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ANALYSIS AND NUMERICAL APPROACH OF A PIEZOELECTRIC CONTACT PROBLEM*

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Abstract

We consider a mathematical model which describes the frictional contact between an electro-viscoelastic body and a conductive foundation. The contact is modelled with normal compliance and a version of Coulomb's law of dry friction, in which the stiffness and friction coefficients depend on the electric potential. We derive a variational formulation of the problem and, under a smallness assumption, we prove an existence and uniqueness result. The proof is based on arguments on evolutionary variational inequalities and fixed point. Then, we introduce the fully discretized problem and present numerical simulations in the study of a two-dimensional test problem which describe the process of contact in a microelectromechanical swich.

keywords: electro-viscoelastic material, normal compliance, Coulomb's law, variational inequality, weak solution, finite element method, numerical simulations.

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1 Introduction

Contact phenomena involving deformable bodies arise in industry and everyday life and play important roles in structural and mechanical systems. Owning to the complicated surface physics involved, they lead to new and nonstandard mathematical models. Considerable progress has been achieved recently in modelling and mathematical analysis of phenomena of contact and, as a result, a general Mathematical Theory of Contact Mechanics is currently emerging as a discipline on its own right. Its aim is to provide a sound, clear and rigorous background to the construction of models, their variational analysis as well as their numerical simulations, see [9, 16] for details.

Currently there is a considerable interest in contact problems involving piezoelectric materials, i.e. materials characterized by the coupling between the mechanical and electrical properties. This coupling, in a piezoelectric material, leads to the appearance of electric potential when mechanical stress is present and, conversely, mechanical stress is generated when electric potential is applied. The first effect is used in mechanical sensors, and the reverse effect is used in actuators, in engineering control equipments. Piezoelectric materials for which the mechanical properties are elastic are called electro-elastic materials and those for which the mechanical properties are viscoelastic are called electro-viscoelastic materials. General models for electro-elastic materials can be found in [6, 10, 14]. Frictional contact problems for electro-elastic or electro-viscoelastic materials were studied in [7, 12, 13, 17], under the assumption that the foundation is insulated. The results in [7, 12] concern mainly the numerical simulation of the problems while the results in [13, 17] concern the variational formulation of the problems and their unique weak solvability.

The study of mathematical models which describe the evolution of the piezoelectric body in frictional or frictionless contact with a conductive foundation is more recent see, for instance, [3, 4, 5, 11]. The problems studied in [3, 4] are frictionless and describe a dynamic and a quasistatic mechanical process for electro-viscoelastic materials, respectively. The problem studied in [5] is frictional and is modeled with normal compliance and a version of Coulomb's law o dry friction, in which the stiffness and friction coefficients depend on the electric potential; the material is assumed to be electro-elastic and the process is static; an existence and uniqueness result was obtained, a discrete scheme was considered, and numerical simulations were provided.

The problem studied in [11] is frictional, too, and is modeled with the standard normal compliance contact condition and the Coulomb's law of dry friction; the material is assumed to be electro-viscoelastic and the process is quasistatic; an existence and uniqueness result was obtained by using arguments of evolutionary variational inequalitiesd and fixed point.

The results in the present paper are related and parallel our previous results obtained in [5, 11]. Nevertheless, there are several major differences between these papers, that we describe in what follows. First, we recall that in [11] we used the standard normal compliance contact condition and the Coulomb's law of dry friction and, as a result, the mechanical and electrical unknowns are decoupled on the frictional contact conditions. Unlike the problem in [11], in the present paper the electric potential is involved in the frictional contact conditions too, which increase the degree of nonlinearity of the problem and requiers the use of new functionals and operators, different to those used in [11]. Moreover, unlike [11], in the present paper we deal with the numerical approach of the problem and provide numerical simulations. In the present paper we use the boundary conditions on the contact surface used recently in [5] in the study a static process for electro-elastic materials. But, unlike [5], in the present paper we consider a quasistatic process for electro-viscoelastic materials, which leads to an evolutionary model, different from the stationnary model studied in [5].

To conclude, the novelty of this paper consists in the study of a frictional contact problem for electro-viscoelastic materials which takes into account the electric conductivity of the foundation. From the physical point of view, the novelty arises in the fact that we let the frictional contact condition to depend on the electric potential; from the mathematical point of view, the novelty arises in the fact that here we provide the unique solvability of a new model, involving new operators and new functionals, together with its numerical approach and numerical simulations.

The manuscript is structured as follows. In Section 2 we describe the physical setting and present the mathematical model of the contact process. In Section 3 we list the assumption on the problem data, derive the variational formulation of the problem and state our main existence and uniqueness result, Theorem 1. The proof of the theorem is provided in Section 4, based on arguments of evolutionary variational inequalities and fixed point. Finally, in Section 5 we introduce the discretized problem, then we present numerical simulations in the study of a two-dimensional test problem.

2 Problem statement

We consider a body made of a piezoelectric material which occupies the domain $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ with a smooth boundary $\partial\Omega = \Gamma$ and a unit outward normal $\boldsymbol{\nu}$. The body is acted upon by body forces of density $\boldsymbol{f}_0$ and has volume free electric charges of density q_0. It is also constrained mechanically and electrically on the boundary. To describe these constraints we assume a partition of Γ into three open disjoint parts Γ_1, Γ_2 and Γ_3, on the one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand. We assume that $meas\,\Gamma_1 > 0$ and $meas\,\Gamma_a > 0$. The body is clamped on Γ_1 and, therefore, the displacement field vanishes there. Surface tractions of density $\boldsymbol{f}_2$ act on Γ_2. We also assume that the electrical potential vanishes on Γ_a and a surface electrical charge of density q_b is prescribed on Γ_b. In the reference configuration the body may come in contact over Γ_3 with an electrically conductive support, the so called foundation. The contact is frictional and we model it with normal compliance and a version of Coulomb's law of dry friction. Also, since the foundation is electrically conductive, we assume that the stiffness coefficient and the friction bound depend on the difference between the electric potential of the body's surface and the electric potential of the foundation. Finally, there may be electrical charges on the part of the body which is in contact with the foundation and which vanish when the contact is lost.

We are interested in the deformation of the body on the time interval $[0,T]$. The process is assumed to be quasistatic, i.e. the inertial effects in the equation of motion are neglected. We denote by $\boldsymbol{x} \in \Omega \cup \Gamma$ and $t \in [0,T]$ the spatial and the time variable, respectively and, to simplify the notation, sometimes we do not indicate the dependence of various functions on $\boldsymbol{x}$ or t. In this paper $i,j,k,l = 1,\dots,d$, summation over two repeated indices is implied, and the index that follows a comma represents the partial derivative with respect to the corresponding component of $\boldsymbol{x}$, i.e. $f_{,i} = \frac{\partial f}{\partial x_i}$. The dot above a variable represents the time derivatives, i.e. $\dot{f} = \frac{\partial f}{\partial t}$.

We use the notation $\mathbb{S}^d$ for the space of second order symmetric tensors on $\mathbb{R}^d$ and "$\cdot$", $\|\cdot\|$ will represent the inner product and the Euclidean norm on $\mathbb{S}^d$ and $\mathbb{R}^d$, respectively, that is

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \quad \|\boldsymbol{v}\| = (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2}$$

for $\boldsymbol{u} = (u_i)$, $\boldsymbol{v} = (v_i) \in \mathbb{R}^d$, and

$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{1/2}$$

for $\boldsymbol{\sigma} = (\sigma_{ij})$, $\boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d$. We also use the usual notation for the normal components and the tangential parts of vectors and tensors, respectively, given by $u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}$, $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu \boldsymbol{\nu}$, $\sigma_\nu = \sigma_{ij}\nu_i\nu_j$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$.

With the notation above, the classical model for the process is as follows.

