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ORLICZ DIFFERENCE TRIPLE LACUNARY IDEAL SEQUENCE SPACES OVER n-NORMED SPACES*

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Abstract

In the present article, we introduce and study some Lacunary $\mathcal{I}-$conver gent and Lacunary $\mathcal{I}-$bounded triple difference sequence spaces defined by Orlicz function over $n-$normed spaces. We shall investigate some algebraic and topological properties of newly formed sequence spaces. We also make an effort to obtain some inclusion results between these spaces.

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keywords: Triple sequence, Orlicz function, difference operator, sequence algebra, $n-$norm, Lacunary sequence

1 Introduction and Preliminaries

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote a set of natural, real and complex numbers respectively. A triple sequence $x = (x_{m,n,k})$ is a function $x : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{R}($ or $\mathbb{C})$. Initially triple sequence spaces were introduced and studied by Sahiner et

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al. [13], Sahiner and Tripathy [14], Tripathy and Goswami [18], Esi and Catalbas [3] and many others. Since then this concept has been studied by several authors. Recently many mathematicians have generalized the concept of triple sequences by introducing triple gai sequences, triple analytic sequences and triple entire sequences associated with statistical convergence, Orlicz function and fuzzy number (see [1], [10], [17]).

A triple sequence $x = (x_{m,n,k})$ is said to convergent to L in the Pringsheim's sense if for every $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $|x_{m,n,k} - L| < \epsilon$ whenever $m \geq N_\epsilon, n \geq N_\epsilon, k \geq N_\epsilon$. A triple sequence convergent in Pringsheim's sense is not necessarily bounded. For example: A triple sequence $x = (x_{m,n,k})$ defined by

$$x_{m,n,k} = \begin{cases} m, & \text{for all } m \in \mathbb{N}, n = 1 = k, \\ \frac{1}{m+n+k}, & \text{otherwise.} \end{cases}$$

Then $x_{m,n,k} \to 0$ in Pringsheim's sense but is unbounded.

A triple sequence $x = (x_{m,n,k})$ is said to be Cauchy sequence if for every $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $|x_{m,n,k} - x_{p,q,l}| < \epsilon$, whenever $m \geq p \geq N_\epsilon, n \geq q \geq N_\epsilon, k \geq l \geq N_\epsilon$.

Ideal convergence is a generalisation of statistical convergence and was first introduced and studied by Kostyrko et al. [7] in metric spaces. Later on Lahiri and Das [8] extended the idea to an arbitrary topological space. A class $\mathcal{I} \subseteq P(X)$(power set of X) is said to be an ideal if it satisfies the following conditions:

(i)$\emptyset \in \mathcal{I}$

(ii) $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$

(iii) $A \in \mathcal{I}$ and $B \subseteq A \Rightarrow B \in \mathcal{I}$.

$\mathcal{I}$ is called a non-trivial ideal if $X \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I} \in X$ is called admissible if $\mathcal{I}$ contains every finite subset of X.

A triple sequence $x = (x_{m,n,k})$ is said to be $\mathcal{I}-$bounded if there exists $D > 0$ such that $\{(m, n, k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{m,n,k}| > D\} \in \mathcal{I}$.

A triple sequence $x = (x_{m,n,k})$ is said to be $\mathcal{I}-$convergent to L, if for every $\epsilon > 0$ the set $\{(m, n, k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{m,n,k} - L| \geq \epsilon\} \in \mathcal{I}$ and we write $\mathcal{I} - \lim x_{m,n,k} = L$.

A triple sequence $x = (x_{m,n,k})$ is said to be $\mathcal{I}-$null, if $L = 0$ and we write $\mathcal{I} - \lim x_{m,n,k} = 0$.

If $\mathcal{I}$ is the ideal of all finite subsets of $\mathbb{N}$, then the ideal convergence coincides with usual convergence.

A triple sequence $x = (x_{m,n,k})$ is said to be $\mathcal{I}-$Cauchy if for every $\epsilon > 0$, there exists $p = p_0, q = q_0$ and $l = l_0$ such that $\{(m, n, k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{m,n,k} - x_{p,q,l}| \geq \epsilon\} \in \mathcal{I}$.

A triple sequence space X is said to be solid if for $m, n, k \in \mathbb{N}, (\alpha_{m,n,k} x_{m,n,k}) \in X$ whenever $(x_{m,n,k}) \in X$ and for all sequences $(\alpha_{m,n,k})$ of scalars with $|\alpha_{m,n,k}| \leq 1$.
A triple sequence space X is said to be monotone if it contains the canonical pre-image of all its step spaces.

