right)^q, \ \forall t \geq 0, \forall \varepsilon \in (0,1]\,;$$

(*x*) *Let* $p \in (1, \infty]$ *and* $f : [0, \infty) \to H$, $f \in W^{1,p}_{\gamma}(0, \infty; H)$. *If* $\gamma > 0$, *then there exist* $C_1 > 0$, $C_2 > 0$ *and* ε_0 *depending on* γ *and* p, *such that*

$$\left\| f(t) - \int_0^\infty K(t,\tau,\varepsilon) f(\tau) d\tau \right\|_H$$
$$\leq C_1\, e^{C_2 t}\, \|f'\|_{L^p_\gamma(0,\infty;H)}\, \varepsilon^{(p-1)/2p},\ \forall t \geq 0,\ \forall \varepsilon \in (0, \varepsilon_0].$$

If $\gamma \leq 0$, *then*

$$\left\| f(t) - \int_0^\infty K(t,\tau,\varepsilon) f(\tau) d\tau \right\|_H$$
$$\leq C(\gamma, p)\, \|f\,'\|_{L^p_\gamma(0,\infty;H)} \left(1 + \sqrt{t}\right)^{\frac{p-1}{p}}\ \varepsilon^{(p-1)/2p},\ \forall t \geq 0,\ \forall \varepsilon \in (0, 1].$$

(*xi*) *For every* $q > 0$ *and* $\alpha \geq 0$, *there exists* $C(q, \alpha) > 0$ *such that*

$$\int_0^t \int_0^\infty K(\tau, \theta, \varepsilon)\, e^{-q\,\theta/\varepsilon}\, |\tau - \theta|^\alpha\, d\theta\, d\tau \leq C(q, \alpha)\, \varepsilon^{1+\alpha},$$

for each $t \geq 0$, *and each* $\varepsilon > 0$.

Now we are ready to establish the relationship between the solution of (P_ε) and the solution of (P_0).

Theorem 11. *Suppose that* $A(\varepsilon)$ *satisfies* **(H1)**. *Let* $f \in L^\infty_c(0, \infty; H)$ *and let* $u_\varepsilon \in W^{2,\infty}_c(0, \infty; H)$ *be the strong solution of the problem* (4.1), *with* $Au_\varepsilon \in L^\infty_c(0, \infty; H)$, *for some* $c \geq 0$. *Then the function* w_ε, *defined by*

$$w_\varepsilon(t) = \int_0^\infty K(t, \tau, \varepsilon)\, u_\varepsilon(\tau)\, d\tau,$$

is the strong solution of the problem

$$\begin{cases} w'_\varepsilon(t) + A(\varepsilon) w_\varepsilon(t) = F_0(t, \varepsilon), \quad t > 0, \\ w_\varepsilon(0) = \varphi_\varepsilon, \end{cases} \tag{5.1}$$

where

$$\varphi_\varepsilon = \int_0^\infty e^{-\tau} u_\varepsilon(2\varepsilon\tau) d\tau, \quad F_0(t, \varepsilon) = f_0(t, \varepsilon) u_1 + \int_0^\infty K(t, \tau, \varepsilon)\, f(\tau)\, d\tau,$$

$$f_0(t, \varepsilon) = \frac{1}{\sqrt{\pi}} \left[2 \exp\left\{ \frac{3t}{4\varepsilon} \right\} \lambda \left(\sqrt{\frac{t}{\varepsilon}} \right) - \lambda \left(\frac{1}{2} \sqrt{\frac{t}{\varepsilon}} \right) \right].$$

Proof. Integrating by parts and using (i),(ii) and (iii) in Lemma 4, we get

$$\left(\omega_\varepsilon'(t),\eta\right)=\left(\int_0^\infty K_t(t,\tau,\varepsilon)u_\varepsilon(\tau)d\tau,\eta\right)$$

$$=\left(\int_0^\infty \left[\varepsilon K_{\tau\tau}(t,\tau,\varepsilon)-K_\tau(t,\tau,\varepsilon)\right]\,u_\varepsilon(\tau)d\tau,\eta\right)$$

$$=-\left(\left[\varepsilon K_\tau(t,0,\varepsilon)-K(t,0,\varepsilon)\right]u_\varepsilon(0),\eta\right)+\left(\varepsilon K(t,0,\varepsilon)u_1,\eta\right)$$

$$+\left(\int_0^\infty K(t,\tau,\varepsilon)\left(\varepsilon u_\varepsilon''(\tau)+u_\varepsilon'(\tau)\right)d\tau,\eta\right)$$

$$=\left(\varepsilon K(t,0,\varepsilon)u_1,\eta\right)+\left(\int_0^\infty K(t,\tau,\varepsilon)\left[f(\tau)-A(\varepsilon)u_\varepsilon(\tau)\right]\,d\tau,\eta\right)$$

$$=\left(\varepsilon K(t,0,\varepsilon)u_1+\int_0^\infty K(t,\tau,\varepsilon)\,f(\tau)d\tau,\eta\right)-\left(A(\varepsilon)w_\varepsilon(t),\eta\right)$$

$$=\left(f_0(t,\varepsilon)u_1+\int_0^\infty K(t,\tau,\varepsilon)\,f(\tau)\,d\tau,\eta\right)-\left(A(\varepsilon)w_\varepsilon(t),\eta\right),$$

for each $\eta\in D(A(\varepsilon))$. Thus

$$\left(w_\varepsilon'(t)+Aw_\varepsilon(t)-F_0(t,\varepsilon),\eta\right)=0,\quad \forall\eta\in D(A(\varepsilon)),\ a.e.\ t>0.$$

Let us observe that $F_0(t,\varepsilon)\in L^\infty_{c_1}(0,\infty;H)$ and from (v) in Lemma 4, we conclude that $w_\varepsilon'\in L^\infty_{c_1}(0,\infty;H)$ (with some $c_1>0$), which implies that $A(\varepsilon)w_\varepsilon\in L^\infty_{c_1}(0,\infty;H)$. Since $\overline{D(A)}=H$, it follows that $w_\varepsilon(t)$ satisfies the first equation in (5.1) a.e. $t>0$.

As the initial condition is a simple consequence of (iv) and (vii) in Lemma 4, the proof is complete. □

6 The limit of the solutions of the problem (P_ε) as $\varepsilon\longrightarrow 0$

In this section we will study the behavior of solutions to the problem (P_ε) as $\varepsilon\to 0$.