Problem $\mathcal{P}$. *Find a displacement field $\boldsymbol{u} = (u_i) : \Omega \times [0,T] \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} = (\sigma_{ij}) : \Omega \times [0,T] \to \mathbb{S}^d$, an electric potential $\varphi : \Omega \times [0,T] \to \mathbb{R}$ and an electric displacement field $\boldsymbol{D} = (D_i) : \Omega \times [0,T] \to \mathbb{R}^d$ such that*

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathcal{E}^*\boldsymbol{E}(\varphi) \quad \text{in } \Omega \times (0,T), \tag{2.1}$$

$$\boldsymbol{D} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\beta}\boldsymbol{E}(\varphi) \quad \text{in } \Omega \times (0,T), \tag{2.2}$$

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega \times (0,T), \tag{2.3}$$

$$\operatorname{div} \boldsymbol{D} - q_0 = 0 \quad \text{in } \Omega \times (0,T), \tag{2.4}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1 \times (0,T), \tag{2.5}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{2.6}$$

$$\varphi = 0 \quad \text{on } \Gamma_a \times (0,T), \tag{2.7}$$

$$\boldsymbol{D} \cdot \boldsymbol{\nu} = q_b \quad \text{on } \Gamma_b \times (0,T), \tag{2.8}$$

$$-\sigma_\nu = h_\nu(\varphi - \varphi_0)\, p_\nu(u_\nu - g) \quad \text{on } \Gamma_3 \times (0,T), \tag{2.9}$$

$$\left.\begin{array}{l} \|\boldsymbol{\sigma}_\tau\| \le h_\tau(\varphi - \varphi_0)\, p_\tau(u_\nu - g), \\ -\boldsymbol{\sigma}_\tau = h_\tau(\varphi - \varphi_0)\, p_\tau(u_\nu - g) \dfrac{\dot{\boldsymbol{u}}_\tau}{\|\dot{\boldsymbol{u}}_\tau\|} \quad \text{if} \quad \dot{\boldsymbol{u}}_\tau \neq \boldsymbol{0} \end{array}\right\} \quad \text{on } \Gamma_3 \times (0,T), \tag{2.10}$$

$$\boldsymbol{D} \cdot \boldsymbol{\nu} = p_e(u_\nu - g)\, h_e(\varphi - \varphi_0) \quad \text{on } \Gamma_3 \times (0,T), \tag{2.11}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega. \tag{2.12}$$

We now describe problem (2.1)–(2.12) and provide explanation of the equations and the boundary conditions.

First, equations (2.1) and (2.2) represent the electro-viscoelastic constitutive law in which $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))$ denotes the linearized strain tensor, $\boldsymbol{E}(\varphi)$ is the electric field, $\mathcal{A}$ and $\mathcal{B}$ are the viscosity and elasticity operators, respectively, $\mathcal{E} = (e_{ijk})$ represents the third-order piezoelectric tensor, $\mathcal{E}^*$ is its transpose and $\boldsymbol{\beta}$ denotes the electric permittivity tensor. We recall that $\boldsymbol{\varepsilon}_{ij}(\boldsymbol{u}) = (u_{i,j} + u_{j,i})/2$ and $\boldsymbol{E}(\varphi) = -\nabla\,\varphi = -(\varphi_{,i})$. Also, the tensors $\mathcal{E}$ and $\mathcal{E}^*$ satisfy the equality

$$\mathcal{E}\boldsymbol{\sigma}\cdot\boldsymbol{v} = \boldsymbol{\sigma}\cdot\mathcal{E}^*\boldsymbol{v} \qquad \forall\boldsymbol{\sigma}\in\mathbb{S}^d,\ \boldsymbol{v}\in\mathbb{R}^d,$$

and the components of the tensor $\mathcal{E}^*$ are given by $e^*_{ijk} = e_{kij}$. Equation (2.1) indicates that the mechanical properties of the materials are described by a viscoelastic Kelvin-Voigt constitutive relation (see [9] for details) which takes into account the dependence of the stress field on the electric field. Relation (2.2) describes a linear dependence of the electric displacement field $\boldsymbol{D}$ on the strain and electric fields; such a relation has been frequently employed in the literature (see, e.g., [6, 7] and the references therein).

Next, equations (2.3) and (2.4) are the balance equations for the stress and electric-displacement fields, respectively, in which “Div” and “div” denote the divergence operators for tensor and vector valued functions, i.e. $\operatorname{Div}\boldsymbol{\sigma} = (\sigma_{ij,j})$, $\operatorname{div}\boldsymbol{D} = (D_{i,i})$. We use these equations since the process is assumed to be quasistatic.

Conditions (2.5) and (2.6) are the displacement and traction boundary conditions, whereas (2.7) and (2.8) represent the electric boundary conditions; these conditions show that the displacement field and the electrical potential vanish on Γ_1 and Γ_a, respectively, while the forces and free electric charges are prescribed on Γ_2 and Γ_b, respectively. Also, (2.12) represents the initial condition in which $\boldsymbol{u}_0$ is the given initial displacement field.

We turn to the boundary conditions (2.9)–(2.11), already used in [5], which describe the mechanical and electrical conditions on the potential contact surface Γ_3; there, g represents the gap in the reference configuration between Γ_3 and the foundation, measured along the direction of $\boldsymbol{\nu}$, and φ_0 denotes the electric potential of the foundation.

First, (2.9) represents the normal compliance contact condition in which p_ν is a prescribed nonnegative function which vanishes when its argument is negative and h_ν is a positive function, the stiffness coefficient. Equality (2.9) shows that when there is no contact (i.e. when $u_\nu < g$) then $\sigma_\nu = 0$ and therefore the normal pressure vanishes; when there is contact (i.e. when $u_\nu \geq g$) then $\sigma_\nu \leq 0$ and therefore the reaction of the foundation is towards the body.

Condition (2.10) is the associated friction law where p_τ is a given function and h_τ is the coefficient of friction. According to (2.10) the tangential shear cannot exceed the maximum frictional resistance $h_\tau(\varphi - \varphi_0)\, p_\tau(u_\nu - g)$, the so-called friction bound. Moreover, when sliding commences, the tangential shear reaches the friction bound and opposes the motion.

Frictional contact conditions of the form (2.9), (2.10) have been used in the study of various piezoelectric contact problems, see, e.g. [11, 17] and the references therein. Unlike these references, we assume here that the stiffness coefficient h_ν and the coefficient of friction h_τ depend on the difference between the potential on the foundation and the body's surface.

Finally, (2.11) is a regularized electrical contact condition on Γ_3, similar to that already used in [3, 4, 5, 11]. Here p_e represents the electrical conductivity coefficient, which vanish when its argument is negative, and h_e is a given function. Thus, condition (2.11) shows that when there is no contact at a point on the surface (i.e. when $u_\nu < g$) then the normal component of the electric displacement field vanishes, and when there is contact (i.e. when $u_\nu \geq g$) then there may be electrical charges which depend on the difference between the potential of the foundation and the body's surface.

Because of the frictional condition (2.10), which is non-smooth, we do not expect the problem to have, in general, any classical solutions. For this reason, we derive in the next section a variational formulation of the problem, then we investigate its weak solvability.

3 Variational formulation

We turn now to the variational formulation of the problem and, to this end, we need additional notation and preliminaries. We use standard notation for the L^p and the Sobolev spaces associated with Ω and Γ; moreover, for a function $\psi \in H^1(\Omega)$ we still write ψ to denote its trace on Γ. Besides the space $L^d(\Omega)^d$ endowed with the canonic inner product $(\cdot,\cdot)_{L^d(\Omega)^d}$ and the associated norm $\|\cdot\|_{L^d(\Omega)^d}$, for the unknowns of Problem $\mathcal{P}$ we use the spaces

$$\begin{aligned} Q &= \{\ \boldsymbol{\tau} = (\tau_{ij}) :\ \tau_{ij} = \tau_{ji} \in L^2(\Omega)\ \},\\ V &= \{\ \boldsymbol{v} \in H^1(\Omega)^d\ :\ \boldsymbol{v} = \mathbf{0}\ \text{ on }\ \Gamma_1\ \},\\ W &= \{\ \psi \in H^1(\Omega) :\ \psi = 0\ \text{ on }\ \Gamma_a\ \}. \end{aligned}$$

The space Q is a real Hilbert space endowed with the inner product

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_Q \int_\Omega \sigma_{ij}\tau_{ij}dx$$

and the associated norm $\|\cdot\|_Q$. Also, since $meas\,\Gamma_1 > 0$ and $meas\,\Gamma_a > 0$, it is well known that V and W are real Hilbert spaces endowed with the inner products

$$(\boldsymbol{u},\boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q, \qquad (\varphi,\psi)_W = (\nabla\varphi,\nabla\psi)_{L^2(\Omega)^d}$$

and the associated norms $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively. Moreover, by the Sobolev trace theorem, there exists two positive constants c_0 and $\widetilde{c}_0$ which depend on Ω, Γ_1 and Γ_3 such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \le c_0\,\|\boldsymbol{v}\|_V \quad \forall\,\boldsymbol{v}\in V, \quad \|\psi\|_{L^2(\Gamma_3)} \le \widetilde{c}_0\,\|\psi\|_W \quad \forall\,\psi\in W. \tag{3.1}$$

Finally, if $(X,\|\cdot\|_X)$ represents a real Banach space, we denote by $C([0,T];X)$ and $C^1([0,T];X)$ the spaces of continuous and continuously differentiable functions on $[0,T]$ with values on X, with the norms

$$\|\boldsymbol{x}\|_{C([0,T];X)} = \max_{t\in[0,T]}\|\boldsymbol{x}(t)\|_X,$$

$$\|\boldsymbol{x}\|_{C^1([0,T];X)} = \max_{t\in[0,T]}\|\boldsymbol{x}(t)\|_X + \max_{t\in[0,T]}\|\dot{\boldsymbol{x}}(t)\|_X.$$

Recall that, here and below, the dot represents the derivative with respect to the time variable.