Remark 1 *[6] A sequence space is solid implies that it is monotone.*

A triple sequence space X is said to be sequence algebra if $(x_{m,n,k} \times y_{m,n,k}) \in X$, whenever $(x_{m,n,k}) \in X$ and $(y_{m,n,k}) \in X$.
In the mid of 1960's Gähler [4] introduced the concept of $2-$normed spaces. Later on Misiak [11] extended the concept to $n-$normed spaces. Let X be a linear space over the field $\mathcal{K}$ of real or complex numbers of dimension d, where $d \geq n \geq 2$. A $n-$norm on X is a real valued function $\|., \cdots, .\|$ on X^n that satisfies following conditions:
(i) $\|x_1, x_2, \cdots, x_n\| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;
(ii) $\|x_1, x_2, \cdots, x_n\|$ is invariant under permutation;
(iii) $\|\alpha x_1, x_2, \cdots, x_n\| = |\alpha| \|x_1, x_2, \cdots, x_n\|$ for any $\alpha \in \mathcal{K}$ and
(iv) $\|x + x', x_2, \cdots, x_n\| \leq \|x, x_2, \cdots, x_n\| + \|x', x_2, \cdots, x_n\|$.
The pair $(X, \|., \cdots, .\|)$ is called $n-$normed space over the field $\mathcal{K}$. A sequence (x_k) in a n-normed space $(X, \|\cdot, \cdots, \cdot\|)$ is said to converge to some $L \in X$ if

$$\lim_{k \to \infty} \|x_k - L, z_1, \cdots, z_{n-1}\| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, \|\cdot, \cdots, \cdot\|)$ is said to be Cauchy if

$$\lim_{k,p \to \infty} \|x_k - x_p, z_1, \cdots, z_{n-1}\| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.
For more details about sequence spaces (see [12], [15], [16]).
Let ω, ω^2 and ω^3 denotes the set of all single, double and triple sequences respectively of real or complex numbers. Kizmaz [5] introduced the notion of difference sequence spaces as follows:

$$Z(\Delta) = \{x = (x_m) \in \omega : (\Delta x_m) \in Z\}$$

for $Z = c, c_0$ and ℓ_∞, the spaces of convergent, null and bounded sequences respectively, where $\Delta(x_k) = x_k - x_{k+1}$ for all $k \in \mathbb{N}$. Later on Tripathy and Sarma [19] introduced and studied difference double sequence spaces as follows:

$$Z(\Delta) = \{x = (x_{m,n}) \in \omega^2 : (\Delta x_{m,n}) \in Z\}$$

for $Z = c^2, c_0^2$ and ℓ_∞^2, the space of convergent, null and bounded double sequences respectively, where $\Delta x_{m,n} = x_{m,n} - x_{m,n+1} - x_{m+1,n} + x_{m+1,n+1}$, for all $m, n \in \mathbb{N}$.
The difference triple sequence space was introduced by Debnath et al. [2] and $\Delta x_{m,n,k} = x_{m,n,k} - x_{m,n+1,k} - x_{m,n,k+1} + x_{m,n+1,k+1} - x_{m+1,n,k} + x_{m+1,n+1,k} + x_{m+1,n,k+1} - x_{m+1,n+1,k+1}$ and $\Delta^0 x_{m,n,k} = (x_{m,n,k})$.
The triple sequence $\theta_{r,s,t} = \{(m_r, n_s, k_t)\}$ is called triple lacunary if there exist three increasing sequences of integers such that

$$m_0 = 0, h_r = m_r - m_{r-1} \to \infty \text{ as } r \to \infty,$$

$$n_0 = 0, \overline{h_s} = n_s - n_{s-1} \to \infty \text{ as } s \to \infty \text{ and}$$

$$k_0 = 0, \overline{h_t} = k_t - k_{t-1} \to \infty \text{ as } t \to \infty.$$

Let $m_{r,s,t} = m_r, n_s, k_t, h_{r,s,t} = h_r \overline{h_s h_t}$ and $\theta_{r,s,t}$ is determined by $I_{r,s,t} = \{(m,n,k) : m_{r-1} < m < m_r, n_{s-1} < n < n_s, k_{t-1} < k < k_t\}$, $q_r = \frac{m_r}{m_{r-1}}, \overline{q_s} = \frac{n_s}{n_{s-1}}, \overline{q_t} = \frac{k_t}{k_{t-1}}$.
A function $M : [0, \infty) \to [0, \infty)$ is said to be an Orlicz function, if it satisfies the following conditions:
(i) M is convex,
(ii) M is continuous,
(iii) M is non-decreasing with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.
Lindenstrauss and Tzafriri [9] used the idea of Orlicz function to define the following sequence space.