Theorem 12. *Let $T > 0$ and $p \in (1, \infty]$. Let us assume that the operators A_0 and A_1 satisfy* (**H1**) *and* (**H2**). *If*

$$u_0, u_{0\,\varepsilon} \in D(A_0), \quad u_{1\,\varepsilon} \in H, \quad f, f_\varepsilon \in W^{1,p}(0,T;H),$$

then there exist $\varepsilon_0 = \varepsilon_0(\omega_0, \omega_1) \in (0,1)$ and $C = C(T, p, \omega_0, \omega_1) > 0$ such that

$$\|u_\varepsilon - v\|_{C([0,T];H)}$$
$$\leq C \left(\mathcal{M}_\varepsilon \, \varepsilon^\beta + |u_{0\,\varepsilon} - u_0| + \|f_\varepsilon - f\|_{L^p(0,T;H)} \right), \tag{6.1}$$

for all $\varepsilon \in (0, \varepsilon_0]$, where u_ε and v are the strong solutions of problems (P_ε) and (P_0) respectively,

$$\beta = \min\{1/4, (p-1)/2p\}$$

and

$$\mathcal{M}_\varepsilon = \left| A_0^{1/2} u_{0\,\varepsilon} \right| + |u_{1\,\varepsilon}| + \|f_\varepsilon\|_{W^{1,p}(0,T;H)}.$$

If, in addition, $u_{1\,\varepsilon} \in D\left(A_0^{1/2}\right)$, then, for each $\varepsilon \in (0, \varepsilon_0]$, we have

$$\|u_\varepsilon - v\|_{C([0,T];H)}$$
$$\leq C \left(\mathcal{M}_{1\,\varepsilon} \, \varepsilon^{(p-1)/2p} + |u_{0\,\varepsilon} - u_0| + \|f_\varepsilon - f\|_{L^p(0,T;H)} \right), \tag{6.2}$$

and

$$\|A_0^{1/2} u_\varepsilon - A_0^{1/2} v\|_{L^2(0,\,T;\,H)}$$
$$\leq C \left(\mathcal{M}_{1\,\varepsilon} \, \varepsilon^\beta + |u_{0\,\varepsilon} - u_0| + \|f_\varepsilon - f\|_{L^p(0,T;H)} \right), \tag{6.3}$$

where $\beta = \min\{1/4, (p-1)/2p\}$ and

$$\mathcal{M}_{1\,\varepsilon} = \left| A_0^{1/2} u_{1\,\varepsilon} \right| + |A_0 u_{0\,\varepsilon}| + |A_1 u_{0\,\varepsilon}| + \|f_\varepsilon\|_{W^{1,p}(0,\,T;\,H)}.$$

Proof. From (**H1**) and (**H2**), it follows that there exists $\gamma = 3\omega_1 > 0$ such that

$$|(A_1 u, v)| \leq |((A_1 + \omega_1 \, A_0) u, v)| + \omega_1 \, |A_0^{1/2} u| \, |A_0^{1/2} v|$$
$$\leq ((A_1 + \omega_1 \, A_0) u, u)^{1/2} \, ((A_1 + \omega_1 \, A_0) v, v)^{1/2} + \omega_1 \, |A_0^{1/2} u| \, |A_0^{1/2} v|$$
$$\leq (2\omega_1 \, (A_0 u, u))^{1/2} \, (2\omega_1 \, (A_0 v, v))^{1/2}$$

$$+\omega_1 \left|A_0^{1/2}u\right| \left|A_0^{1/2}v\right| \leq \gamma \left|A_0^{1/2}u\right| \left|A_0^{1/2}v\right|, \quad \forall u,v \in D(A_0). \tag{6.4}$$

If $f_\varepsilon \in W^{l,p}(0,T;H)$ with $p \in (1,\infty]$ and $l \in \mathbb{N}^\star$, then, due to Theorems 1 and 2, we have that $f_\varepsilon \in C([0,T];H)$ and there exists an extension $\tilde{f}_\varepsilon \in W^{l,p}(0,\infty;H)$ such that

$$\|\tilde{f}_\varepsilon\|_{C([0,\infty);H)} + \|\tilde{f}_\varepsilon\|_{W^{l,p}(0,\infty;H)} \leq C(T,p,l)\, \|f_\varepsilon\|_{W^{l,p}(0,T;H)}. \tag{6.5}$$

Let us denote by $\tilde{u}_\varepsilon$ the unique strong solution to the problem (P_ε) and by $\tilde{v}$ the unique strong solution to the problem (P_0), defined on $(0,\infty)$ instead of $(0,T)$, and f_ε by $\tilde{f}_\varepsilon$. From Theorem 10, we have

$$\begin{cases} \tilde{u}_\varepsilon \in W^{2,\infty}(0,T;H),\ A^{1/2}(\varepsilon)\tilde{u}_\varepsilon \in W^{1,\infty}(0,T;H), \\ A(\varepsilon)\tilde{u}_\varepsilon \in L^\infty(0,T;H),\ \forall T \in (0,\infty). \end{cases}$$

From Lemma 2 and (6.4), it follows that

$$\begin{cases} \tilde{u}_\varepsilon \in W^{2,\infty}(0,\infty;H),\ A_0^{1/2}\tilde{u}_\varepsilon \in W^{1,2}(0,\infty;H), \\ A(\varepsilon)\tilde{u}_\varepsilon \in L^\infty(0,\infty;H). \end{cases}$$

Moreover, due to the same lemma and to (6.4) and (6.5), we get

$$\|A_0^{1/2}\tilde{u}_\varepsilon\|_{C([0,t];\,H)} + \|\tilde{u}'_\varepsilon\|_{L^2(0,t;\,H)} \leq C\,\mathcal{M}_\varepsilon, \forall t \geq 0,\ \forall \varepsilon \in (0,\varepsilon_0]. \tag{6.6}$$

If in addition, $u_{1\,\varepsilon} \in D\left(A_0^{1/2}\right)$, then

$$\|\tilde{u}'_\varepsilon\|_{C([0,t];\,H)} + \|A_0^{1/2}\,\tilde{u}'_\varepsilon\|_{L^2(0,t;\,H)} \leq C\,\mathcal{M}_{1\,\varepsilon}, \tag{6.7}$$

for all $t \in [0,T]$ and all $\varepsilon \in (0,\varepsilon_0]$.