In the study of the mechanical problem (2.1)–(2.12) we assume that the viscosity operator $\mathcal{A}$, the elasticity operator $\mathcal{B}$, the piezoelectric tensor $\mathcal{E}$ and the electric permittivity tensor $\boldsymbol{\beta}$ satisfy

$$\left\{\begin{array}{l}
\text{(a) } \mathcal{A}:\Omega\times\mathbb{S}^d\to\mathbb{S}^d.\\
\text{(b) There exists } L_{\mathcal{A}}>0 \text{ such that}\\
\qquad \|\mathcal{A}(\boldsymbol{x},\boldsymbol{\xi}_1)-\mathcal{A}(\boldsymbol{x},\boldsymbol{\xi}_2)\| \le L_{\mathcal{A}}\|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\|\\
\qquad \forall\,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2\in\mathbb{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega.\\
\text{(c) There exists } m_{\mathcal{A}}>0 \text{ such that}\\
\qquad (\mathcal{A}(\boldsymbol{x},\boldsymbol{\xi}_1)-\mathcal{A}(\boldsymbol{x},\boldsymbol{\xi}_2))\cdot(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2) \ge m_{\mathcal{A}}\|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\|^2\\
\qquad \forall\,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2\in\mathbb{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega.\\
\text{(d) The mapping } \boldsymbol{x}\mapsto\mathcal{A}(\boldsymbol{x},\boldsymbol{\xi}) \text{ is Lebesgue measurable on } \Omega,\\
\qquad \text{for any } \boldsymbol{\xi}\in\mathbb{S}^d.\\
\text{(e) The mapping } \boldsymbol{x}\mapsto\mathcal{A}(\boldsymbol{x},\boldsymbol{0}) \text{ belongs to } Q.
\end{array}\right. \tag{3.2}$$

$$
\left\{
\begin{array}{l}
\text{(a) } \mathcal{B} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\
\text{(b) There exists } L_{\mathcal{B}} > 0 \text{ such that} \\
\qquad \|\mathcal{B}(\boldsymbol{x}, \boldsymbol{\xi}_1) - \mathcal{B}(\boldsymbol{x}, \boldsymbol{\xi}_2)\| \le L_{\mathcal{B}} \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\| \\
\qquad \forall\, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \quad \text{a.e. } \boldsymbol{x} \in \Omega. \\
\text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{B}(\boldsymbol{x}, \boldsymbol{\xi}) \text{ is measurable on } \Omega, \\
\qquad \text{for any } \boldsymbol{\xi} \in \mathbb{S}^d. \\
\text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{B}(\boldsymbol{x}, \boldsymbol{0}) \text{ belongs to } Q.
\end{array}
\right.
\tag{3.3}
$$

$$
\left\{
\begin{array}{l}
\text{(a) } \mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{R}^d. \\
\text{(b) } \mathcal{E}(\boldsymbol{x}, \boldsymbol{\tau}) = (e_{ijk}(\boldsymbol{x}) \tau_{jk}) \quad \forall \boldsymbol{\tau} = (\boldsymbol{\tau}_{ij}) \in \mathbb{S}^d, \quad \text{a.e. } \boldsymbol{x} \in \Omega. \\
\text{(c) } e_{ijk} = e_{ikj} \in L^\infty(\Omega).
\end{array}
\right.
\tag{3.4}
$$

$$
\left\{
\begin{array}{l}
\text{(a) } \boldsymbol{\beta} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d. \\
\text{(b) } \boldsymbol{\beta}(\boldsymbol{x}, \boldsymbol{E}) = (\beta_{ij}(\boldsymbol{x}) E_j) \quad \forall \boldsymbol{E} = (E_i) \in \mathbb{R}^d, \quad \text{a.e. } \boldsymbol{x} \in \Omega. \\
\text{(c) } \beta_{ij} = \beta_{ji} \in L^\infty(\Omega). \\
\text{(d) There exists } m_\beta > 0 \text{ such that } \beta_{ij}(\boldsymbol{x}) E_i E_j \ge m_\beta \|\boldsymbol{E}\|^2 \\
\qquad \forall \boldsymbol{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Omega.
\end{array}
\right.
\tag{3.5}
$$

The functions p_r and h_r (for $r = \nu,\ \tau,\ e$) are such that

$$
\left\{
\begin{array}{l}
\text{(a) } p_r : \Gamma_3 \times \mathbb{R} \to \mathbb{R}. \\
\text{(b) There exists } L_r > 0 \text{ such that} \\
\qquad |p_r(\boldsymbol{x}, u_1) - p_r(\boldsymbol{x}, u_2)| \le L_r |u_1 - u_2| \ \forall u_1,\ u_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\
\text{(c) There exists } \overline{p}_r \ge 0 \text{ such that} \\
\qquad 0 \le p_r(\boldsymbol{x}, u) \le \overline{p}_r \ \forall u \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\
\text{(d) The mapping } \boldsymbol{x} \mapsto p_r(\boldsymbol{x}, u) \text{ is measurable on } \Gamma_3, \text{ for any } u \in \mathbb{R}. \\
\text{(e) } p_r(\boldsymbol{x}, u) = 0 \quad \forall u < 0, \text{ a.e. } \boldsymbol{x} \in \Gamma_3.
\end{array}
\right.
\tag{3.6}
$$

$$
\left\{
\begin{array}{l}
\text{(a) } h_r : \Gamma_3 \times \mathbb{R} \to \mathbb{R}, \text{ for } r = \nu,\ \tau,\ e. \\
\text{(b) There exists } l_r > 0 \text{ such that} \\
\qquad |h_r(\boldsymbol{x}, \varphi_1) - h_r(\boldsymbol{x}, \varphi_2)| \le l_r |\varphi_1 - \varphi_2| \\
\qquad \forall \varphi_1,\ \varphi_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3, \text{ for } r = \nu,\ \tau,\ e. \\
\text{(c) There exists } \overline{h}_r \ge 0 \text{ such that} \\
\qquad 0 \le h_r(\boldsymbol{x}, \varphi) \le \overline{h}_r \ \forall \varphi \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3, \text{ for } r = \nu,\ \tau. \\
\text{(d) There exists } \overline{h}_e \ge 0 \text{ such that} \\
\qquad |h_e(\boldsymbol{x}, \varphi)| \le \overline{h}_e \ \forall \varphi \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\
\text{(e) The mapping } \boldsymbol{x} \mapsto h_r(\boldsymbol{x}, u) \text{ is measurable on } \Gamma_3, \\
\qquad \text{for any } \varphi \in \mathbb{R}, \text{ for } r = \nu,\ \tau,\ e.
\end{array}
\right.
\tag{3.7}
$$

The forces, tractions, volume and surface free charge densities satisfy

$$\boldsymbol{f}_0 \in C([0,T];L^2(\Omega)^d), \qquad \boldsymbol{f}_2 \in C([0,T];L^2(\Gamma_2)^d), \tag{3.8}$$

$$q_0 \in C([0,T];L^2(\Omega)), \qquad q_b \in C([0,T];L^2(\Gamma_b)). \tag{3.9}$$

Finally, we assume that the gap function, the potential of the foundation and the initial displacement satisfy

$$g \in L^2(\Gamma_3), \quad g \geq 0 \quad \text{a.e. on } \Gamma_3, \tag{3.10}$$

$$\varphi_0 \in L^2(\Gamma_3), \tag{3.11}$$

$$\boldsymbol{u}_0 \in V. \tag{3.12}$$

Next, we define the four mappings $J : W \times V \times V \to \mathbb{R}$, $G : V \times W \times W \to \mathbb{R}$, $\boldsymbol{f} : [0,T] \to V$ and $q : [0,T] \to W$, respectively, by

$$J(\varphi, \boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} h_\nu(\varphi - \varphi_0)\, p_\nu(u_\nu - g) v_\nu \, da \tag{3.13}$$
$$+ \int_{\Gamma_3} h_\tau(\varphi - \varphi_0)\, p_\tau(u_\nu - g) \|\boldsymbol{v}_\tau\| \, da,$$

$$G(\boldsymbol{u}, \varphi, \psi) = \int_{\Gamma_3} p_e(u_\nu - g)\, h_e(\varphi - \varphi_0) \psi \, da, \tag{3.14}$$