$$l_M = \left\{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

which is called as an Orlicz sequence space.
An Orlicz function M is said to satisfy Δ_2 condition for all values of v, if there exist a constant $R > 0$, such that $M(2v) \le RM(v), v \ge 0$. Note that if $0 < \lambda < 1$, then $M(\lambda x) \le \lambda M(x)$, for all $x \ge 0$.
Let M be an Orlicz function, $u = (u_{m,n,k})$ be a triple sequence of strictly positive real numbers, $p = (p_{m,n,k})$ be a triple sequence of bounded real numbers, $\theta_{r,s,t} = \{(m_r, n_s, k_t)\}$ be a triple lacunary sequence. In the present paper, we define the following sequence spaces:
$[M, u, \Delta, c_{0\mathcal{I}}^3, p, \|., \cdots, .\|, \theta_{r,s,t}] =$

$$\Big\{_{x\in\omega^3:\mathcal{I}-}\lim_{r,s,t}\frac{1}{h_{r,s,t}}\sum_{m\in I_{r,s,t}}\sum_{n\in I_{r,s,t}}\sum_{k\in I_{r,s,t}}[M\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho},z_1,\cdots,z_{n-1}\|]^{p_{m,n,k}}$$

$$=0,\text{ for some }\rho>0\Big\}\in\mathcal{I}.$$

$[M,u,\Delta,c^3_{\mathcal{I}},p,\|.,\cdots,.\|,\theta_{r,s,t}]=$

$$\Big\{_{x\in\omega^3:\mathcal{I}-}\lim_{r,s,t}\frac{1}{h_{r,s,t}}\sum_{m\in I_{r,s,t}}\sum_{n\in I_{r,s,t}}\sum_{k\in I_{r,s,t}}[M\|\frac{u_{m,n,k}\Delta(x_{m,n,k})-L}{\rho},z_1,\cdots,z_{n-1}\|]^{p_{m,n,k}}$$

$$=0,\text{ for some }L\text{ and }\rho>0\Big\}\in\mathcal{I}.$$

$[M,u,\Delta,\ell^3_{\infty\mathcal{I}},p,\|.,\cdots,.\|,\theta_{r,s,t}]=$

$$\Big\{x\in\omega^3:\Big(\frac{1}{h_{r,s,t}}\sum_{m\in I_{r,s,t}}\sum_{n\in I_{r,s,t}}\sum_{k\in I_{r,s,t}}[M\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho},z_1,\cdots,z_{n-1}\|]^{p_{m,n,k}}\Big)$$

$$\text{is }\mathcal{I}-\text{ bounded for some }\rho>0\Big\}\in\mathcal{I}.$$

$[M,u,\Delta,m^3_{\mathcal{I}},p,\|.,\cdots,.\|,\theta_{r,s,t}]=$

$$[M,u,\Delta,c^3_{\mathcal{I}},p,\|.,\cdots,.\|,\theta_{r,s,t}]\cap[M,u,\Delta,\ell^3_{\infty\mathcal{I}},p,\|.,\cdots,.\|,\theta_{r,s,t}].$$

$[M,u,\Delta,m^3_{0\mathcal{I}},p,\|.,\cdots,.\|,\theta_{r,s,t}]=$

$$[M,u,\Delta,c^3_{0\mathcal{I}},p,\|.,\cdots,.\|,\theta_{r,s,t}]\cap[M,u,\Delta,\ell^3_{\infty\mathcal{I}},p,\|.,\cdots,.\|,\theta_{r,s,t}].$$

The following inequality will be used throughout the paper. If $0\le p_{m,n,k}\le \sup p_{m,n,k}=H, K=\max(1,2^{H-1})$, then

$$|a_{m,n,k}+b_{m,n,k}|^{p_{m,n,k}}\le K\{|a_{m,n,k}|^{p_{m,n,k}}+|b_{m,n,k}|^{p_{m,n,k}}\}. \tag{1}$$

for all m,n,k and $a_{m,n,k},b_{m,n,k}\in\mathbb{C}$. Also

$$|a|^{p_{m,n,k}}\le\max(1,|a|^H)\text{ for all }a\in\mathbb{C}. \tag{2}$$

The main purpose of this paper is to introduce the idea of Lacunary $\mathcal{I}-$convergence in triple difference sequence spaces by means of Orlicz function over $n-$normed spaces. We discuss some algebraic properties, topological properties and inclusion relations between the concerning spaces.