Proof of (6.1). According to Theorem 4, the function

$$w_\varepsilon(t) = \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{u}_\varepsilon(\tau)\,d\tau,$$

is the strong solution to the problem

$$\begin{cases} w'_\varepsilon(t) + A(\varepsilon)w_\varepsilon(t) = F(t,\varepsilon), \quad t > 0, \quad \hat{\i}\text{n} \quad H, \\ w_\varepsilon(0) = w_0, \end{cases}$$

for $0 < \varepsilon \leq \varepsilon_0$, where

$$\begin{cases} F(t,\varepsilon) = f_0(t,\varepsilon)\, u_{1\,\varepsilon} + \displaystyle\int_0^\infty K(t,\tau,\varepsilon)\, \tilde{f}_\varepsilon(\tau)\, d\tau, \\[2ex] f_0(t,\varepsilon) = \dfrac{1}{\sqrt{\pi}} \left[2 \exp\left\{ \dfrac{3t}{4\varepsilon} \right\} \lambda \left(\sqrt{\dfrac{t}{\varepsilon}} \right) - \lambda \left(\dfrac{1}{2} \sqrt{\dfrac{t}{\varepsilon}} \right) \right], \\[2ex] w_0 = \displaystyle\int_0^\infty e^{-\tau}\, \tilde{u}_\varepsilon(2\varepsilon\tau) d\tau. \end{cases}$$

Using Hölder's inequality, (vi), $(viii)$, (ix) (x) in Lemma 4, and (6.6), we obtain

$$\begin{aligned} \|\tilde{u}_\varepsilon(t) - w_\varepsilon(t)\|_H &= \left\| \tilde{u}_\varepsilon(t) - \int_0^\infty K(t,\tau,\varepsilon)\, \tilde{u}_\varepsilon(\tau)\, d\tau \right\|_H \\ &\leq \int_0^\infty K(t,\tau,\varepsilon)\, \|\tilde{u}_\varepsilon(t) - \tilde{u}_\varepsilon(\tau)\|_H \, d\tau \\ &\leq \int_0^\infty K(t,\tau,\varepsilon) \left\| \int_t^\tau \|\tilde{u}'_\varepsilon(s)\|_H \, ds \right\| d\tau \\ \leq \|\widetilde{u}'_\varepsilon\|_{L^2(0,\,\infty;\,H)} & \int_0^\infty K(t,\tau,\varepsilon)\, |t-\tau|^{1/2}\, d\tau \leq C\, \mathcal{M}_\varepsilon\, \varepsilon^{1/4}, \end{aligned}$$

for all $t \in [0,T]$ and all $\varepsilon \in (0,\varepsilon_0]$. It then follows

$$\|\tilde{u}_\varepsilon - w_\varepsilon\|_{C([0,\,T];\,H)} \leq C\, \mathcal{M}_\varepsilon\, \varepsilon^{1/4}, \ \forall \varepsilon \in (0,\varepsilon_0]. \tag{6.8}$$

Let us denote by $R(t,\varepsilon) = \tilde{v}(t) - w_\varepsilon(t)$ which clearly is the strong solution of the problem

$$\begin{cases} R'(t,\varepsilon) + A_0 R(t,\varepsilon) = \varepsilon\, A_1 w_\varepsilon(t) + \mathcal{F}(t,\varepsilon), \quad t > 0, \\ R(0,\varepsilon) = R_0, \end{cases} \tag{6.9}$$

where $R_0 = u_0 - w_0$ and

$$\mathcal{F}(t,\varepsilon) = \tilde{f}(t) - \int_0^\infty K(t,\tau,\varepsilon) \tilde{f}_\varepsilon(\tau)\, d\tau - f_0(t,\varepsilon)\, u_{1\,\varepsilon}. \tag{6.10}$$

Taking the inner product by R in the equation in (6.9) and then integrating, we obtain

$$|R(t,\varepsilon)|^2 + 2 \int_0^t \left| A_0^{1/2} R(s,\varepsilon) \right|^2 ds$$

$$= |R_0|^2 + 2\int_0^t |\mathcal{F}(s,\varepsilon)|\,|R(s,\varepsilon)|\,ds + 2\,\varepsilon\int_0^t (A_1 w_\varepsilon(s), R(s,\varepsilon))\,ds,$$

for all $t \geq 0$. Using (6.4), from the last equality, we get

$$|R(t,\varepsilon)|^2 + \int_0^t \left|A_0^{1/2} R(s,\varepsilon)\right|^2 ds \leq |R_0|^2$$

$$+2\int_0^t |\mathcal{F}(s,\varepsilon)|\,|R(s,\varepsilon)|\,ds + \gamma^2\,\varepsilon^2\int_0^t \left|A_0^{1/2} w_\varepsilon(s)\right|^2 ds, \tag{6.11}$$

for all $t \geq 0$. Applying Lemma 1 to (6.11), we obtain

$$|R(t,\varepsilon)| + \left(\int_0^t \left|A_0^{1/2} R(s,\varepsilon)\right|^2 ds\right)^{1/2} \leq |R_0|$$

$$+\int_0^t |\mathcal{F}(s,\varepsilon)|\,ds + \gamma\,\varepsilon\left(\int_0^t \left|A_0^{1/2} w_\varepsilon(s)\right|^2 ds\right)^{1/2},\ \forall t \geq 0. \tag{6.12}$$

From (6.6), we deduce

$$|R_0| \leq |u_{0\,\varepsilon} - u_0| + \int_0^\infty e^{-s}\,|\tilde{u}_\varepsilon(2\varepsilon s) - u_{0\,\varepsilon}|\,ds \leq |u_{0\,\varepsilon} - u_0|$$

$$+\int_0^\infty e^{-s}\int_0^{2\varepsilon s} \left|\tilde{u}_\varepsilon'(\tau)\right| d\tau\,ds \leq |u_{0\,\varepsilon} - u_0| + C\,\mathcal{M}_\varepsilon\,\varepsilon^{1/2}, \tag{6.13}$$

for all $\varepsilon \in (0, \varepsilon_0]$. Using (x) in Lemma 4 and (6.5), we get

$$\left|\tilde{f}(t) - \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{f}_\varepsilon(\tau)\,d\tau\right|$$

$$\leq \left|\tilde{f}(t) - \tilde{f}_\varepsilon(t)\right| + \int_0^\infty K(t,\tau,\varepsilon)\,\left|\tilde{f}_\varepsilon(t) - \tilde{f}_\varepsilon(\tau)\right| d\tau \leq \left|\tilde{f}(t) - \tilde{f}_\varepsilon(t)\right|$$

$$+C(T,p)\,\|f_\varepsilon'\|_{L^p(0,T;H)}\,\varepsilon^{(p-1)/2p},\ \forall t \geq 0,\ \forall \varepsilon \in (0, \varepsilon_0]. \tag{6.14}$$

