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TOEPLITZ OPERATORS WITH BOUNDED HARMONIC SYMBOLS*

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Abstract

In this paper we have shown that if $\phi \in h^\infty(\mathbb{D})$ and $T_\phi^{(\alpha)}$ is the Toeplitz operator with symbol ϕ defined on the weighted Bergman space $L_a^2(dA_\alpha)$ and if the set $\left\{\left(T_\phi^{(\alpha)}\right)^* T_\phi^{(\alpha)} f, \left(T_\phi^{(\alpha)}\right)^* f, T_\phi^{(\alpha)} f, f\right\}$ is linearly dependent for all $f \in L_a^2(dA_\alpha)$ then either ϕ is a constant function or there exists $\lambda_\alpha, \mu_\alpha \in \mathbb{C}$ such that $\frac{\phi-\mu_\alpha}{\lambda_\alpha}$ is a real-valued function in $h^\infty(\mathbb{D})$. Here $h^\infty(\mathbb{D})$ is the set of all bounded harmonic functions on the open unit disk $\mathbb{D}$.

MSC: 47B38, 47B32

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1 Introduction

Let $dA(z) = \frac{1}{\pi}dxdy$ be the area measure on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$. It is normalized so that the area of $\mathbb{D}$ is 1. For $\alpha > -1$, let $L^2(\mathbb{D}, dA_\alpha)$ be the space consisting of all absolutely square-integrable, Lebesgue measurable functions on $\mathbb{D}$ with respect to the measure $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z),\ z \in \mathbb{D}$. The measure dA_α is a probability measure on $\mathbb{D}$. Let $L_a^2(dA_\alpha)$ be the subspace of all analytic functions of $L^2(\mathbb{D}, dA_\alpha)$. The space $L_a^2(dA_\alpha)$ is called the weighted

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Bergman space of the disk $\mathbb{D}$. The space $L_a^2(dA_\alpha)$ is a closed subspace of the Hilbert space $L^2(\mathbb{D}, dA_\alpha)$ with respect to the inner product defined by $\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}dA_\alpha(z),\ f, g \in L^2(\mathbb{D}, dA_\alpha)$. We shall denote $L_a^2(dA_0) = L_a^2(dA)$, the unweighted Bergman space. The reproducing kernel of $L_a^2(\mathbb{D})$ is given by $K(z,w) = \frac{1}{(1-z\overline{w})^2}$. Let $K_z(w) = \overline{K(z,w)}$. The reproducing kernel of $L_a^2(dA_\alpha)$ is given by $K^{(\alpha)}(z,w) = \frac{1}{(1-z\overline{w})^{\alpha+2}}$ for $z, w \in \mathbb{D}$. Thus $K^{(\alpha)}(z,w) = [K(z,w)]^{1+\frac{\alpha}{2}}$. Let $K_z^{(\alpha)}(w) = [K_z(w)]^{1+\frac{\alpha}{2}} = \overline{K^{(\alpha)}(z,w)}$. The orthogonal projection P_α from space $L^2(\mathbb{D}, dA_\alpha)$ onto the space $L_a^2(dA_\alpha)$ is given by $P_\alpha f(z) = \int_{\mathbb{D}} K^{(\alpha)}(z,w)f(w)dA_\alpha(w)$. Let $k_z(w) = \frac{(1-|z|^2)}{(1-w\overline{z})^2},\ z, w \in \mathbb{D}$. The functions $k_z^{1+\frac{\alpha}{2}}(w) = \left[\frac{(1-|z|^2)}{(1-w\overline{z})^2}\right]^{1+\frac{\alpha}{2}} = \frac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-w\overline{z})^{2+\alpha}},\ z, w \in \mathbb{D}$ are the normalized reproducing kernels of the space $L_a^2(dA_\alpha)$. Let $L^\infty(\mathbb{D}, dA)$ denote the Banach space of Lebesgue measurable functions f on $\mathbb{D}$ with $\|f\|_\infty = ess\ sup\{|f(z)| : z \in \mathbb{D}\} < \infty$ and $H^\infty(\mathbb{D})$ be the space of bounded analytic functions on $\mathbb{D}$. Let $h^\infty(\mathbb{D})$ be the space of all bounded harmonic functions on $\mathbb{D}$.

For any $z \in \mathbb{D}$, let ϕ_z be the analytic mapping on $\mathbb{D}$ defined by $\phi_z(w) = \frac{z-w}{1-\overline{z}w}, w \in \mathbb{D}$. An easy calculation shows [5] that the derivative of ϕ_z at w is equal to $-k_z(w)$. It follows that the real Jacobian determinant of ϕ_z at w is $J_{\phi_z}(w) = |k_z(w)|^2 = \frac{(1-|z|^2)^2}{|1-\overline{z}w|^4}$. Given a function $\phi \in L^\infty(\mathbb{D})$, we define the Toeplitz operator $T_\phi^{(\alpha)}$ on the space $L_a^2(dA_\alpha)$ by $T_\phi^{(\alpha)} f = P_\alpha(\phi f),\ f \in L_a^2(dA_\alpha)$. The operator $T_\phi^{(\alpha)}$ is called the Toeplitz operator with symbol ϕ. Since $\|P_\alpha\| \leq 1$, hence $\|T_\phi^{(\alpha)}\| \leq \|\phi\|_\infty$. The Toeplitz operator $T_\phi^{(\alpha)}$ can also be written as,

$$T_\phi^{(\alpha)} f(z) = \int_{\mathbb{D}} \phi(w)K^{(\alpha)}(z,w)f(w)dA_\alpha(w) = \int_{\mathbb{D}} \frac{\phi(w)f(w)}{(1-z\overline{w})^{\alpha+2}}dA_\alpha(w).$$

Let $\mathcal{L}\left(L_a^2(dA_\alpha)\right)$ be the space of all bounded linear operators from the weighted Bergman space $L_a^2(dA_\alpha)$ into itself. An operator $T \in \mathcal{L}\left(L_a^2(dA_\alpha)\right)$, the numerical range $W(T)$ of T is defined by

$$W(T) = \left\{\langle Tf, f\rangle : f \in L_a^2(dA_\alpha), \|f\| = 1\right\}.$$