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v} \, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v} \, da, \tag{3.15}$$

$$(q(t), \psi)_W = \int_\Omega q_0(t) \psi \, dx - \int_{\Gamma_b} q_b(t) \psi \, da, \tag{3.16}$$

for all $\boldsymbol{u}$, $\boldsymbol{v} \in V$, φ, $\psi \in W$ and $t \in [0,T]$. We note that the definitions of $\boldsymbol{f}$ and q are based on the Riesz representation theorem; moreover, it follows from assumptions (3.6)–(3.11) that the integrals in (3.13)–(3.16) are well-defined and, in addition

$$\boldsymbol{f} \in C([0,T];V), \tag{3.17}$$

$$q \in C([0,T];W). \tag{3.18}$$

Finally, assumptions (3.6) and (3.7) combined with (3.1) yield

$$J(\varphi_1, \boldsymbol{u}_1, \boldsymbol{v}_2) - J(\varphi_1, \boldsymbol{u}_1, \boldsymbol{v}_2) + J(\varphi_2, \boldsymbol{u}_2, \boldsymbol{v}_1) - J(\varphi_2, \boldsymbol{u}_2, \boldsymbol{v}_2) \tag{3.19}$$
$$\leq c\,(\|\varphi_1 - \varphi_2\|_W + \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V) \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V),$$

$$G(\boldsymbol{u}_1, \varphi_1, \psi) - G(\boldsymbol{u}_2, \varphi_2, \psi) \tag{3.20}$$
$$\leq (c_0 \tilde{c}_0 L_p \overline{h}_e \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V + \tilde{c}_0^2 l_e \overline{p}_e \|\varphi_1 - \varphi_2\|_W) \|\psi\|_W,$$

for all $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{v}_1$, $\boldsymbol{v}_2 \in V$, φ_1, φ_2, $\psi \in W$, where $c > 0$.

Using integration by parts, it is straightforward to see that if $(\boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \boldsymbol{D})$ are sufficiently regular functions which satisfy (2.3)–(2.11) then

$$\begin{aligned}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))_Q + J(\varphi(t), \boldsymbol{u}(t), \boldsymbol{v}) - J(\varphi(t), \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) \\ \geq (\boldsymbol{f}(t), \dot{\boldsymbol{u}}(t) - \boldsymbol{v})_V,\end{aligned} \tag{3.21}$$

$$(\boldsymbol{D}(t), \nabla\psi)_{L^2(\Omega)^d} + (q(t), \psi)_W = G(\boldsymbol{u}(t), \varphi(t)), \psi), \tag{3.22}$$

for all $\boldsymbol{v} \in V$, $\psi \in W$ and $t \in [0, T]$. We substitute (2.1) in (3.21), (2.2) in (3.22), note that $\boldsymbol{E}(\varphi) = -\nabla\varphi$ and use the initial condition (2.12). As a result we obtain the following variational formulation of problem $\mathcal{P}$.

Problem $\mathcal{P}_V$. *Find a displacement field $\boldsymbol{u} : [0, T] \to V$ and an electric potential $\varphi : [0, T] \to W$ such that*

$$\begin{aligned}(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \varepsilon(\dot{\boldsymbol{u}}(t)))_Q + (\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_Q \\ +(\mathcal{E}^*\nabla\varphi(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_Q + J(\varphi(t), \boldsymbol{u}(t), \boldsymbol{v}) - J(\varphi(t), \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) \\ \geq (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V,\end{aligned} \tag{3.23}$$

for all $\boldsymbol{v} \in V$ and $t \in [0, T]$,

$$\begin{aligned}(\boldsymbol{\beta}\nabla\varphi(t), \nabla\psi)_{L^2(\Omega)^d} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \nabla\psi)_{L^2(\Omega)^d} \\ +G(\boldsymbol{u}(t), \varphi(t)), \psi) = (q(t), \psi)_W,\end{aligned} \tag{3.24}$$

for all $\psi \in W$ and $t \in [0, T]$, and

$$\boldsymbol{u}(0) = \boldsymbol{u}_0. \tag{3.25}$$

To study problem $\mathcal{P}_V$ we make the smallness assumption

$$\tilde{c}_0^2 l_e \overline{p}_e < m_\beta, \tag{3.26}$$

where $\tilde{c}_0$, l_e, $\overline{p}_e$ and m_β are given in (3.1) (3.7), (3.6) and (3.5), respectively. We note that only the trace constant, the Lipshitz contant of h_e, the bound of p_e and the coercivity constant of $\boldsymbol{\beta}$ and are involved in (3.26); therefore, this smallness assumption involves only the geometry and the electrical part, and does not depend on the mechanical data of the problem. Moreover, it is satisfied when the obstacle is insulated, since then $p_e \equiv 0$ and so $\overline{p}_e = 0$.

Our main existence and uniqueness result that we state now and prove in the next section is the following.

Theorem 1. *Assume that* (3.2)–(3.12) *and* (3.26) *hold. Then Problem* $\mathcal{P}_V$ *has a unique solution which satisfies*

$$\boldsymbol{u} \in C^1([0,T];V), \quad \varphi \in C([0,T];W). \tag{3.27}$$

A quadruple of functions $(\boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \boldsymbol{D})$ which satisfies (2.1), (2.2), (3.23)–(3.25) is called a *weak solution* of the piezoelectric contact problem $\mathcal{P}$. It follows from Theorem 1 that, under the assumptions (3.2)–(3.12), (3.26), there exists a unique weak solution of Problem $\mathcal{P}$. To precise the regularity of the weak solution we note that the constitutive relations (2.1) and (2.2), assumptions (3.2)–(3.5) and regularity (3.27) imply that

$$\boldsymbol{\sigma} \in C([0,T];Q), \qquad \boldsymbol{D} \in C([0,T];L^2(\Omega)^d). \tag{3.28}$$

Moreover, using again (2.1), (2.2) combined with (3.23), (3.24) and the notation (3.13)–(3.16), after standard arguments we obtain that $\mathrm{Div}\,\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0}$ and $\mathrm{div}\,\boldsymbol{D}(t) = q_0(t)$, for all $t \in [0,T]$. It follows now from the regularity (3.8) and (3.9) that

$$\mathrm{Div}\,\boldsymbol{\sigma} \in C([0,T];L^2(\Omega)^d), \qquad \mathrm{div}\,\boldsymbol{D} \in C([0,T];L^2(\Omega)). \tag{3.29}$$

We conclude that the weak solution $(\boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \boldsymbol{D})$ of the piezoelectric contact problem $\mathcal{P}$ has the regularity (3.27)–(3.29).

4 Proof of Theorem 1

We turn now to the proof of Theorem 1 which will be carried out in several steps. We assume in what follows that (3.2)–(3.12) and (3.26) hold and, everywhere below, we denote by c various positive constants which are independent on time and whose value may change from line to line. We consider the product space $X = V \times W$ together with the inner product

$$(\boldsymbol{x}, \boldsymbol{y})_X = (\boldsymbol{u}, \boldsymbol{v})_V + (\varphi, \psi)_W \qquad \forall \boldsymbol{x} = (\boldsymbol{u}, \varphi),\ \boldsymbol{y} = (\boldsymbol{v}, \psi) \in X$$

and the associated norm $\|\cdot\|_X$. Let $\boldsymbol{\eta} = (\boldsymbol{\eta}_1, \eta_2) \in C([0,T], X)$ be given. In the first step, we consider the following intermediate problem.

Problem $\mathcal{P}_\eta^{disp}$. *Find a displacement field* $\boldsymbol{u}_\eta : [0,T] \to V$ *such that*

$$\begin{aligned}&(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\eta(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\eta(t)))_Q + (\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{\eta}_1(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\eta(t)))_Q \\ &\quad + (\mathcal{E}^*\nabla\eta_2(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\eta(t)))_Q + J(\eta_2(t), \boldsymbol{\eta}_1(t), \boldsymbol{v}) \\ &\qquad - J(\eta_2(t), \boldsymbol{\eta}_1(t), \dot{\boldsymbol{u}}_\eta(t)) \ge (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_\eta(t))_V \quad \forall \boldsymbol{v} \in V,\ t \in [0,T],\end{aligned} \tag{4.1}$$

$$\boldsymbol{u}_\eta(0) = \boldsymbol{u}_0. \tag{4.2}$$

In the study of the variational problem $\mathcal{P}_\eta^{disp}$ we have the following result.