As $e^\tau \lambda(\sqrt{\tau}) \leq C$ for all $\tau \geq 0$, we have

$$\int_0^t \exp\left\{\frac{3\tau}{4\varepsilon}\right\}\lambda\left(\sqrt{\frac{\tau}{\varepsilon}}\right) d\tau \leq C\,\varepsilon\int_0^{\frac{t}{\varepsilon}} e^{-\tau/4}\,d\tau \leq C\,\varepsilon\int_0^\infty e^{-\tau/4}\,d\tau \leq C\varepsilon$$

and

$$\int_0^t \lambda\left(\frac{1}{2}\sqrt{\frac{\tau}{\varepsilon}}\right) d\tau \leq \varepsilon \int_0^\infty \lambda\left(\frac{1}{2}\sqrt{\tau}\right) d\tau \leq C\,\varepsilon,$$

for all $t \geq 0$. Hence

$$\left|\int_0^t f_0(\tau,\varepsilon)\, d\tau\, u_{1\,\varepsilon}\right| \leq C\,\varepsilon\,|u_{1\,\varepsilon}|, \quad \forall t \geq 0. \tag{6.15}$$

Using (6.10), from (6.14) and (6.15), we get

$$\int_0^t |\mathcal{F}(s,\varepsilon)|\, ds \leq C\left(\mathcal{M}_\varepsilon\, \varepsilon^{(p-1)/2p} + \|f_\varepsilon - f\|_{L^p(0,T;H)}\right), \tag{6.16}$$

for every $t \in [0,T]$ and every $\varepsilon \in (0,\varepsilon_0]$.

As $A_0^{1/2}$ is closed, using (6.6), we obtain

$$\left|A_0^{1/2} w_\varepsilon(t)\right| \leq \int_0^\infty K(t,\tau,\varepsilon)\left|A_0^{1/2}\tilde{u}_\varepsilon(\tau)\right| d\tau \leq C\,\mathcal{M}_\varepsilon, \tag{6.17}$$

for every $t \in [0,T]$ and every $\varepsilon \in (0,\varepsilon_0]$.

Thanks to (6.13), (6.16) and (6.17), from (6.12), it follows that

$$\|R\|_{C([0,T];\,H)} + \left\|A_0^{1/2} R\right\|_{L^2(0,T;\,H)}$$

$$\leq C\left(\mathcal{M}_\varepsilon\, \varepsilon^{(p-1)/2\,p} + |u_{0\,\varepsilon} - u_0| + \|f_\varepsilon - f\|_{L^p(0,T;H)}\right), \tag{6.18}$$

for every $\varepsilon \in (0,\varepsilon_0]$. Finally, from (6.8) and (6.18), it follows that

$$\|\tilde{u}_\varepsilon - \tilde{v}\|_{C([0,T];H)} \leq \|\tilde{u}_\varepsilon - w_\varepsilon\|_{C([0,T];H)} + \|R\|_{C([0,T];H)}$$

$$\leq C\left(\mathcal{M}_\varepsilon\, \varepsilon^{\beta} + |u_{0\,\varepsilon} - u_0| + \|f_\varepsilon - f\|_{L^p(0,T;H)}\right), \tag{6.19}$$

for every $\varepsilon \in (0,\varepsilon_0]$. According to Theorems 9 and 10, we have that $u_\varepsilon(t) = \tilde{u}_\varepsilon(t)$ and $\tilde{v}(t) = v(t)$ for $t \in [0,T]$. Therefore, from (6.19), we deduce (6.1).

Proof of (6.2). If $u_{1\,\varepsilon} \in D\left(A_0^{1/2}\right)$, then, using (vi), $(viii)$, (x) in Lemma 4 and (6.7), we get

$$\|\tilde{u}_\varepsilon(t) - w_\varepsilon(t)\|_H = \left\|\tilde{u}_\varepsilon(t) - \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{u}_\varepsilon(\tau)\, d\tau\right\|_H$$

$$\leq \int_0^\infty K(t,\tau,\varepsilon)\, \|\tilde{u}_\varepsilon(t) - \tilde{u}_\varepsilon(\tau)\|_H \, d\tau$$

$$\leq \int_0^\infty K(t,\tau,\varepsilon) \left| \int_t^\tau \|\tilde{u}'_\varepsilon(s)\|_H \, ds \right| d\tau$$

$$\leq \|\tilde{u}'_\varepsilon\|_{C([0,\infty);\,H)} \int_0^\infty K(t,\tau,\varepsilon)\, |t-\tau| \, d\tau \leq C\, \mathcal{M}_{1\,\varepsilon}\, \varepsilon^{1/2},$$

for every $t \in [0,T]$ and every $\varepsilon \in (0,\varepsilon_0]$. This yields

$$\|\tilde{u}_\varepsilon - w_\varepsilon\|_{C([0,T];\,H)} \leq C\, \mathcal{M}_{1\,\varepsilon}\, \varepsilon^{1/2}, \quad \forall \varepsilon \in (0,\varepsilon_0].$$

As, for $p \in (1,\infty]$, we have $(p-1)/2p \leq 1/2$, the proof of (6.2) follows in the same way as the proof of (6.1).

Proof of (6.3). Using (vi), $(viii)$, (x) in Lemma 4 and (6.7), we get

$$\left| A_0^{1/2} (\tilde{u}_\varepsilon(t) - w_\varepsilon(t) \right| \leq \int_0^\infty K(t,\tau,\varepsilon) \left| A_0^{1/2} (\tilde{u}_\varepsilon(t) - \tilde{u}_\varepsilon(\tau) \right| d\tau$$

$$\leq \int_0^\infty K(t,\tau,\varepsilon) \left| \int_\tau^t \left\| A_0^{1/2} \tilde{u}'_\varepsilon(s) \right\|_H ds \right| d\tau$$

$$\leq \int_0^\infty K(t,\tau,\varepsilon)\, |t-\tau|^{1/2} \left| \int_\tau^t \left\| A_0^{1/2} \tilde{u}'_\varepsilon(s) \right\|_H^2 ds \right|^{1/2} d\tau$$

$$\leq C\, \mathcal{M}_{1\,\varepsilon}\, \varepsilon^{1/4}, \quad \forall t \geq 0, \quad \forall \varepsilon \in (0,\varepsilon_0].$$