The numerical radius of $T \in \mathcal{L}\left(L_a^2(dA_\alpha)\right)$, denoted by $w(T)$, is defined by $w(T) = \sup\{|\lambda| : \lambda \in W(T)\}$. It is well-known that $w(\cdot)$ defines a norm on $L_a^2(dA_\alpha)$, and is equivalent to the usual operator norm given by, $\|T\| =$

$\sup\{\|Tf\| : f \in L^2_a(dA_\alpha), \|f\| = 1\}$. In fact, for every $T \in \mathcal{L}\left(L^2_a(dA_\alpha)\right)$,

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|. \tag{1.1}$$

For details see [3]. Define $B_\alpha : \mathcal{L}(L^2_a(dA_\alpha)) \longrightarrow L^\infty(\mathbb{D})$ by $B_\alpha(T)(z) = \left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\rangle$, $z \in \mathbb{D}$. Since

$$|B_\alpha(T)(z)| = \left|\left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\rangle\right| \leq \|T\| \left\|k_z^{1+\frac{\alpha}{2}}\right\|^2 = \|T\|,$$

hence $\|B_\alpha(T)\| \leq \|T\|$. The map B_α is linear and one-one. For more details refer [4].

It is not so difficult to see that if $\phi \in H^\infty(\mathbb{D})$, $z \in \mathbb{D}$ then $\left(T_\phi^{(\alpha)}\right)^* k_z^{1+\frac{\alpha}{2}}$ and $k_z^{1+\frac{\alpha}{2}}$ are linearly dependent. In this work, we have established that if $\phi \in h^\infty(\mathbb{D})$ and if the set $\left\{\left(T_\phi^{(\alpha)}\right)^* T_\phi^{(\alpha)} f, \left(T_\phi^{(\alpha)}\right)^* f, T_\phi^{(\alpha)} f, f\right\}$ is linearly dependent for all $f \in L^2_a(dA_\alpha)$ then either ϕ is a constant function or there exists $\lambda_\alpha, \mu_\alpha \in \mathbb{C}$ such that $\frac{\phi - \mu_\alpha}{\lambda_\alpha}$ is a real-valued harmonic function in $h^\infty(\mathbb{D})$.

The plan layout of this paper is as follows. In section 2, we proved some preliminary lemmas. We showed that if $z \in \mathbb{D}$ and Θ_z is the projection onto *span* $\left\{k_z^{1+\frac{\alpha}{2}}\right\}$ then the operator $T \in \mathcal{L}\left(L^2_a(dA_\alpha)\right)$ is normal if and only if $\|(I - \Theta_z)T\Theta_z\| = \|(I - \Theta_z)T^*\Theta_z\|$ for all $z \in \mathbb{D}$. Further, we have shown that if $T \in \mathcal{L}\left(L^2_a(dA_\alpha)\right)$ is a normal operator, $\Theta_{z,T}$ is the projection onto *span* $\left\{k_z^{1+\frac{\alpha}{2}}, Tk_z^{1+\frac{\alpha}{2}}\right\}$ and $\|(I - \Theta_{z,T})T\Theta_{z,T}\| = \|(I - \Theta_{z,T})T^*\Theta_{z,T}\|$ for all $z \in \mathbb{D}$, then the set $\{T^*Tk_z^{1+\frac{\alpha}{2}}, T^*k_z^{1+\frac{\alpha}{2}}, Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\}$ is linearly dependent for all $z \in \mathbb{D}$. In section 3, we established the main results of the paper. We have shown that if $T \in \mathcal{L}\left(L^2_a(dA_\alpha)\right)$ is normal and if the set of vectors $\{T^*Tf, T^*f, Tf, f\}$ is linearly dependent for all $f \in L^2_a(dA_\alpha)$, then $B_\alpha(T)(z) = \lambda_\alpha\psi_\alpha(z) + \mu_\alpha$ where $\lambda_\alpha, \mu_\alpha \in \mathbb{C}$ and either $\psi_\alpha = B_\alpha(R)$, $R \in \mathcal{L}\left(L^2_a(dA_\alpha)\right)$ is a self-adjoint operator or $\psi_\alpha = B_\alpha(e^{iK})$ where $K \in \mathcal{L}\left(L^2_a(dA_\alpha)\right)$ is a self-adjoint operator. As a consequence of this result we showed that if $\phi \in h^\infty(\mathbb{D})$, $T_\phi^{(\alpha)}$ is the Toeplitz operator with symbol ϕ defined on $L^2_a(dA_\alpha)$ and if the set $\left\{\left(T_\phi^{(\alpha)}\right)^* T_\phi^{(\alpha)} f, \left(T_\phi^{(\alpha)}\right)^* f, T_\phi^{(\alpha)} f, f\right\}$ is linearly dependent for all $f \in L^2_a(dA_\alpha)$ then either ϕ is a constant function or there exists $\lambda_\alpha, \mu_\alpha \in \mathbb{C}$ such that $\frac{\phi - \mu_\alpha}{\lambda_\alpha}$ is a real-valued harmonic function in $h^\infty(\mathbb{D})$.

2 Preliminaries

In this section, we proved some preliminary lemmas that are needed to prove the main result of the paper. In the following lemma, we shall show that if $\phi \in H^\infty(\mathbb{D})$, then $T^{(\alpha)}_{\overline{\phi}} k_z^{1+\frac{\alpha}{2}}$ and $k_z^{1+\frac{\alpha}{2}}$ are linearly dependent.