Lemma 1. *There exists a unique solution $\boldsymbol{u}_\eta \in C^1([0,T],V)$ to the problem* (4.1)–(4.2)*. Moreover, if $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two solutions of problem* (4.1)–(4.2) *corresponding to the data $\boldsymbol{\eta}^1 = (\boldsymbol{\eta}_1^1, \eta_2^1)$, $\boldsymbol{\eta}^2 = (\boldsymbol{\eta}_1^2, \eta_2^2) \in C([0,T],X)$ then there exists $c > 0$ such that*

$$\|\dot{\boldsymbol{u}}_1(t) - \dot{\boldsymbol{u}}_2(t)\|_V \le c\,\|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_X \quad \forall\, t \in [0,T]. \tag{4.3}$$

Proof. We use classical results on elliptic variational inequalities (see [9, p. 60]) to deduce that, for each $t \in [0,T]$, there exists a unique element $\boldsymbol{v}_\eta(t) \in V$ such that

$$\begin{aligned}
&(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_\eta(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{v}_\eta(t)))_Q + (\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{\eta}_1(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{v}_\eta(t)))_Q \\
&\quad + (\mathcal{E}^*\nabla\eta_2(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{v}_\eta(t)))_Q + J(\eta_2(t), \boldsymbol{\eta}_1(t), \boldsymbol{v}) \\
&\qquad - J(\eta_2(t), \boldsymbol{\eta}_1(t), \boldsymbol{v}_\eta(t)) \ge (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{v}_\eta(t))_V \quad \forall\, \boldsymbol{v} \in V.
\end{aligned} \tag{4.4}$$

Let $t_1,\ t_2 \in [0,T]$; using (4.4) for $t = t_1$ and $t = t_2$, we easily derive the inequality

$$\begin{aligned}
&(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_\eta(t_1)) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_\eta(t_2)), \boldsymbol{\varepsilon}(\boldsymbol{v}_\eta(t_1)) - \boldsymbol{\varepsilon}(\boldsymbol{v}_\eta(t_2)))_Q \\
&\quad \le (\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{\eta}_1(t_1)) - \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{\eta}_1(t_2)), \boldsymbol{\varepsilon}(\boldsymbol{v}_\eta(t_2)) - \boldsymbol{\varepsilon}(\boldsymbol{v}_\eta(t_1)))_Q + \\
&\qquad + (\mathcal{E}^*\nabla\eta_2(t_1) - \mathcal{E}^*\nabla\eta_2(t_2), \boldsymbol{\varepsilon}(\boldsymbol{v}_\eta(t_2)) - \boldsymbol{\varepsilon}(\boldsymbol{v}_\eta(t_1)))_Q \\
&\qquad\quad + J(\eta_2(t_1), \boldsymbol{\eta}_1(t_1), \boldsymbol{v}_\eta(t_2)) - J(\eta_2(t_1), \boldsymbol{\eta}_1(t_1), \boldsymbol{v}_\eta(t_1)) \\
&\qquad\qquad + J(\eta_2(t_2), \boldsymbol{\eta}_1(t_2), \boldsymbol{v}_\eta(t_1)) - J(\eta_2(t_2), \boldsymbol{\eta}_1(t_2), \boldsymbol{v}_\eta(t_2)) \\
&\qquad\qquad\quad + (\boldsymbol{f}(t_1) - \boldsymbol{f}(t_2), \boldsymbol{v}_\eta(t_1) - \boldsymbol{v}_\eta(t_2))_V.
\end{aligned}$$

Then, we use assumptions (3.2), (3.3), (3.4) and (3.19) to obtain

$$\begin{aligned}
\|\boldsymbol{v}_\eta(t_1) - \boldsymbol{v}_\eta(t_2)\|_V &\le c\,(\|\boldsymbol{\eta}_1(t_1) - \boldsymbol{\eta}_1(t_2)\|_V \\
&\quad + \|\eta_2(t_1) - \eta_2(t_2)\|_W + \|\boldsymbol{f}(t_1) - \boldsymbol{f}(t_2)\|_V).
\end{aligned} \tag{4.5}$$

From (4.5), (3.17) and the regularity of $\boldsymbol{\eta}$ it follows that $\boldsymbol{v}_\eta \in C([0,T];V)$. Let $\boldsymbol{u}_\eta : [0,T] \to V$ be the function defined by

$$\boldsymbol{u}_\eta(t) = \int_0^t \boldsymbol{v}_\eta(s)\,ds + \boldsymbol{u}_0, \qquad \forall t \in [0,T]. \tag{4.6}$$

It follows from (4.6) and (4.4) that $\boldsymbol{u}_\eta$ is a solution of Problem $\mathcal{P}_\eta^{disp}$ and, moreover, $\boldsymbol{u}_\eta \in C^1([0,T];V)$. This proves the existence part of Lemma 1. The uniqueness part follows from the unique solvability of the variational inequality (4.4) at each $t \in [0,T]$.

Let now denote by $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ the solutions of problem (4.1)–(4.2) corresponding to the data $\boldsymbol{\eta}^1 = (\boldsymbol{\eta}_1^1, \eta_2^1)$, $\boldsymbol{\eta}^2 = (\boldsymbol{\eta}_1^2, \eta_2^2) \in C([0,T],X)$ and let $\dot{\boldsymbol{u}}_1 = \boldsymbol{v}_1$, $\dot{\boldsymbol{u}}_2 = \boldsymbol{v}_2$. Arguments similar to those used in the proof of (4.5) lead to

$$\|\dot{\boldsymbol{u}}_1(t) - \dot{\boldsymbol{u}}_2(t)\|_V \le c\,(\|\boldsymbol{\eta}_1^1(t) - \boldsymbol{\eta}_1^2(t)\|_V + \|\eta_2^1(t) - \eta_2^2(t)\|_V) \quad \forall\, t \in [0,T],$$

which shows that (4.3) holds. □

In the next step we use the solution $\boldsymbol{u}_\eta \in C^1([0,T],V)$ obtained in Lemma 1 to construct the following variational problem.

Problem $\mathcal{P}_\eta^{pot}$. *Find an electric potential field $\varphi_\eta : [0,T] \to W$ such that*

$$\begin{aligned}(\boldsymbol{\beta}\nabla\varphi_\eta(t), \nabla\psi)_{L^2(\Omega)^d} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t)), \nabla\psi)_{L^2(\Omega)^d}\\ +G(\boldsymbol{u}_\eta(t), \eta_2(t), \psi) = (q(t), \psi)_W \qquad \forall\, \psi \in W,\, t \in [0,T]\end{aligned} \tag{4.7}$$

The well-posedness of the problem $\mathcal{P}_\eta^{pot}$ is given by the following result.

Lemma 2. *There exists a unique solution $\varphi_\eta \in C([0,T];W)$ which satisfies* (4.7)*. Moreover, if $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and φ_1, φ_2 are two solutions of of* (4.1)–(4.2) *and* (4.7)*, respectively, corresponding to $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2 \in C([0,T];X)$, then there exists $c > 0$ such that*

$$\begin{aligned}\|\varphi_1(t) - \varphi_2(t)\|_W \le c\,\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V\\ +\frac{\tilde{c}_0^2 l_e \overline{p}_e}{m_\beta}\,\|\boldsymbol{\eta}^1(t) - \boldsymbol{\eta}^2(t)\|_X \quad \forall\, t \in [0,T].\end{aligned} \tag{4.8}$$

Proof. It follows from (3.5) that the bilinear form

$$a(\varphi, \psi) = (\boldsymbol{\beta}\nabla\varphi, \nabla\psi)_{L^2(\Omega)^d} \tag{4.9}$$

is continuous, symmetric and coercive on W. Moreover, using (3.18), (3.20), assumption (3.4) on the piezoelectric tensor $\mathcal{E}$ and the regularity $\boldsymbol{u}_\eta \in C^1([0,T];V)$, it follows that the function $q_\eta : [0,T] \to W$, given by

$$\begin{aligned}(q_\eta(t), \psi)_W = (q(t), \psi)_W + (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t)), \nabla\psi)_{L^2(\Omega)^d}\\ -G(\boldsymbol{u}_\eta(t), \eta_2(t), \psi) \qquad \forall\, \psi \in W,\ t \in [0,T],\end{aligned} \tag{4.10}$$

is continuous. The existence and uniqueness part in Lemma 4.3 is now a straight consequence of the well-known Lax-Milgram theorem applied to the time-dependent variational equation

$$a(\varphi(t), \psi) = (q_\eta(t), \psi) \qquad \forall \psi \in W,\ t \in [0, T],$$

combined with the equalities (4.9), (4.10). Moreover, the estimate (4.8) follows from (4.7), (3.4), (3.5) and (3.20). □

We now consider the operator $\Lambda : C([0, T]; X) \to C([0, T]; X)$ defined by

$$\Lambda \boldsymbol{\eta}(t) = (\boldsymbol{u}_\eta(t), \varphi_\eta(t)) \qquad \forall\, \boldsymbol{\eta} \in C([0, T]; X)),\ t \in [0, T]. \tag{4.11}$$

The next step consists in the following result.