Hence $u_\varepsilon(t) = \tilde{u}_\varepsilon(t)$, for $t \in [0,T]$, and therefore

$$\left\| A_0^{1/2} (u_\varepsilon - w_\varepsilon) \right\|_{C([0,T];\,H)} \leq C\, \mathcal{M}_{1\,\varepsilon}\, \varepsilon^{1/4}, \quad \forall \varepsilon \in (0,\varepsilon_0]. \tag{6.20}$$

From (6.18), it follows that

$$\left\| A_0^{1/2} R \right\|_{L^2(0,T)\,;H)} \leq C \Big(\mathcal{M}_\varepsilon\, \varepsilon^{(p-1)/2\,p}$$

$$+ |u_{0\,\varepsilon} - u_0| + C\, \|f_\varepsilon - f\|_{L^p(0,T;H)} \Big), \quad \forall \varepsilon \in (0,\varepsilon_0]. \tag{6.21}$$

Finally, (6.20) and (6.21) imply (6.3) and this completes the proof. □

Remark 6.1. If, in the conditions of Theorem 12, we assume that $f, f_\varepsilon \in W^{1,\infty}(0,T,H)$, then (6.1), (6.2) and (6.3) take the form

$$\|u_\varepsilon - v\|_{C([0,T];H)} \leq C\,\left(\mathcal{M}_\varepsilon\,\varepsilon^{1/4} + |u_{0\,\varepsilon} - u_0| + \|f_\varepsilon - f\|_{L^\infty(0,T;H)}\right),$$

where

$$\mathcal{M}_\varepsilon = \left|A_0^{1/2} u_{0\,\varepsilon}\right| + |u_{1\,\varepsilon}| + \|f_\varepsilon\|_{W^{1,\infty}(0,T;H)},$$

$$\|u_\varepsilon - v\|_{C([0,T];H)} \leq C\,\left(\mathcal{M}_{1\,\varepsilon}\,\varepsilon^{1/2} + |u_{0\,\varepsilon} - u_0| + \|f_\varepsilon - f\|_{L^\infty(0,T;H)}\right),$$

and

$$\|A_0^{1/2} u_\varepsilon - A_0^{1/2} v\|_{L^2(0,T;\,H)}$$
$$\leq C\,\left(\mathcal{M}_{1\,\varepsilon}\,\varepsilon^{1/4} + |u_{0\,\varepsilon} - u_0| + |f_\varepsilon - f\|_{L^\infty(0,T;H)}\right),$$

with

$$\mathcal{M}_{1\,\varepsilon} = \left|A_0^{1/2} u_{1\,\varepsilon}\right| + |A_0 u_{0\,\varepsilon}| + |A_1 u_{0\,\varepsilon}| + \|f\|_{W^{1,\infty}(0,T;\,H)}.$$

for all $\varepsilon \in (0, \varepsilon_0]$.

Theorem 13. *Let $T > 0$ and $p \in (1, \infty]$. Suppose that the operators A_0 and A_1 satisfy* **(H1)** *and* **(H2)**. *If*

$$u_0, u_{0\varepsilon}, A_0 u_0, A_1 u_{0\varepsilon}, A_0 u_{0\varepsilon}, u_{1\varepsilon}, f(0), f_\varepsilon(0) \in D(A_0)$$

and

$$f, f_\varepsilon \in W^{2,p}(0,T;H),$$

then there exist $\varepsilon_0 = \varepsilon_0(\omega_0, \omega_1) \in (0,1)$ and $C = C(T, p, \omega_0, \omega_1) > 0$ such that

$$\left\|u_\varepsilon' - v' + \alpha_\varepsilon\, e^{-\frac{t}{\varepsilon}}\right\|_{C([0,\,T];\,H)} \leq C\,\left(\mathcal{M}_{2\,\varepsilon}\,\varepsilon^{(p-1)/2p} + D_\varepsilon\right), \tag{6.22}$$

$$\left\|A_0^{1/2}\left(u_\varepsilon' - v' + \alpha_\varepsilon\, e^{-\frac{t}{\varepsilon}}\right)\right\|_{L^2(0,\,T;\,H)} \leq C\,\left(\mathcal{M}_{2\,\varepsilon}\,\varepsilon^{\beta} + D_\varepsilon\right), \tag{6.23}$$

where v and u_ε are the strong solutions of the problems (P_0) and (P_ε) respectively, $\beta = \min\{1/4, (p-1)/2p\}$, $\alpha_\varepsilon = f_\varepsilon(0) - u_{1\,\varepsilon} - A(\varepsilon) u_{0\,\varepsilon}$,

$$D_\varepsilon = \|f_\varepsilon - f\|_{W^{1,\,p}(0,\,T;\,H)} + |A_0(u_{0\varepsilon} - u_0)|\,,$$

$$\mathcal{M}_{2\,\varepsilon} = |A(\varepsilon) u_{1\,\varepsilon}| + \|f_\varepsilon\|_{W^{2,p}(0,\,T;\,H)} + |A_1 u_{0\,\varepsilon}| + |A(\varepsilon)\alpha_\varepsilon|\,.$$

Proof. Within this proof, for $\tilde{u}_\varepsilon$, $\tilde{v}$, $\tilde{f}$ and $\tilde{f}_\varepsilon$, we will use the same notations as in the proof of Theorem 12.

Let us denote by

$$\tilde{z}_\varepsilon(t) = \tilde{u}'_\varepsilon(t) + \alpha_\varepsilon e^{-\frac{t}{\varepsilon}}, \quad \alpha_\varepsilon = f_\varepsilon(0) - u_{1\,\varepsilon} - A(\varepsilon)u_{0\,\varepsilon}.$$

If $u_{1\,\varepsilon} + \alpha_\varepsilon \in D\,(A_0)$ and $f \in W^{2,1}(0,T;H)$, then, due to (6.4) and (6.5), $u_{1\,\varepsilon} + \alpha_\varepsilon \in D\,(A(\varepsilon))$ and $\tilde{f} \in W^{2,1}(0,\infty;H)$. According to Theorem 10, $\tilde{z}_\varepsilon$ is the strong solution in H to the problem

$$\begin{cases} \varepsilon \tilde{z}''_\varepsilon(t) + \tilde{z}'_\varepsilon(t) + A(\varepsilon)\tilde{z}_\varepsilon(t) = \tilde{\mathcal{F}}(t,\varepsilon), \quad t > 0, \\ \tilde{z}_\varepsilon(0) = f_\varepsilon(0) - A(\varepsilon)u_{0\,\varepsilon}, \quad \tilde{z}'_\varepsilon(0) = 0, \end{cases}$$

where

$$\tilde{\mathcal{F}}(t,\varepsilon) = \tilde{f}'_\varepsilon(t) + e^{-t/\varepsilon}\, A(\varepsilon)\alpha_\varepsilon.$$