Lemma 2.1. *For any $\phi \in H^\infty(\mathbb{D})$ and $z \in \mathbb{D}$, $T^{(\alpha)}_{\overline{\phi}} k_z^{1+\frac{\alpha}{2}} = \overline{\phi(z)} k_z^{1+\frac{\alpha}{2}}$.*

Proof. Notice that for any $g \in L^2_a(dA_\alpha)$ and $z \in \mathbb{D}$, we have $g(z) = \int_{\mathbb{D}} K^{(\alpha)}(z,w) g(w) dA_\alpha(w)$. Further the Toeplitz operator $T^{(\alpha)}_{\overline{\phi}}$ is an integral operator and is given by

$$\left(T^{(\alpha)}_{\overline{\phi}} f\right)(z) = \int_{\mathbb{D}} K^{(\alpha)}(z,w)\overline{\phi(w)} f(w) dA_\alpha(w)$$

for $f \in L^2_a(dA_\alpha)$. Now since $\phi \in H^\infty(\mathbb{D})$, we obtain

$$\begin{aligned}
\left(T^{(\alpha)}_{\overline{\phi}} K^{(\alpha)}(\cdot, z)\right)(w) &= \int_{\mathbb{D}} K^{(\alpha)}(w,v) K^{(\alpha)}(v,z) \overline{\phi}(v) dA_\alpha(v) \\
&= \overline{\int_{\mathbb{D}} \overline{K^{(\alpha)}(w,v) K^{(\alpha)}(v,z)} \phi(v) dA_\alpha(v)} \\
&= \overline{\int_{\mathbb{D}} K^{(\alpha)}(v,w) K^{(\alpha)}(z,v) \phi(v) dA_\alpha(v)} \\
&= \overline{K^{(\alpha)}(z,w)\phi(z)} \\
&= \overline{\phi(z)} K^{(\alpha)}(w,z),
\end{aligned}$$

and so $T^{(\alpha)}_{\overline{\phi}} K^{(\alpha)}(\cdot, z) = \overline{\phi(z)} K^{(\alpha)}(\cdot, z)$
Dividing both sides by $\sqrt{K^{(\alpha)}(z,z)}$, we obtain $T^{(\alpha)}_{\overline{\phi}} k_z^{1+\frac{\alpha}{2}} = \overline{\phi(z)} k_z^{1+\frac{\alpha}{2}}$. The result follows. □

Lemma 2.2. *Fix $\alpha > -1$. Let $z \in \mathbb{D}$ and Θ_z be the projection onto the span $\left\{k_z^{1+\frac{\alpha}{2}}\right\}$. If $T \in \mathcal{L}\left(L^2_a(dA_\alpha)\right)$ then*

(i) $\|(I - \Theta_z) T \Theta_z\|^2 = \left\|T k_z^{1+\frac{\alpha}{2}}\right\|^2 - \left|\left\langle T k_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\rangle\right|^2 = \left\|T k_z^{1+\frac{\alpha}{2}}\right\|^2 - |B_\alpha(T)(z)|^2.$

(ii) If $w(T) \leq \frac{1}{2}$, then

$$\|(I-\Theta_z)T\Theta_z\| \leq \left(\sqrt{1-\left|\left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\rangle\right|^2}+1\right)^2 = \left(1+\sqrt{1-|B_\alpha(T)(z)|^2}\right)^2.$$

Proof. (i) Notice that

$$\|(I-\Theta_z)T\Theta_z\| = \sup_{\|f\|=1} \|(I-\Theta_z)T\Theta_z f\|.$$

Let $f = \delta k_z^{1+\frac{\alpha}{2}} + h,\ \delta \in \mathbb{C}$ and where $\left\langle h, k_z^{1+\frac{\alpha}{2}}\right\rangle = 0$. Then

$$\|f\|^2 = \langle f, f\rangle = \left\langle \delta k_z^{1+\frac{\alpha}{2}} + h, \delta k_z^{1+\frac{\alpha}{2}} + h\right\rangle = |\delta|^2 + \|h\|^2$$

as $\left\|k_z^{1+\frac{\alpha}{2}}\right\| = 1$. Thus

$$\|(I-\Theta_z)T\Theta_z f\| = |\delta| \left\|(I-\Theta_z)Tk_z^{1+\frac{\alpha}{2}}\right\|.$$

For any vector $f_1 \in L_a^2(dA_\alpha),\ \Theta_z f_1 = \langle f_1, k_z^{1+\frac{\alpha}{2}}\rangle k_z^{1+\frac{\alpha}{2}}$ and hence

$$\|(I-\Theta_z)T\Theta_z f\|^2 = |\delta|^2 \left\|Tk_z^{1+\frac{\alpha}{2}} - \left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\rangle k_z^{1+\frac{\alpha}{2}}\right\|^2.$$

Let $Tk_z^{1+\frac{\alpha}{2}} = \beta k_z^{1+\frac{\alpha}{2}} + b$ where $\left\langle b, k_z^{1+\frac{\alpha}{2}}\right\rangle = 0$. Then

$$\left\|Tk_z^{1+\frac{\alpha}{2}}\right\|^2 = |\beta|^2 + \|b\|^2 \ \text{ or } \ \|b\|^2 = \left\|Tk_z^{1+\frac{\alpha}{2}}\right\|^2 - |\beta|^2.$$

But $|\beta|^2 = \left|\left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\rangle\right|^2$ and $b = Tk_z^{1+\frac{\alpha}{2}} - \left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\rangle k_z^{1+\frac{\alpha}{2}}$. So

$$\begin{aligned}\|(I-\Theta_z)T\Theta_z\|^2 &= \sup_{|\delta|\leq 1}\left(\left\|Tk_z^{1+\frac{\alpha}{2}}\right\|^2 - \left|\left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\rangle\right|^2\right)\\ &= \left\|Tk_z^{1+\frac{\alpha}{2}}\right\|^2 - \left|\left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\rangle\right|^2\\ &= \left\|Tk_z^{1+\frac{\alpha}{2}}\right\|^2 - |B_\alpha(T)(z)|^2.\end{aligned}$$

(ii) Multiplying T by a unimodular scalar, we may assume that

$$\left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle \geq 0.$$

Let $C = \frac{T+T^*}{2}$ and $D = \frac{T-T^*}{2i}$. Then by using triangle inequality, we obtain

$$\begin{aligned}\left\| Tk_z^{1+\frac{\alpha}{2}} - \left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle k_z^{1+\frac{\alpha}{2}} \right\| &\leq \left\| Ck_z^{1+\frac{\alpha}{2}} - \left\langle Ck_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle k_z^{1+\frac{\alpha}{2}} \right\| \\ &+ \left\| Dk_z^{1+\frac{\alpha}{2}} - \left\langle Dk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle k_z^{1+\frac{\alpha}{2}} \right\|.\end{aligned}$$