Lemma 3. *There exists a unique* $\boldsymbol{\eta}^* \in C([0, T]; X)$ *such that* $\Lambda \boldsymbol{\eta}^* = \boldsymbol{\eta}^*$.

Proof. Let $\boldsymbol{\eta}^1 = (\boldsymbol{\eta}^1_1, \eta^1_2)$, $\boldsymbol{\eta}^2 = (\boldsymbol{\eta}^2_1, \eta^2_2) \in C([0, T]; X)$ and, for simplicity, we use the notation $\boldsymbol{u}_i$ and φ_i for the functions $\boldsymbol{u}_{\eta_i}$ and φ_{η_i} obtained in Lemmas 1 and 2, for $i = 1, 2$. Let $t \in [0, T]$. Using (4.11) and (4.8) we obtain

$$\|\Lambda \boldsymbol{\eta}^1(t) - \Lambda \boldsymbol{\eta}^2(t)\|_Q \le c\, \|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V + \frac{\tilde{c}_0^2 l_e \overline{p}_e}{m_\beta}\, \|\boldsymbol{\eta}^1(t) - \boldsymbol{\eta}^2(t)\|_X. \tag{4.12}$$

On the other hand, since

$$\boldsymbol{u}_i(t) = \boldsymbol{u}_0 + \int_0^t \dot{\boldsymbol{u}}_i(s)\, ds$$

we have

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V \le \int_0^t \|\dot{\boldsymbol{u}}_1(s) - \dot{\boldsymbol{u}}_2(s)\|_V\, ds$$

and, combining this inequality with (4.3), we find

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V \le c \int_0^t \|\boldsymbol{\eta}^1(t) - \boldsymbol{\eta}^2(t)\|_X\, ds. \tag{4.13}$$

We use now (4.12) and (4.13) to obtain

$$\|\Lambda \boldsymbol{\eta}^1(t) - \Lambda \boldsymbol{\eta}^2(t)\|_Q \le c \int_0^t \|\boldsymbol{\eta}^1(s) - \boldsymbol{\eta}^2(s)\|_X\, ds + \frac{\tilde{c}_0^2 l_e \overline{p}_e}{m_\beta}\, \|\boldsymbol{\eta}^1(t) - \boldsymbol{\eta}^2(t)\|_X.$$

The last inequality combined with the smallness assumption (3.26) allows the use of Corollary 2.1 in [18]; as a result it follows that the operator Λ has a unique fixed point, which concludes the proof. $\square$

We have now all the ingredients to prove the Theorem 1.

Existence. Let $\boldsymbol{\eta}^* = (\boldsymbol{\eta}_1^*, \eta_2^*) \in C([0,T];X)$ be the fixed point of the operator Λ, and let $\boldsymbol{u}_{\eta^*}$, φ_{η^*} be the solutions of problems $\mathcal{P}_\eta^{disp}$ and $\mathcal{P}_\eta^{pot}$, respectively, for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$. It follows from (4.11) that $\boldsymbol{u}_{\eta^*} = \boldsymbol{\eta}_1^*$, $\varphi_{\eta^*} = \eta_2^*$ and therefore (4.1), (4.2) and (4.7) imply that $(\boldsymbol{u}_{\eta^*}, \varphi_{\eta^*})$ is a solution of problem $\mathcal{P}_V$. The regularity (3.27) follows from Lemmas 4.2 and 4.3.

Uniqueness. The uniqueness of the solution follows from the uniqueness of the fixed point of the operator Λ, given by Lemma 3. $\square$

5 Numerical approach

Discretized problem. Everywhere below we assume that (3.2)–(3.12) and (3.26) hold. We now introduce a fully discrete scheme to approximate the solution of Problem $\mathcal{P}_V$, provided by Theorem 1. First, we consider two finite dimensional spaces $V^h \subset V$ and $W^h \subset W$ approximating the spaces V and W, respectively, in which $h > 0$ denotes the spatial discretization parameter. In the numerical simulations presented below, V^h and W^h consist of continuous and piecewise affine functions, that is,

$$V^h = \{\boldsymbol{w}^h \in [C(\overline{\Omega})]^d \,:\, \boldsymbol{w}^h_{|Tr} \in [P_1(Tr)]^d \;\forall\, Tr \in \mathcal{T}^h,\; \boldsymbol{w}^h = \boldsymbol{0} \text{ on } \Gamma_1\}, \tag{5.1}$$

$$W^h = \{\zeta^h \in C(\overline{\Omega}) \,:\, \zeta^h_{|Tr} \in P_1(Tr) \;\;\forall\, Tr \in \mathcal{T}^h,\; \zeta^h = 0 \text{ on } \Gamma_a\}, \tag{5.2}$$

where Ω is assumed to be a polygonal domain, $\mathcal{T}^h$ denotes a finite element triangulation of $\overline{\Omega}$, and $P_1(Tr)$ represents the space of polynomials of global degree less or equal to one in Tr. In addition, we consider a uniform partition of $[0,T]$, $0 = t_0 < t_1 < \ldots < t_N = T$, that we use to discretize the time derivatives and, everywhere in this section, we use the notation k for the time step size, i.e. $k = T/N$. Finally, for a continuous function $f(t)$ we denote $f_n = f(t_n)$ and for a sequence $\{w_n\}_{n=0}^N$ we use $\delta w_n = (w_n - w_{n-1})/k$ for the divided differences.

Let $\boldsymbol{u}_0^{hk}$ be an appropriate approximation of the initial condition $\boldsymbol{u}_0$. Then using the backward Euler scheme, the fully discrete approximation of Problem $\mathcal{P}_V$ is the following.

Problem $\mathcal{P}_V^{hk}$. *Find a discrete displacement field $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_n^{hk}\}_{n=0}^N \subset K^h$ and a discrete electric potential $\varphi^{hk} = \{\varphi_n^{hk}\}_{n=0}^N \subset W^h$ such that*

$$\begin{aligned}
&(\mathcal{A}\boldsymbol{\varepsilon}(\delta \boldsymbol{u}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}))_Q + (\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}))_Q \\
&\quad + (\mathcal{E}^* \nabla \varphi_n^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{w}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}))_Q + J(\varphi_n^{hk}, \boldsymbol{u}_n^{hk}, \boldsymbol{w}^h) - J(\varphi_n^{hk}, \boldsymbol{u}_n^{hk}, \delta \boldsymbol{u}_n^{hk}) \\
&\quad\quad \geq (\boldsymbol{f}_n, \boldsymbol{w}^h - \boldsymbol{u}_n^{hk})_V \quad \forall\, \boldsymbol{w}^h \in V^h, \text{ for all } n = 1, \ldots, N,
\end{aligned}$$

$$\begin{aligned}
&(\boldsymbol{\beta} \nabla \varphi_n^{hk}, \nabla \psi^h)_{L^2(\Omega)^d} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}), \nabla \psi^h)_{L^2(\Omega)^d} + G(\boldsymbol{u}_n^{hk}, \varphi_n^{hk}, \psi^h) \\
&\quad\quad = (q_n, \psi^h)_W \quad \forall\, \psi^h \in W^h, \text{ for all } n = 0, \ldots, N.
\end{aligned}$$

The existence of a unique solution to Problem $\mathcal{P}_V^{hk}$ can be obtained by arguments similar to those presented in Section 4. The solution algorithm in solving Problem $\mathcal{P}_V^{hk}$ combines the finite differences method (the backward Euler difference method) with the linear iterations method (the Newton method). Details on these methods can be find in the monograph [19] and, therefore, we omit them. Nevertheless, we note that the numerical treatment of the frictional contact term is based on the use of a penalization method for the contact part and an augmented Lagrangean method for the non-smooth friction part, see [19] and [2], respectively.

Numerical simulations. We now present numerical simulations in the study of a real-world example of Problem $\mathcal{P}$, the microelectromechanical switches, see [15] for details. Microelectromechanical systems (MEMS) are being recognized as enabling components to switch or tune radio frequency (rf) components, modules or systems in manufacturing and operation. In short, they are referred to as rf-MEMS. Most rf-MEMS involve the manipulation of air as the dielectric materials. Various designs of capacitive rf-MEMS switches made out of nickel, aluminium, gold or zinc oxide have so far been reported in literature, see for instance [1, 8]. The mechanical simulation of switch consists in the following design concept: the switch design is based on a suspended metal bridge (zinc oxyde in our example) which connects two grounds of a coplanar wave-guide and crosses over a signal line on which a dielectric foundation is deposited. When an external force is acting, this action pulls the metal bridge down and contacts the dielectric, which results in a low impedance between signal line and ground line for shunting high-frequency signal transmission.