From Lemma 3 and (6.4), it follows that

$$\tilde{z}_\varepsilon \in W^{2,\infty}(0,\infty;H),\ A_0^{1/2}\tilde{z}_\varepsilon \in W^{1,2}(0,\infty;H),\ A(\varepsilon)\tilde{z}_\varepsilon \in L^\infty(0,\infty;H).$$

Moreover, from the same lemma, (6.4) and (6.5), we get

$$\begin{aligned} &\|A_0^{1/2}\tilde{z}_\varepsilon\|_{C([0,\,\infty];\,H)} + \|\tilde{z}'_\varepsilon\|_{C([0,\,\infty);\,H)} \\ &+ \left\|A_0^{1/2}\tilde{z}'_\varepsilon\right\|_{L^2(0,\,\infty;\,H)} \le C\,\mathcal{M}_{2\,\varepsilon}, \quad \forall \varepsilon \in (0,\varepsilon_0]. \end{aligned} \tag{6.24}$$

According to Theorem 4, the function

$$w_{1\,\varepsilon}(t) = \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{z}_\varepsilon(\tau)d\tau$$

is a strong solution of

$$\begin{cases} w'_{1\,\varepsilon}(t) + A(\varepsilon)w_{1\varepsilon}(t) = \mathcal{F}_1(t,\varepsilon), \quad t > 0, \\ w_{1\,\varepsilon}(0) = \int_0^\infty e^{-\tau}\,\tilde{z}_\varepsilon(2\,\varepsilon\,\tau)d\tau, \end{cases}$$

where

$$\mathcal{F}_1(t,\varepsilon) = \int_0^\infty K(t,\tau,\varepsilon)\left(\tilde{f}'_\varepsilon(\tau) + e^{-\frac{\tau}{\varepsilon}}\,A(\varepsilon)\alpha_\varepsilon\right)\,d\tau.$$

Moreover,

$$\left|A_0^{1/2}w_{1\,\varepsilon}(t)\right| \le \int_0^\infty K(t,\tau,\varepsilon)\,\left|A_0^{1/2}\tilde{z}_\varepsilon(\tau)\right|\,d\tau \le C\mathcal{M}_{2\,\varepsilon}, \tag{6.25}$$

for all $t \geq 0$. Using (vi), $(viii)$, (x) in Lemma 4 and (6.24), we get

$$\|\tilde{z}_\varepsilon(t) - w_{1\,\varepsilon}(t)|_H = \left\| \tilde{z}_\varepsilon(t) - \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{z}_\varepsilon(\tau)\,d\tau \right\|_H$$

$$\leq \int_0^\infty K(t,\tau,\varepsilon)\,\|\tilde{z}_\varepsilon(t) - \tilde{z}_\varepsilon(\tau)\|_H\;d\tau$$

$$\leq \int_0^\infty K(t,\tau,\varepsilon) \left| \int_t^\tau \|\tilde{z}'_\varepsilon(s)\|_H\,ds \right| d\tau$$

$$\leq \|\tilde{z}'_\varepsilon\|_{C([0,\infty);\,H)} \int_0^\infty K(t,\tau,\varepsilon)\,|t-\tau|\,d\tau \leq C\,\mathcal{M}_{2\,\varepsilon}\,\varepsilon^{1/2},$$

for all $t \in [0,T]$ and all $\varepsilon \in (0,\varepsilon_0]$,

$$\left\| A_0^{1/2}\left(\tilde{z}_\varepsilon(t) - w_{1\,\varepsilon}(t)\right) \right\|_H$$

$$= \left\| A_0^{1/2}\tilde{z}_\varepsilon(t) - \int_0^\infty K(t,\tau,\varepsilon)\,A_0^{1/2}\tilde{z}_\varepsilon(\tau)\,d\tau \right\|_H$$

$$\leq \int_0^\infty K(t,\tau,\varepsilon) \left\| A_0^{1/2}\left(\tilde{z}_\varepsilon(t) - \tilde{z}_\varepsilon(\tau)\right) \right\|_H d\tau$$

$$\leq \int_0^\infty K(t,\tau,\varepsilon) \left| \int_t^\tau \|A_0^{1/2}\tilde{z}'_\varepsilon(s)\|_H\,ds \right| d\tau$$

$$\leq \|A_0^{1/2}\tilde{z}'_\varepsilon\|_{L^2(0,\infty;\,H)} \int_0^\infty K(t,\tau,\varepsilon)\,|t-\tau|^{1/2}\,d\tau \leq C\,\mathcal{M}_{2\,\varepsilon}\,\varepsilon^{1/4},$$

for all $t \in [0,T]$ and all $\varepsilon \in (0,\varepsilon_0]$. It then follows that

$$\|\tilde{z}_\varepsilon - w_{1\,\varepsilon}\|_{C([0,T];\,H)} \leq C\,\mathcal{M}_{2\,\varepsilon}\,\varepsilon^{1/2}, \quad \forall \varepsilon \in (0,\varepsilon_0], \tag{6.26}$$

$$\left\| A_0^{1/2}\left(\tilde{z}_\varepsilon - w_{1\,\varepsilon}\right) \right\|_{L^2(0,T;\,H)} \leq C\,\mathcal{M}_{2\,\varepsilon}\,\varepsilon^{1/4}, \quad \forall \varepsilon \in (0,\varepsilon_0]. \tag{6.27}$$

Let $R_1(t,\varepsilon) = \tilde{v}'(t) - w_{1\varepsilon}(t)$. If $f(0) - A_0 u_0 \in D(A_0)$ and $f \in W^{2,1}(0,T;H)$, then, according to Theorem 3.1, $\tilde{v} \in W^{2,\infty}(0,\infty;H)$, $A_0^{1/2}\tilde{v} \in W^{1,2}(0,\infty;H)$. Therefore $R_1 \in W^{1,\infty}(0,\infty;H)$ and

$$\begin{cases} R_1'(t,\varepsilon) + A_0 R_1(t,\varepsilon) = \tilde{f}'(t) - \mathcal{F}_1(t,\varepsilon) + \varepsilon A_1 w_{1\,\varepsilon}(t), \quad t > 0, \\ R_1(0,\varepsilon) = f(0) - A_0 u_0 - w_{1\,\varepsilon}(0). \end{cases}$$