But

$$\begin{aligned}&\left\| Tk_z^{1+\frac{\alpha}{2}} - \left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle k_z^{1+\frac{\alpha}{2}} \right\|^2 \\ &= \left\langle Tk_z^{1+\frac{\alpha}{2}} - \left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle k_z^{1+\frac{\alpha}{2}}, Tk_z^{1+\frac{\alpha}{2}} - \left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle k_z^{1+\frac{\alpha}{2}} \right\rangle \\ &= \left\| Tk_z^{1+\frac{\alpha}{2}} \right\|^2 - \overline{\left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle} \left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle \\ &- \left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle \left\langle k_z^{1+\frac{\alpha}{2}}, Tk_z^{1+\frac{\alpha}{2}} \right\rangle + \left| \left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle \right|^2 \left\| k_z^{1+\frac{\alpha}{2}} \right\|^2 \\ &= \left\| Tk_z^{1+\frac{\alpha}{2}} \right\|^2 - \left| \left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle \right|^2 - \left| \left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle \right|^2 + \left| \left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle \right|^2 \\ &= \left\| Tk_z^{1+\frac{\alpha}{2}} \right\|^2 - \left| \left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle \right|^2\end{aligned}$$

and this also holds for the operators C and D. Further, the operators $C = ReT$, $D = ImT$ are self-adjoint operators and have numerical radius at most $\frac{1}{2}$. This implies by (1.1), $\|T\| \leq 1$. Thus $\|C\| \leq 1$ and $\|D\| \leq 1$. Again $\left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle \geq 0$ implies that $\left\langle Ck_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle = \left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle$ and $\left\langle Dk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle = 0$. Hence

$$\begin{aligned}\left\| Ck_z^{1+\frac{\alpha}{2}} - \left\langle Ck_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle k_z^{1+\frac{\alpha}{2}} \right\|^2 &= \left\| Ck_z^{1+\frac{\alpha}{2}} \right\|^2 - \left| \left\langle Ck_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle \right|^2 \\ &\leq 1 - \left| \left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle \right|^2\end{aligned}$$

and

$$\left\| Dk_z^{1+\frac{\alpha}{2}} - \left\langle Dk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle k_z^{1+\frac{\alpha}{2}} \right\|^2 = \left\| Dk_z^{1+\frac{\alpha}{2}} \right\|^2 - \left| \left\langle Dk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle \right|^2 \leq 1.$$

Combining these inequalities, we obtain

$$\sqrt{\left\|Tk_z^{1+\frac{\alpha}{2}}\right\|^2 - \left|\left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\rangle\right|^2} = \sqrt{\left\|Tk_z^{1+\frac{\alpha}{2}} - \left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\rangle k_z^{1+\frac{\alpha}{2}}\right\|^2}$$
$$= \left\|Tk_z^{1+\frac{\alpha}{2}} - \left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\rangle k_z^{1+\frac{\alpha}{2}}\right\|$$
$$\leq \left\|Ck_z^{1+\frac{\alpha}{2}} - \left\langle Ck_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\rangle k_z^{1+\frac{\alpha}{2}}\right\| + \left\|Dk_z^{1+\frac{\alpha}{2}} - \left\langle Dk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\rangle k_z^{1+\frac{\alpha}{2}}\right\|$$
$$\leq \sqrt{1 - \left|\left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\rangle\right|^2} + 1$$
$$= 1 + \sqrt{1 - |B_\alpha(T)(z)|^2}.$$

□

Lemma 2.3. *Let for $z \in \mathbb{D}$, Θ_z be the projection onto span $\left\{k_z^{1+\frac{\alpha}{2}}\right\}$. The operator $T \in \mathcal{L}\left(L_a^2(dA_\alpha)\right)$ is normal if and only if*

$$\|(I - \Theta_z)T\Theta_z\| = \|(I - \Theta_z)T^*\Theta_z\|$$

for all $z \in \mathbb{D}$.

Proof. By Lemma 2.2,

$$\|(I - \Theta_z)T\Theta_z\| = \|(I - \Theta_z)T^*\Theta_z\| \qquad \text{for all } z \in \mathbb{D} \tag{2.1}$$

if and only if

$$\left\|Tk_z^{1+\frac{\alpha}{2}}\right\|^2 - \left|\left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\rangle\right|^2 = \left\|T^*k_z^{1+\frac{\alpha}{2}}\right\|^2 - \left|\left\langle T^*k_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\rangle\right|^2$$

for all $z \in \mathbb{D}$. That is, if and only if $\left\|Tk_z^{1+\frac{\alpha}{2}}\right\|^2 = \left\|T^*k_z^{1+\frac{\alpha}{2}}\right\|^2$ for all $z \in \mathbb{D}$. But $k_z^{1+\frac{\alpha}{2}} = \frac{K_z^{(\alpha)}}{\|K_z^{(\alpha)}\|}$. Thus (2.1) holds if and only if $\left\|TK_z^{(\alpha)}\right\| = \left\|T^*K_z^{(\alpha)}\right\|$ for all $z \in \mathbb{D}$. That is, if and only if

$$\left\langle T^*TK_z^{(\alpha)}, K_z^{(\alpha)}\right\rangle = \left\langle TT^*K_z^{(\alpha)}, K_z^{(\alpha)}\right\rangle \qquad \text{for all } z \in \mathbb{D}. \tag{2.2}$$

But (2.2) holds if and only if

$$\left\langle T^*T\left(\sum_{j=1}^n c_jK_{z_j}^{(\alpha)}\right), \sum_{j=1}^n c_jK_{z_j}^{(\alpha)}\right\rangle = \left\langle TT^*\left(\sum_{j=1}^n c_jK_{z_j}^{(\alpha)}\right), \sum_{j=1}^n c_jK_{z_j}^{(\alpha)}\right\rangle$$

for all $z_1, z_2, ..., z_n \in \mathbb{D}$ and $c_1, c_2, ..., c_n \in \mathbb{C}$. Since the set of vectors $\left\{\sum_{j=1}^{n} c_j K_{z_j}^{(\alpha)} : c_1, c_2, ..., c_n \in \mathbb{C}, z_1, z_2, ..., z_n \in \mathbb{D}\right\}$ is dense in $L_a^2(dA_\alpha)$, hence (2.2) holds if and only if

$$\langle T^*Tf, f\rangle = \langle TT^*f, f\rangle \qquad \text{for all } f \in L_a^2(dA_\alpha). \tag{2.3}$$