To describe this example, we consider an electro-viscoelastic body extended indefinitely in the direction X_1 of a cartesian coordinate system

(O, X_1, X_2, X_3). The material used is assumed to be a linearly isotropic piezoceramic with hexagonal symmetry like zinc oxyde material (class $6mm$ in the international classification [10]) which presents a viscous behavior. In the crystallographic frame, the X_3-direction is a six-fold revolution symmetry axis and the $(X_1 O X_3)$ and $(X_2 O X_3)$ planes are mirrors. The electrical and mechanical loads applied to the body are supposed to be constant along the X_1 direction. As a consequence, the fields $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ turn out to be constant along X_1. In addition, we suppose that $\boldsymbol{\varepsilon}_{11} = 0$, $\boldsymbol{\varepsilon}_{12} = 0$, $\boldsymbol{\varepsilon}_{13} = 0$ and $D_1 = 0$, i.e. we consider a plane problem. Under these assumptions, the unknown of our electro-viscoelastic contact problem is the pair $(\boldsymbol{u}, \varphi)$ where the displacement field $\boldsymbol{u} = (u_2, u_3)$ belongs to the plane (O, X_2, X_3).

Assume that the viscosity and elasticity operators are linear and denote by a_{ijkl} and b_{ijkl} their components, i.e. $\mathcal{A} = (a_{ijkl})$ and $\mathcal{B} = (b_{ijkl})$. Then, in the system (O, X_2, X_3), the constitutive equations (2.1) and (2.2) can be written by using the following compressed matrix notation,

$$\begin{bmatrix} \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} b_{22} & b_{23} & 0 & 0 & e_{32} \\ b_{23} & b_{33} & 0 & 0 & e_{33} \\ 0 & 0 & b_{44} & e_{24} & 0 \\ 0 & 0 & e_{24} & -\beta_{22} & 0 \\ e_{32} & e_{33} & 0 & 0 & -\beta_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ -E_2 \\ -E_3 \end{bmatrix} + \begin{bmatrix} a_{22} & a_{23} & 0 & 0 & 0 \\ a_{23} & a_{33} & 0 & 0 & 0 \\ 0 & 0 & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\varepsilon}_{22} \\ \dot{\varepsilon}_{33} \\ 2\dot{\varepsilon}_{23} \\ -E_2 \\ -E_3 \end{bmatrix}. \tag{5.3}$$

Here $\dot{\varepsilon}_{ij} = \dfrac{1}{2}\left(\dfrac{\partial \dot{u}_i}{\partial x_j} + \dfrac{\partial \dot{u}_j}{\partial x_i}\right)$ and note that equation (5.3) is obtained by the identification

$$b_{ijkl} \equiv b_{pq} = \begin{pmatrix} b_{22} & b_{23} & 0 \\ b_{23} & b_{33} & 0 \\ 0 & 0 & b_{44} \end{pmatrix}, \quad a_{ijkl} \equiv a_{pq} = \begin{pmatrix} a_{22} & a_{23} & 0 \\ a_{23} & a_{33} & 0 \\ 0 & 0 & a_{44}, \end{pmatrix},$$

with the rule

$$\begin{aligned} ij \text{ or } kl = 22 \quad &\longrightarrow \quad p \text{ or } q = 2, \\ ij \text{ or } kl = 33 \quad &\longrightarrow \quad p \text{ or } q = 3, \\ ij \text{ or } kl = 23 \text{ or } 32 \quad &\longrightarrow \quad p \text{ or } q = 4. \end{aligned}$$

This rule, which allows to describe the link between the fourth-order tensors of components b_{ijkl} and a_{ijkl} and the corresponding second-order tensors of components b_{pq} and a_{pq}, respectively, is obtained by using the symmetries of the various tensors involved in the constitutive law. In the same way, for the third order piezoelectric tensor we have

$$e_{ijk} \equiv e_{iq} = \begin{pmatrix} 0 & 0 & e_{24} \\ e_{32} & e_{33} & 0 \end{pmatrix} \quad \text{with} \quad \begin{array}{rcl} jk = 22 & \longrightarrow & q = 2, \\ jk = 33 & \longrightarrow & q = 3, \\ jk = 23 \text{ or } 32 & \longrightarrow & q = 4. \end{array}$$

We use the material constants given in Tables 1 and 2, in which $\epsilon_0 8.885 \times 10^{-12} C^2/Nm^2$ represents the permittivity constant of the vacuum.

Elastic (GPa)				Viscoelastic $(GPa \cdot s)$			
b_{22}	b_{23}	b_{33}	b_{44}	a_{22}	a_{23}	a_{33}	a_{44}
210	105	211	42.5	2.1	1.05	2.11	0.425

Table 1: Elastic and viscoelastic constants of the piezoelectric body.

Piezoelectric (C/m^2)			Permittivity (C^2/Nm^2)	
e_{32}	e_{33}	e_{24}	β_{22}/ϵ_0	β_{33}/ϵ_0
-0.61	1.14	-0.59	-8.3	-8.8

Table 2: Electric constants of the piezoelectric body.

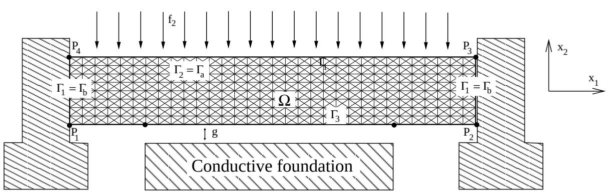

Figure 1: Physical setting of MEMS : an electroelastic body in contact with a conductive obstacle.

As a two-dimensional example, we consider the physical setting depicted in Figure 1, where $\Omega = [0, 12] \times [0, 2]$, $\Gamma_1 = \Gamma_b = (\{0\} \times [0, 2]) \cup (\{12\} \times [0, 2])$, $\Gamma_2 = \Gamma_a = ([0, 12] \times \{2\}) \cup ([0, 2] \times \{0\}) \cup ([10, 12] \times \{0\})$, and the potential contact surface is $\Gamma_3 = [2, 10] \times \{0\}$. The body is subjected to the action of surface pression $\boldsymbol{f}_2 = (0, -5) N/\mu m$ which acts on the top of the bridge, i.e. on $[0, 12] \times \{2\}$; the body forces and electric charges vanish, i.e. $\boldsymbol{f}_0 = \mathbf{0}\, N/\mu m^2$, $q_0 = 0\, C/\mu m^2$ and $q_b = 0\, C/\mu m$; and the gap between the body and the foundation is $g = 0.5 \mu m$. The functions h_r and p_r ($r = \nu, \tau$) in the frictional contact conditions (2.9) and (2.10) are given by

$$h_r(s) = c_r \times \begin{cases} \alpha_r & \text{if } |s| > 128, \\ 1 + (\alpha_r - 1) \times \frac{|s|}{128} & \text{if } |s| \leq 128, \end{cases}$$

$$p_r(s) = \begin{cases} 0 & \text{if } s < 0, \\ s & \text{if } 0 \leq s \leq n_\nu, \\ n_r & \text{if } s > n_\nu, \end{cases}$$

where c_r, α_r and n_r are positive constants, $\alpha_r > 1$. And, finally, for the regularized electrical condition (2.11) we take

$$h_e(s) = \begin{cases} -m_e & \text{if } s < -m_e, \\ s & \text{if } -m_e \leq s \leq m_e, \\ m_e & \text{if } s > m_e \end{cases} \qquad p_e(s) = k_e \times \begin{cases} 0 & \text{if } s < 0, \\ \frac{s}{\epsilon_e} & \text{if } 0 \leq s \leq \epsilon_e, \\ 1 & \text{if } s > \epsilon_e, \end{cases}$$

where m_e, k_e and ϵ_e are positive constants.

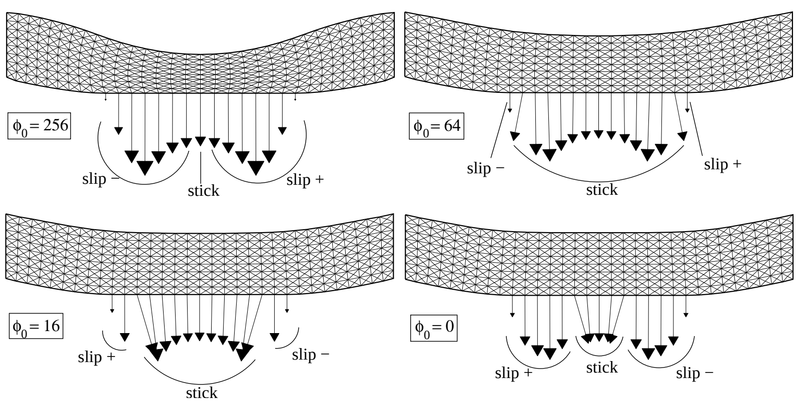

Figure 2: Sequence of deformed meshes and corresponding contact forces.