Similarly to (6.12), we deduce

$$|R_1(t,\varepsilon)| + \left(\int_0^t \left| A_0^{1/2} R_1(s,\varepsilon) \right|^2 ds \right)^{1/2} \leq |R_1(0,\varepsilon)|$$

$$+ \int_0^t \left| \tilde{f}'(s) - \mathcal{F}_1(s,\varepsilon) \right| ds + \gamma\, \varepsilon \left(\int_0^t \left| A_0^{1/2} w_{1\,\varepsilon}(s) \right|^2 ds \right)^{1/2}, \qquad (6.28)$$

for all $t \geq 0$. Using (6.24), we get:

$$|R_1(0,\varepsilon)| \leq |f(0) - f_\varepsilon(0)| + |A_0(u_0 - u_{0\,\varepsilon})| + \varepsilon\, |A_1 u_{0\,\varepsilon}|$$

$$+ \int_0^\infty e^{-s} \, |\tilde{z}_\varepsilon(2\,\varepsilon\, s) - \tilde{z}_\varepsilon(0)| \, ds$$

$$\leq C\, D_\varepsilon + \varepsilon\, |A_1 u_{0\,\varepsilon}| + \mathcal{M}_{2\,\varepsilon}\, \varepsilon \leq C\, D_\varepsilon + \mathcal{M}_{2\,\varepsilon}\, \varepsilon, \quad \forall \varepsilon \in (0, \varepsilon_0]. \qquad (6.29)$$

As

$$\left| \tilde{f}'(s) - \mathcal{F}_1(s,\varepsilon) \right| \leq \left| \tilde{f}'(s) - \tilde{f}'_\varepsilon(s) \right| + \int_0^\infty K(s,\tau,\varepsilon) \left| \tilde{f}'_\varepsilon(\tau) - \tilde{f}'_\varepsilon(s) \right| d\tau$$

$$+ \int_0^\infty K(s,\tau,\varepsilon)\, e^{-\frac{\tau}{\varepsilon}} \, d\tau \; | A(\varepsilon) \alpha_\varepsilon |,$$

then, due to (ix), (xi) in Lemma 4, we obtain

$$\int_0^t \left| \tilde{f}'(s) - \mathcal{F}_1(s,\varepsilon) \right| ds \leq C \left(D_\varepsilon + \mathcal{M}_{2\,\varepsilon}\, \varepsilon^{(p-1)/2p} + |\, A(\varepsilon)\alpha_\varepsilon |\; \varepsilon \right)$$

$$\leq C \left(D_\varepsilon + \mathcal{M}_{2\,\varepsilon}\, \varepsilon^{(p-1)/2p} \right), \quad \forall t \in [0,T], \quad \forall \varepsilon \in (0,\varepsilon_0]. \qquad (6.30)$$

Using (6.25), (6.29), (6.30), from (6.28) we get

$$\|R_1\|_{C([0,T];\,H)} + \left\| A_0^{1/2} R_1 \right\|_{L^2(0,T;\,H)} \leq C \left(D_\varepsilon + \mathcal{M}_{2\,\varepsilon}\, \varepsilon^{(p-1)/2p} \right), \qquad (6.31)$$

for all $\varepsilon \in (0,1]$.

Finally, as (6.26), (6.27) and (6.31) imply (6.22) and (6.23), the proof is complete. □

Remark 6.2. If, in the conditions of Theorem 13, we assume that $f, f_\varepsilon \in W^{2,\infty}(0,T,H)$, then (6.22) and (6.23) take the form

$$\left\| u'_\varepsilon - v' + \alpha_\varepsilon\, e^{-\frac{t}{\varepsilon}} \right\|_{C([0,T];\,H)} \leq C\,\left(\mathcal{M}_{2\,\varepsilon}\,\varepsilon^{1/2} + D_\varepsilon\right),$$

$$\left\| A_0^{1/2}\left(u'_\varepsilon - v' + \alpha_\varepsilon\, e^{-\frac{t}{\varepsilon}}\right) \right|_{L^2(0,T;\,H)} \leq C\,\left(\mathcal{M}_{2\,\varepsilon}\,\varepsilon^{1/4} + D_\varepsilon\right),$$

$$D_\varepsilon = \| f_\varepsilon - f|_{W^{1,\infty}(0,T;\,H)} + |A_0(u_{0\varepsilon} - u_0)|\,,$$

$$\mathcal{M}_{2\,\varepsilon} = |A(\varepsilon)u_{1\,\varepsilon}| + \| f_\varepsilon|_{W^{2,\infty}(0,T;\,H)} + |A_1 u_{0\,\varepsilon}| + |A(\varepsilon)\alpha_\varepsilon|\,.$$

7 An Example

Let $\Omega \subset R^n$ be an open bounded set with C^1 boundary $\partial\Omega$. In the real Hilbert space $L^2(\Omega)$, with the usual inner product

$$(u,v) = \int_\Omega u(x)\,v(x)\,dx,$$

we consider the following Cauchy problem

$$\begin{cases} \varepsilon\partial_t^2\, u_\varepsilon(x,t) + \partial_t\, u_\varepsilon(x,t) + A_0 u_\varepsilon(x,t) + \varepsilon A_1 u_\varepsilon(x,t) = f(x,t), \\ x \in \Omega,\ t > 0, \\ u_\varepsilon(x,0) = u_{0\varepsilon}(x), \quad \partial_t\, u_\varepsilon(x,0) = u_{1\varepsilon}(x) \end{cases} \tag{7.1}$$

where $D(A_i) = H^2(\Omega) \cap H_0^1(\Omega)$, $i = 0,1$,

$$A_0 u(x) = -\sum_{i,j=1}^n \partial_{x_i}\left(a_{ij}(x)\partial_{x_j} u(x)\right) + a(x)u(x), \quad u \in D(A_0),$$

$$a_{ij} \in C^1(\overline{\Omega}),\ a \in C(\overline{\Omega}),\ a(x) \geq 0,\ a_{ij}(x) = a_{ji}(x), \quad x \in \overline{\Omega}, \tag{7.2}$$

and

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\,\xi_j \geq a_0\,|\xi|^2, \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^n, \quad a_0 > 0. \tag{7.3}$$

$$A_1 u(x) = -\sum_{i,j=1}^n \partial_{x_i}\left(b_{ij}(x)\partial_{x_j} u(x)\right) + b(x)u(x) + \int_\Omega K(x,y)u(y)dy,$$

for $u \in D(A_1)$,

$$K : \Omega \times \Omega \mapsto R, \quad K \in L^2(\Omega \times \Omega), \tag{7.4}$$