But (2.3) holds if and only if T is normal. □

Lemma 2.4. *Let $f_1, f_2 \in L_a^2(dA_\alpha)$ be such that $\|f_1\| = \|f_2\| = 1$ and $\langle f_1, f_2\rangle = 0$. Let $T \in \mathcal{L}\left(L_a^2(dA_\alpha)\right)$. Let Θ_{f_1,f_2} be the projection onto span$\{f_1, f_2\}$. Let $(I - \Theta_{f_1,f_2})Tf_1 = p$ and $(I - \Theta_{f_1,f_2})Tf_2 = q$. Let $A = \|p\|^2$, $B = \|q\|^2$ and $C = |\langle p, q\rangle|$. Then*

$$\|(I - \Theta_{f_1,f_2})T\Theta_{f_1,f_2}\|^2 = \frac{1}{2}[(A + B) + \sqrt{4C^2 + (A - B)^2}].$$

Proof. We shall calculate the norm $\|(I - \Theta_{f_1,f_2})T\Theta_{f_1,f_2}\|$. This is obviously attained by vectors in *span*$\{f_1, f_2\}$. Let $g = \alpha_1 f_1 + \beta_1 f_2 \in span\{f_1, f_2\} = Range\ \Theta_{f_1,f_2}$. Assume $\|g\| = 1$. That is, $|\alpha_1|^2 + |\beta_1|^2 = 1$. Then $\Theta_{f_1,f_2} g = g$ and

$$\begin{aligned}
&(I - \Theta_{f_1,f_2})T\Theta_{f_1,f_2} g = (I - \Theta_{f_1,f_2})Tg = Tg - \langle Tg, f_1\rangle f_1 - \langle Tg, f_2\rangle f_2 \\
&= \alpha_1[Tf_1 - \langle Tf_1, f_1\rangle f_1 - \langle Tf_1, f_2\rangle f_2] + \beta_1[Tf_2 - \langle Tf_2, f_1\rangle f_1 - \langle Tf_2, f_2\rangle f_2] \\
&= \alpha_1 p + \beta_1 q.
\end{aligned}$$

Hence

$$\begin{aligned}
\|(I - \Theta_{f_1,f_2})T\Theta_{f_1,f_2}\|^2 &= \sup_{|\alpha_1|^2+|\beta_1|^2=1} \|\alpha p + \beta q\|^2 \\
&= \sup_{|\alpha_1|^2+|\beta_1|^2=1} \left[|\alpha_1|^2\|p\|^2 + |\beta_1|^2\|q\|^2 + \alpha_1\overline{\beta_1}\langle p, q\rangle + \overline{\alpha_1}\beta_1\langle q, p\rangle\right] \\
&= \sup_{|\alpha_1|^2+|\beta_1|^2=1} \left[|\alpha_1|^2\|p\|^2 + |\beta_1|^2\|q\|^2 + 2Re(\alpha_1\overline{\beta_1}\langle p, q\rangle)\right].
\end{aligned}$$

Now the supremum will be attained for those values of α_1 and β_1 for which $\alpha_1\overline{\beta_1}\langle p, q\rangle$ is real and positive. Thus

$$\begin{aligned}
&\|(I - \Theta_{f_1,f_2})T\Theta_{f_1,f_2}\|^2 \\
&= \sup_{|\alpha_1|^2+|\beta_1|^2=1} \left[\, |\alpha_1|^2\|p\|^2 + |\beta_1|^2\|q\|^2 + 2|\alpha_1|\, |\beta_1|\, |\langle p, q\rangle|\, \right]
\end{aligned}$$

$$= \sup_{|\alpha_1|^2+|\beta_1|^2=1} \left[\ A|\alpha_1|^2\ + B|\beta_1|^2 + 2C|\alpha_1|\ |\beta_1|\ \right]$$

where $A = \|p\|^2, B = \|q\|^2$ and $C = |\langle p, q\rangle|$. Assume $A \geq B$. For if $A \ngeq B$, then interchange the role of f_1 and f_2 to make it so. Let $t = |\alpha_1|^2$. Then $|\beta_1|^2 = 1 - t$. We then have

$$\begin{aligned} \|(I - \Theta_{f_1,f_2})T\Theta_{f_1,f_2}\|^2 &= \sup_{0\leq t\leq 1} tA + (1-t)B + 2C\sqrt{t(1-t)} \\ &= B + \sup_{0\leq t\leq 1} (A-B)t + 2C\sqrt{t(1-t)}. \end{aligned}$$

If $\langle p, q\rangle = 0$, then $C = 0$ and we have

$$\|(I - \Theta_{f_1,f_2})T\Theta_{f_1,f_2}\|^2 = \sup_{0\leq t\leq 1} B + (A-B)t = A,$$

since we have assumed $A \geq B$. If $C \neq 0$, then

$$\begin{aligned} \|(I - \Theta_{f_1,f_2})T\Theta_{f_1,f_2}\|^2 &= B + C \sup_{0\leq t\leq 1} \frac{A-B}{C}t + 2\sqrt{t(1-t)} \\ &= B + C \sup_{0\leq t\leq 1} rt + 2\sqrt{t(1-t)} \end{aligned}$$

where $r = \frac{A-B}{C} \geq 0$. The maximum value of the continuous function $\gamma(t) = rt + 2\sqrt{t(1-t)},\ 0 \leq t \leq 1$ is attained at $m_1 = \frac{1}{2}\left(1 + \frac{r}{\sqrt{4+r^2}}\right)$ and $\gamma(m_1) = \frac{1}{2}\left(r + \sqrt{4+r^2}\right)$. Thus

$$\begin{aligned} \|(I - \Theta_{f_1,f_2})T\Theta_{f_1,f_2}\|^2 &= B + C\gamma(m_1) \\ &= B + \frac{C}{2}\left(r + \sqrt{4+r^2}\right) \\ &= B + \frac{C}{2}\left[\frac{A-B}{C} + \sqrt{4 + \left(\frac{A-B}{C}\right)^2}\right] \\ &= \frac{1}{2}\left[(B+A) + \sqrt{4C^2 + (A-B)^2}\right]. \end{aligned}$$