Our interest in this piezoelectric contact model is to study the influence of the electric potential of the foundation on the process. Our results are presented in Figures 2–6, in which we use the notation $\phi_0 = -\varphi_0$ and $k = k_e$.

In Figure 2 we plot a sequence of deformed meshes with the corresponding contact interface forces and the contact status, obtained for four different values of the electric potential of the foundation: $\varphi_0 = -\phi_0$, where ϕ_0 takes successively the values 256, 64, 16 and 0. It results from the figure that the deformations and the magnitude of the contact forces decrease when ϕ_0 decreases, i.e. when the magnitude of the electric potential of the foundation decreases.

According to Figure 3 we note that, for k given, the magnitude of the normal electric displacement increases with ϕ_0. A similar behavior follows from Figure 4 which shows that, for a given ϕ_0, the magnitude of the normal electric displacement increases with the electrical conductivity coefficient k. These results are compatible with the electrical boundary condition we use on the contact surface and show the effect of the conductivity of the foundation on the process.

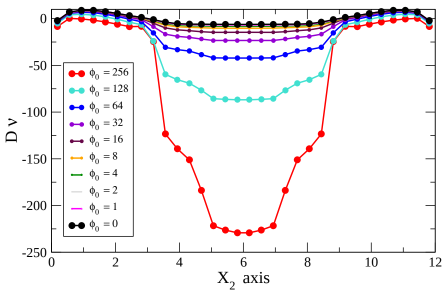

Figure 3: Dependence of the normal electric displacement $\boldsymbol{D}\cdot\boldsymbol{\nu}$ with respect to ϕ_0, for $k = 1$.

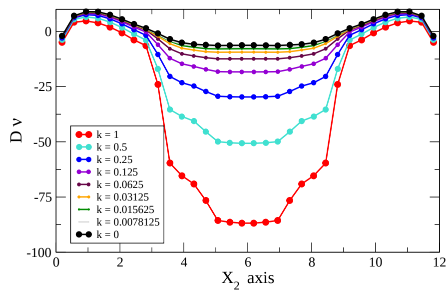

Figure 4: Dependence of the normal electric displacement $\boldsymbol{D}\cdot\boldsymbol{\nu}$ with respect to k, for $\phi_0 = 128$.

Finally, Figure 5 shows the electric potential in the body whereas Figure 6 represents the electric displacement fields in the deformed configuration, for four different values of the potential of the foundation, corresponding to $\phi_0 = 256$, $\phi_0 = 64$, $\phi_0 = 16$ and $\phi_0 = 0$. According to Figures 5 and 6, we note that the magnitude of the electric potential and the magnitude of the electric displacement increase on the contact interface, when the magnitude ϕ_0 of the potential of the foundation increases.

We conclude that our simulations above underline the effects of the electrical conductivity of the foundation on the frictional contact process. Performing these simulations we found that the numerical solution worked well

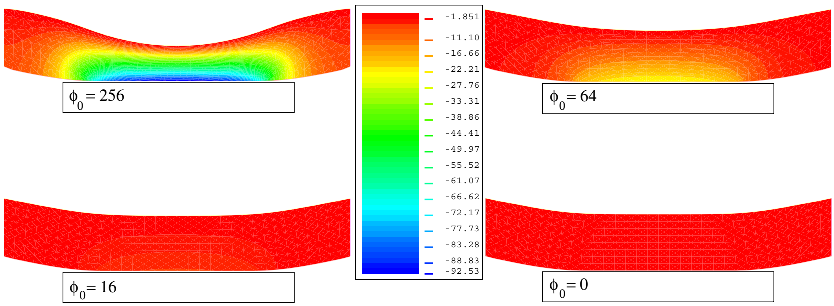

Figure 5: Sequence of deformed meshes and corresponding electric potential.

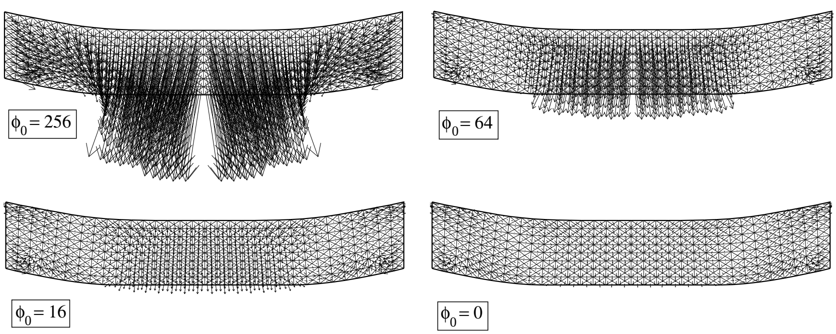

Figure 6: Sequence of deformed meshes and corresponding electric displacement fields.

and the convergence was rapid.

References

[1] A. Arora, A. Arora, V.K. Dwivedi, P.J. George, K. Sreenivas, V. Gupta. Zinc oxide thin film-based MEMS acoustic sensor with tunnel for pressure compensation. *Sensors and Actuators A: Physical.* 141:256-261, 2008.

[2] P. Alart, A. Curnier. A mixed formulation for frictional contact problems prone to Newton like solution methods. *Comput. Methods Appl. Mech. Engrg.* 92:353-375, 1991.

[3] M. Barboteu, M. Sofonea. Solvability of a Dynamic Contact Problem between a Piezoelectric Body and a Conductive Foundation. Submitted in *Applied Mathematics and Computation.*

[4] M. Barboteu, M. Sofonea. Modelling and Analysis of the Unilateral Contact of a Piezoelectric Body with a Conductive Support. *Journal of Mathematical Analysis and Applications*, 2009 (to appear).

[5] M. Barboteu, M. Sofonea. Modelling and analysis of piezoelectric contact. In *Prceedings of International Conference on Systems Theory : Modelling, Analysis and Control*, Fes, Maroc, 2009.

[6] R.C. Batra, J.S. Yang. Saint-Venant's principle in linear piezoelectricity. *Journal of Elasticity.* 38:209-218, 1995.

[7] P. Bisegna, F. Lebon, F. Maceri, The unilateral frictional contact of a piezoelectric body with a rigid support. In *Contact Mechanics*, J.A.C. Martins and Manuel D.P. Monteiro Marques (Eds.) Kluwer, 2002, 347-354.

[8] F. M. Guo, Z. Q. Zhu, Y. F. Long, W. M. Wang, S. Z. Zhu, Z. S. Lai, N. Li, G. Q. Yang, W. Lu. Study on low voltage actuated MEMS rf capacitive switches. *Sensors and Actuators A: Physical.* 108:128-133, 2003.

[9] W. Han, M. Sofonea. *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity.* Americal Mathematical Society, Providence, RI–International Press, Somerville, MA, 2002.

[10] T. Ikeda. *Fundamentals of Piezoelectricity.* Oxford University Press, Oxford, 1990.

[11] Z. Lerguet, M. Shillor, M. Sofonea. A Frictional Contact Problem for an Electro-Viscoelastic Body. *Electronic Journal of Differential Equations.* 170:1-16, 2007.

[12] F. Maceri, P. Bisegna. The unilateral frictionless contact of a piezoelectric body with a rigid support. *Math. Comp. Modelling.* 28:19-28, 1998.

[13] S. Migórski. Hemivariational inequality for a frictional contact problem in elasto-piezoelectric. *Discrete and Continous Dynamical Systems - Series B.* 6:1339–1356, 2006.

[14] V. Z. Patron, B. A. Kudryavtsev. *Electromagnetoelasticity, Piezoelectrics and Electrically Conductive Solids*, Gordon & Breach, London, 1988.

[15] B. Schauwecker, J. Mehner, K. M. Strohm, H. Haspeklo, J.-F. Luy. Investigations of rf shunt airbridge switches among different environmental conditions. *Sensors and Actuators A: Physical.* 114:49-58, 2004.

[16] M. Shillor, M. Sofonea, J.J. Telega. *Models and Analysis of Quasistatic Contact.* Lect. Notes Phys. **655**, Springer, Berlin Heidelberg, 2004.

[17] M. Sofonea, El H. Essoufi. Quasistatic frictional contact of a viscoelastic piezoelectric body. *Adv. Math. Sci. Appl.* 14:613-631, 2004.

[18] M. Sofonea, A. Matei. Fixed Point Result for Operators Defined on Spaces of Vector Valued Continuous Functions. *Ann. Univ. Craiova, Math–Info.* 29:19-22, 2002.

[19] P. Wriggers. *Computational Contact Mechanics.* Wiley, Chichester, 2002.