$$b_{ij} \in C^1(\overline{\Omega}),\ b \in C(\overline{\Omega}),\ b_{ij}(x) = b_{ji}(x), \quad x \in \overline{\Omega}, \tag{7.5}$$

$$|b(x)| \le b_1 a(x), \quad \left| \sum_{i,j=1}^{n} b_{ij}(x)\xi_i\, \xi_j \right| \le b_0 \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\, \xi_j \tag{7.6}$$

for $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^n$. Under the hypotheses (7.2)-(7.3), the operator A_0 is positive and self-adjoint with $D(A_0^{1/2}) = H_0^1(\Omega)$ and

$$\|A_0^{1/2} u\|^2_{L^2(\Omega)} = \int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}(x)\, \partial_{x_i} u(x)\, \partial_{x_j} u(x) + a(x) u^2(x) \right) dx,$$

for $u \in H_0^1(\Omega)$. If (7.5) holds, the operator A_1 is self-adjoint with

$$\|A_1^{1/2} u\|^2_{L^2(\Omega)} = \int_\Omega \left(\sum_{i,j=1}^{n} b_{ij}(x)\, \partial_{x_i} u(x)\, \partial_{x_j} u(x) + b(x) u^2(x) \right) dx$$

$$+ \int_\Omega \int_\Omega K(x,y) u(x) u(y) dy\, dx, \quad \forall u \in H_0^1(\Omega).$$

Moreover, (7.2)-(7.6) imply (**H2**) with

$$\omega_1 = \max\{b_0, b_1\} + \|K\|_{L_2(\Omega\times\Omega)}/\omega_0.$$

Let us now consider the unperturbed problem associated to (7.1)

$$\begin{cases} \partial_t\, v(x,t) + A_0 v(x,t) = f(x,t), \quad x \in \Omega,\ t > 0, \\ v(x,0) = u_0(x). \end{cases} \tag{7.7}$$

Using Theorem 12, we obtain:

Theorem 14. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with C^1 boundary $\partial\Omega$. Let $T > 0$ and $p \in (1, \infty]$. Let us assume that (7.2)-(7.6) are satisfied. If*

$$u_0, u_{0\,\varepsilon} \in H^2(\Omega) \cap H_0^1(\Omega), \quad u_{1\,\varepsilon} \in L^2(\Omega), \quad f, f_\varepsilon \in W^{1,p}(0,T; L^2(\Omega)),$$

then there exist $\varepsilon_0 = \varepsilon_0(\omega_0, \omega_1) \in (0,1)$ and $C = C(T, p, n, \omega_0, \omega_1) > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0]$, we have

$$\|u_\varepsilon - v\|_{C([0,T]; L^2(\Omega))}$$

$$\leq C \left(\widetilde{\mathcal{M}}_{\varepsilon}\, \varepsilon^{\beta} + |u_{0\,\varepsilon} - u_0| + \|f_\varepsilon - f\|_{L^p(0,T;L^2(\Omega))} \right),$$

where u_ε and v are the strong solutions of (7.1) *and* (7.7) *respectively,*

$$\beta = \min\{1/4, (p-1)/2p\}$$

and

$$\widetilde{\mathcal{M}}_{\varepsilon} = \left| A_0^{1/2} u_{0\,\varepsilon} \right| + |u_{1\,\varepsilon}| + \|f_\varepsilon\|_{W^{1,p}(0,T;L^2(\Omega))}.$$

If, in addition, $u_{1\,\varepsilon} \in H_0^1(\Omega)$, then

$$\|u_\varepsilon - v\|_{C([0,T];L^2(\Omega))}$$

$$\leq C \left(\widetilde{\mathcal{M}}_{1\,\varepsilon}\, \varepsilon^{(p-1)/2p} + |u_{0\,\varepsilon} - u_0| + \|f_\varepsilon - f\|_{L^p(0,T;L^2(\Omega))} \right),$$

for each $\varepsilon \in (0, \varepsilon_0]$, where $\beta = \min\{1/4, (p-1)/2p\}$ and

$$\widetilde{\mathcal{M}}_{1\,\varepsilon} = \left| A_0^{1/2} u_{1\,\varepsilon} \right| + |A_0 u_{0\,\varepsilon}| + |A_1 u_{0\,\varepsilon}| + \|f_\varepsilon\|_{W^{1,p}(0,\,T;\,L^2(\Omega))}.$$

Using Theorem 13, we deduce:

Theorem 15. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with C^1 boundary $\partial\Omega$. Let $T > 0$ and $p \in (1, \infty]$. Let us assume that* (7.2)-(7.6) *are satisfied. If*

$$u_0, u_{0\varepsilon}, A_0 u_0, A_1 u_{0\varepsilon}, A_0 u_{0\varepsilon}, u_{1\varepsilon}, f(0), f_\varepsilon(0) \in H^2(\Omega) \cap H_0^1(\Omega),$$

and

$$f, f_\varepsilon \in W^{2,p}(0,T;L^2(\Omega)),$$

then there exist $\varepsilon_0 = \varepsilon_0(\omega_0, \omega_1) \in (0,1)$ and $C = C(T, p, n, \omega_0, \omega_1) > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0]$, we have

$$\left\| u_\varepsilon' - v' + \alpha_\varepsilon\, e^{-\frac{t}{\varepsilon}} \right\|_{C([0,\,T];\,L^2(\Omega))} \leq C \left(\widetilde{\mathcal{M}}_{2\,\varepsilon}\, \varepsilon^{(p-1)/2p} + \widetilde{D}_\varepsilon \right),$$

where v and u_ε are the strong solutions of (7.1) *and* (7.7) *respectively,*
$\beta = \min\{1/4, (p-1)/2p\}$, $\alpha_\varepsilon = f_\varepsilon(0) - u_{1\,\varepsilon} - A(\varepsilon) u_{0\,\varepsilon}$,

$$\widetilde{D}_\varepsilon = \|f_\varepsilon - f\|_{W^{1,\,p}(0,\,T;\,H_0^1(\Omega))} + |A_0(u_{0\varepsilon} - u_0)|,$$

$$\widetilde{\mathcal{M}}_{2\,\varepsilon} = |A(\varepsilon) u_{1\,\varepsilon}| + \|f_\varepsilon\|_{W^{2,p}(0,\,T;\,H_0^1(\Omega))} + |A_1 u_{0\,\varepsilon}| + |A(\varepsilon)\alpha_\varepsilon|.$$

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