If $C = 0$, then $\|(I - \Theta_{f_1,f_2})T\Theta_{f_1,f_2}\| = A$. Thus the last equality holds for any $C \geq 0$. Notice that the expression is symmetric of B and A, so we can drop the restriction $A \geq B$. $\square$

Lemma 2.5. *Let* $T \in \mathcal{L}\left(L^2_a(dA_\alpha)\right)$ *be a normal operator. Let* $\Theta_{z,T}$ *be* *the projection onto span* $\left\{k_z^{1+\frac{\alpha}{2}}, Tk_z^{1+\frac{\alpha}{2}}\right\}$. *If* $\|(I-\Theta_{z,T})T\Theta_{z,T}\| = \|(I-\Theta_{z,T})T^*\Theta_{z,T}\|$ *for all* $z\in\mathbb{D}$, *then the set* $\{T^*Tk_z^{1+\frac{\alpha}{2}}, T^*k_z^{1+\frac{\alpha}{2}}, Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\}$ *is linearly dependent for all* $z\in\mathbb{D}$.

Proof. Let $z\in\mathbb{D}$. Assume that $\|(I-\Theta_{z,T})T\Theta_{z,T}\| = \|(I-\Theta_{z,T})T^*\Theta_{z,T}\|$. We shall show that the set $\left\{T^*Tk_z^{1+\frac{\alpha}{2}}, T^*k_z^{1+\frac{\alpha}{2}}, Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\}$ is linearly dependent. If $Tk_z^{1+\frac{\alpha}{2}} = \lambda k_z^{1+\frac{\alpha}{2}}$ for some $\lambda\in\mathbb{C}$, then the set $\left\{k_z^{1+\frac{\alpha}{2}}, Tk_z^{1+\frac{\alpha}{2}}\right\}$ is linearly dependent. If $k_z^{1+\frac{\alpha}{2}}$ is not an eigenvector of T, then let $g\in L_a^2(dA_\alpha)$, $\|g\|=1$ and $g\perp k_z^{1+\frac{\alpha}{2}}$ and suppose $g\in$ *span* $\left\{k_z^{1+\frac{\alpha}{2}}, Tk_z^{1+\frac{\alpha}{2}}\right\} = \mathcal{N}$. We shall first show that the components of $T^*k_z^{1+\frac{\alpha}{2}}$ and T^*g in $\mathcal{N}^\perp$ are linearly dependent. Let

$$Tk_z^{1+\frac{\alpha}{2}} = \alpha k_z^{1+\frac{\alpha}{2}} + \beta g; \quad T^*k_z^{1+\frac{\alpha}{2}} = \overline{\alpha} k_z^{1+\frac{\alpha}{2}} + \lambda g + l';$$
$$Tg = \overline{\lambda} k_z^{1+\frac{\alpha}{2}} + \delta g + h; \quad T^*g = \overline{\beta} k_z^{1+\frac{\alpha}{2}} + \overline{\delta} g + h'.$$

This is so, since $\left\langle Tk_z^{1+\frac{\alpha}{2}}, g\right\rangle = \overline{\left\langle T^*g, k_z^{1+\frac{\alpha}{2}}\right\rangle}$. Since T is normal, we have $\left\|Tk_z^{1+\frac{\alpha}{2}}\right\|^2 = \left\|T^*k_z^{1+\frac{\alpha}{2}}\right\|^2$. That is, $|\alpha|^2+|\beta|^2 = |\alpha|^2+|\lambda|^2+\|l'\|^2$ or $|\lambda|^2+\|l'\|^2 = |\beta|^2$. Similarly, since $\|T^*g\| = \|Tg\|$, we obtain $|\lambda|^2+\|h\|^2 = |\beta|^2+\|h'\|^2$, and hence $\|l'\|^2+\|h'\|^2 = \|h\|^2$. Let $A' = \|l'\|^2$, $B' = \|h'\|^2$, $A = \|p\|^2 = 0$ where $p = (I-\Theta_{z,T})Tk_z^{1+\frac{\alpha}{2}}$, $C = |\langle p,h\rangle| = 0$ and $B = \|h\|^2$ where $h = (I-\Theta_{z,T})Tg$, $l' = (I-\Theta_{z,T})T^*k_z^{1+\frac{\alpha}{2}}$, $h' = (I-\Theta_{z,T})T^*g$, $C' = |\langle l',h'\rangle|$. Then $A'+B' = \|l'\|^2+\|h'\|^2 = \|h\|^2 = B$. Now the equation

$$\|(I-\Theta_{z,T})T\Theta_{z,T}\| = \|(I-\Theta_{z,T})T^*\Theta_{z,T}\|$$

can be written as

$$\frac{1}{2}(A'+B') + \frac{1}{2}\sqrt{4C'^2+(A'-B')^2} = \frac{1}{2}(A+B) + \frac{1}{2}\sqrt{4C^2+(A-B)^2} = B$$

as $A=C=0$. Since $A'+B'=B$, hence we obtain $(A'+B')^2 = 4C'^2+(A'-B')^2$. Thus $A'B' = C'^2$. That is, $\|l'\|\|h'\| = |\langle l',h'\rangle|$. Equality in Cauchy-Schwartz inequality can occur only if the vectors l' and h' are linearly dependent. Thus $T^*k_z^{1+\frac{\alpha}{2}}$ and T^*g are linearly dependent. If $l'=0$, the vectors $T^*k_z^{1+\frac{\alpha}{2}}, Tk_z^{1+\frac{\alpha}{2}}$ and $k_z^{1+\frac{\alpha}{2}}$ are linearly dependent. If $l'\neq 0$, then $h' = \alpha_1 l'$ for some $\alpha_1\in\mathbb{C}$. From the decomposition of the vectors

$Tk_z^{1+\frac{\alpha}{2}}, T^*k_z^{1+\frac{\alpha}{2}}, Tg$ and T^*g we have

$$T^*Tk_z^{1+\frac{\alpha}{2}} = \alpha T^*k_z^{1+\frac{\alpha}{2}} + \beta T^*g = \alpha T^*k_z^{1+\frac{\alpha}{2}} + |\beta|^2 k_z^{1+\frac{\alpha}{2}} + \beta\overline{\delta}g + \beta\alpha_1 l'.$$

Writing l' in terms of $k_z^{1+\frac{\alpha}{2}}, T^*k_z^{1+\frac{\alpha}{2}}$, and g and writing g in terms of $k_z^{1+\frac{\alpha}{2}}$ and $Tk_z^{1+\frac{\alpha}{2}}$, gives the linear dependence relation. Thus the set of vectors $\left\{T^*Tk_z^{1+\frac{\alpha}{2}}, T^*k_z^{1+\frac{\alpha}{2}}, Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\}$ is linearly dependent. □

Let $\sigma(A)$ be the spectrum of the operator $A \in \mathcal{L}(L_a^2(dA_\alpha))$ and $B(\Omega)$ be the set of all bounded, complex-valued measurable functions on $\Omega \subset \mathbb{C}$.

Lemma 2.6. *Let $T \in \mathcal{L}(L_a^2(dA_\alpha))$ be a normal operator. Let $q : \mathbb{C} \to \mathbb{C}$ be a continuous function. Then $(q \circ f)(T) = q(f(T))$ for all $f \in B(\sigma(T))$.*

Proof. The result is easy to verify if q is a polynomial in z and $\overline{z}$. Now suppose $q : \mathbb{C} \to \mathbb{C}$ is an arbitrary continuous function. Then by applying the Stone-Weierstrass theorem [2] to $C(\Delta)$, the space of continuous complex-valued functions on Δ, we obtain q is the uniform limit of polynomials q_n in z and $\overline{z}$ on the disk $\Delta = \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|f\|_\infty\}$. Here $C(\Delta)$ is equipped with the supremum norm. □

Lemma 2.7. *Let $V \in \mathcal{L}(L_a^2(dA_\alpha))$ be unitary. Then there exists a Hermitian operator S in $\mathcal{L}(L_a^2(dA_\alpha))$ such that $V = e^{iS}$ and $\|S\| \leq 2\pi$.*

Proof. Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$. Then the function $h : [0, 2\pi) \to \mathbb{T}$ defined by $h(t) = e^{it}$ is a continuous, bijective function with Borel measurable inverse θ. Since $\sigma(V) \subset \mathbb{T}$, we can set $S = \theta(V)$. The operator S is self-adjoint as θ is a real-valued function. Further, $\|S\| \leq \|\theta\|_\infty \leq 2\pi$. By Lemma 2.6, $(h \circ \theta)(V) = h(\theta(V)) = h(S) = e^{iS}$. But $(h \circ \theta)(\lambda) = \lambda$ for all $\lambda \in \mathbb{T}$. Hence $(h \circ \theta)(V) = V$. Therefore, $V = e^{iS}$. The proof is complete. □

3 Main results

In this section, we established the main results of the paper. We have shown that if $T \in \mathcal{L}\left(L_a^2(dA_\alpha)\right)$ is normal and if the set of vectors $\{T^*Tf, T^*f, Tf, f\}$ is linearly dependent for all $f \in L_a^2(dA_\alpha)$, then $B_\alpha(T)(z) = \lambda_\alpha\psi_\alpha(z) + \mu_\alpha$ where $\lambda_\alpha, \mu_\alpha \in \mathbb{C}$ and either $\psi_\alpha = B_\alpha(R)$, $R \in \mathcal{L}\left(L_a^2(dA_\alpha)\right)$ is a self-adjoint operator or $\psi_\alpha = B_\alpha(e^{iK})$ where $K \in \mathcal{L}\left(L_a^2(dA_\alpha)\right)$ is a self-adjoint operator. As a consequence of this result we showed that if $\phi \in h^\infty(\mathbb{D})$,

$T_\phi^{(\alpha)}$ is the Toeplitz operator with symbol ϕ defined on $L_a^2(dA_\alpha)$ and if the set $\left\{\left(T_\phi^{(\alpha)}\right)^* T_\phi^{(\alpha)} f, \left(T_\phi^{(\alpha)}\right)^* f, T_\phi^{(\alpha)} f, f\right\}$ is linearly dependent for all $f \in L_a^2(dA_\alpha)$ then either ϕ is a constant function or there exists $\lambda_\alpha, \mu_\alpha \in \mathbb{C}$ such that $\frac{\phi-\mu_\alpha}{\lambda_\alpha}$ is a real-valued harmonic function in $h^\infty(\mathbb{D})$.

Theorem 3.1. *If $T \in \mathcal{L}\left(L_a^2(dA_\alpha)\right)$ is normal and if the set of vectors $\{T^*Tf, T^*f, Tf, f\}$ is linearly dependent for all $f \in L_a^2(dA_\alpha)$, then*

$$B_\alpha(T)(z) = \lambda_\alpha \psi_\alpha(z) + \mu_\alpha$$

where $\lambda_\alpha, \mu_\alpha \in \mathbb{C}$ and either $\psi_\alpha = B_\alpha(R)$, $R \in \mathcal{L}\left(L_a^2(dA_\alpha)\right)$ is a self-adjoint operator or $\psi_\alpha = B_\alpha(e^{iK})$ where $K \in \mathcal{L}\left(L_a^2(dA_\alpha)\right)$ is a self-adjoint operator.

Proof. The operator T is normal. Hence by the spectral theorem [2], there is a function $\phi_\alpha \in L^\infty(\mathbb{D})$ such that T is unitarily equivalent to a multiplication operator M_{ϕ_α}, on $L_a^2(dA_\alpha)$. Now for all $f_\alpha \in L_a^2(dA_\alpha)$, the function $|\phi_\alpha|^2 f_\alpha, \overline{\phi_\alpha} f_\alpha, \phi_\alpha f_\alpha$ and f_α are linearly dependent as $T^*Tf_\alpha, T^*f_\alpha, Tf_\alpha$ and f_α are linearly dependent for all $f_\alpha \in L_a^2(dA_\alpha)$. Thus there exists complex numbers $\beta_{1\alpha}, \beta_{2\alpha}, \beta_{3\alpha}, \beta_{4\alpha}$ not all zero (depending on f_α) such that $\beta_{1\alpha}|\phi_\alpha|^2 f_\alpha + \beta_{2\alpha}\overline{\phi_\alpha} f_\alpha + \beta_{3\alpha}\phi_\alpha f_\alpha + \beta_{4\alpha} f_\alpha = 0$. This implies there exists a square integrable function $g_\alpha \in L_a^2(dA_\alpha)$ such that $g_\alpha(z)$ is never zero for any $z \in \mathbb{D}$. For this g_α, we have

$$\left(\beta_{1\alpha}|\phi_\alpha|^2 + \beta_{2\alpha}\overline{\phi_\alpha} + \beta_{3\alpha}\phi_\alpha + \beta_{4\alpha}\right) g_\alpha = 0$$

or

$$\left(\beta_{1\alpha}|\phi_\alpha|^2 + \beta_{2\alpha}\overline{\phi_\alpha} + \beta_{3\alpha}\phi_\alpha + \beta_{4\alpha}\right) = 0.$$

Since $g_\alpha(z) \neq 0$ for all $z \in \mathbb{D}$, hence $\beta_{1\alpha}T^*T + \beta_{2\alpha}T^* + \beta_{3\alpha}T + \beta_{4\alpha} = 0$. If $\beta_{1\alpha} = 0$, then $T = \lambda_\alpha S + \mu_\alpha$ where S is a self-adjoint operator. If $\beta_{1\alpha} \neq 0$, we can assume that $\beta_{1\alpha} = 1$. Then

$$\beta_{1\alpha}T^*T + \beta_{2\alpha}T^* + \beta_{3\alpha}T + \beta_{4\alpha} = 0. \tag{3.1}$$

Further

$$\beta_{1\alpha}T^*T + \overline{\beta_{2\alpha}}T + \overline{\beta_{3\alpha}}T^* + \overline{\beta_{4\alpha}} = 0. \tag{3.2}$$

Subtracting (3.2) from (3.1) we obtain $(\beta_{2\alpha} - \overline{\beta_{3\alpha}})T^* + (\beta_{3\alpha} - \overline{\beta_{2\alpha}})T + 2i\ Im\ \beta_{4\alpha} = 0$. If $\beta_{2\alpha} \neq \overline{\beta_{3\alpha}}$; then $T = \lambda_\alpha S + \mu_\alpha$ where S is self-adjoint. If $\beta_{3\alpha} = \overline{\beta_{2\alpha}}$ and $\beta_{4\alpha}$ is real then we have $T^*T + \beta_{2\alpha}T^* + \overline{\beta_{2\alpha}}T + \beta_{4\alpha} = 0$. Let $R = T + \beta_{2\alpha}$. Then R is normal and $R^*R = |\beta_{2\alpha}|^2 - \beta_{4\alpha}$, a multiple of a unitary operator S and $T = \lambda_\alpha S + \mu_\alpha$ where S is unitary. Thus $B_\alpha T(z) = \lambda_\alpha \psi_\alpha(z) + \mu_\alpha$ where $\lambda_\alpha, \mu_\alpha \in \mathbb{C}$ and ψ_α is the Berezin transform of a self-adjoint or a unitary operator. The Theorem follows from Lemma 2.7. □

Theorem 3.2. *Let $\phi \in h^\infty(\mathbb{D})$ and $T_\phi^{(\alpha)}$ be the Toeplitz operator with symbol ϕ defined on $L_a^2(dA_\alpha)$. If the set $\left\{\left(T_\phi^{(\alpha)}\right)^* T_\phi^{(\alpha)} f, \left(T_\phi^{(\alpha)}\right)^* f, T_\phi^{(\alpha)} f, f\right\}$ is linearly dependent for all $f \in L_a^2(dA_\alpha)$ then either ϕ is a constant function or there exists $\lambda_\alpha, \mu_\alpha \in \mathbb{C}$ such that $\frac{\phi - \mu_\alpha}{\lambda_\alpha}$ is a real-valued harmonic function in $h^\infty(\mathbb{D})$.*

Proof. Assume that the set of vectors $\left\{\left(T_\phi^{(\alpha)}\right)^* T_\phi^{(\alpha)} f, \left(T_\phi^{(\alpha)}\right)^* f, T_\phi^{(\alpha)} f, f\right\}$ is linearly dependent for all $f \in L_a^2(dA_\alpha)$. From Theorem 3.1 it follows that $T_\phi^{(\alpha)} = \lambda_\alpha S + \mu_\alpha$ where S is either a self-adjoint or a unitary operator. This implies that $T^{(\alpha)}_{\frac{\phi - \mu_\alpha}{\lambda_\alpha}} = S$ is either a self-adjoint or a unitary operator. If S is unitary, then $T^{(\alpha)}_{\frac{\phi - \mu_\alpha}{\lambda_\alpha}}$ is unitary. It follows from [1] that $\frac{\phi - \mu_\alpha}{\lambda_\alpha}$ is a constant function and $\left|\frac{\phi - \mu_\alpha}{\lambda_\alpha}\right| = 1$. This implies that $\frac{\phi - \mu_\alpha}{\lambda_\alpha} = e^{i\theta_\alpha}$ for some $\theta_\alpha \in \mathbb{R}$. Hence ϕ is a constant function. If $T^{(\alpha)}_{\frac{\phi - \mu_\alpha}{\lambda_\alpha}}$ is self-adjoint, then it follows that $\frac{\phi - \mu_\alpha}{\lambda_\alpha}$ is a real-valued harmonic function in $h^\infty(\mathbb{D})$. □

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