2 Main Results

Theorem 1 *Let M be an Orlicz function, $p = (p_{m,n,k})$ be a bounded triple sequence of positive real numbers, $\theta_{r,s,t} = \{(m_r, n_s, k_t)\}$ be a triple lacunary sequence and $u = (u_{m,n,k})$ be a triple sequence of positive real numbers. Then the sequence spaces $[M, u, \Delta, c^3_{0\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}]$, $[M, u, \Delta, c^3_{\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}]$, $[M, u, \Delta, \ell^3_{\infty\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}]$, $[M, u, \Delta, m^3_{\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}]$ and $[M, u, \Delta, m^3_{0\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}]$ are linear spaces over the complex field $\mathbb{C}$.*

Proof 1 *Here, we only prove for the sequence space $[M, u, \Delta, c^3_{0\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}]$. For other spaces it follows in the same manner.*
Let $(x_{m,n,k})$ and $(y_{m,n,k}) \in [M, u, \Delta, c^3_{0\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}]$. Then for $\rho_1, \rho_2 > 0$, we have

$$\mathcal{I}\text{-}\lim_{r,s,t} \frac{1}{h_{r,s,t}} \sum_{m\in I_{r,s,t}} \sum_{n\in I_{r,s,t}} \sum_{k\in I_{r,s,t}} [M\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho_1}, z_1, \cdots, z_{n-1}\|]^{p_{m,n,k}} = 0$$

and

$$\mathcal{I}\text{-}\lim_{r,s,t} \frac{1}{h_{r,s,t}} \sum_{m\in I_{r,s,t}} \sum_{n\in I_{r,s,t}} \sum_{k\in I_{r,s,t}} [M\|\frac{u_{m,n,k}\Delta(y_{m,n,k})}{\rho_2}, z_1, \cdots, z_{n-1}\|]^{p_{m,n,k}} = 0.$$

Now, for every $\epsilon > 0$, consider

$$T_1 = \left\{(r,s,t) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \tfrac{1}{h_{r,s,t}} \sum_{m\in I_{r,s,t}} \sum_{n\in I_{r,s,t}} \sum_{k\in I_{r,s,t}} [M\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho_1}, z_1, \cdots, z_{n-1}\|]^{p_{m,n,k}} \geq \frac{\epsilon}{2K}\right\} \in \mathcal{I}$$

and

$$T_2 = \left\{(r,s,t) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \tfrac{1}{h_{r,s,t}} \sum_{m\in I_{r,s,t}} \sum_{n\in I_{r,s,t}} \sum_{k\in I_{r,s,t}} [M\|\frac{u_{m,n,k}\Delta(y_{m,n,k})}{\rho_2}, z_1, \cdots, z_{n-1}\|]^{p_{m,n,k}} \geq \frac{\epsilon}{2K}\right\} \in \mathcal{I}.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Suppose $(r,s,t) \notin T_1 \cup T_2$ and α, β be scalars such that $|\alpha| \leq 1$ and $|\beta| \leq 1$. Then by using inequality (1), we have

$$\frac{1}{h_{r,s,t}} \sum_{m\in I_{r,s,t}} \sum_{n\in I_{r,s,t}} \sum_{k\in I_{r,s,t}} [M\| \frac{u_{m,n,k}\Delta(\alpha x_{m,n,k} + \beta y_{m,n,k})}{\rho_3}, z_1, \cdots, z_{n-1}\|]^{p_{m,n,k}}$$

$$\leq K\Big\{ \frac{1}{h_{r,s,t}} \sum_{m\in I_{r,s,t}} \sum_{n\in I_{r,s,t}} \sum_{k\in I_{r,s,t}} [M\| \frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho_1}, z_1, \cdots, z_{n-1}\|]^{p_{m,n,k}}$$

$$+ \frac{1}{h_{r,s,t}} \sum_{m\in I_{r,s,t}} \sum_{n\in I_{r,s,t}} \sum_{k\in I_{r,s,t}} [M\| \frac{u_{m,n,k}\Delta(y_{m,n,k})}{\rho_2}, z_1, \cdots, z_{n-1}\|]^{p_{m,n,k}} \Big\}$$

$$< K\Big\{ \frac{\epsilon}{2K} + \frac{\epsilon}{2K} \Big\}$$

$$= \epsilon.$$

Since M is non-decreasing and convex function. Hence,

$$(r,s,t) \notin T_0 = \Big\{ (r,s,t) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s,t}} \sum_{m\in I_{r,s,t}} \sum_{n\in I_{r,s,t}} \sum_{k\in I_{r,s,t}} \times$$

$$\Big[M \Big\| \frac{u_{m,n,k}\Delta(\alpha x_{m,n,k} + \beta y_{m,n,k})}{\rho_3}, z_1, \cdots, z_{n-1} \Big\| \Big]^{p_{m,n,k}} \geq \epsilon \Big\}.$$

Hence $T_0 \subset T_1 \cup T_2$. By additivity and heritability of $\mathcal{I}$, we have $T_0 \in \mathcal{I}$. Therefore, $[M, u, \Delta, c^3_{0\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}]$ is a linear space.

Theorem 2 *Let M be an Orlicz function, $p = (p_{m,n,k})$ be a bounded triple sequence of positive real numbers, $\theta_{r,s,t} = \{(m_r, n_s, k_t)\}$ be a triple lacunary sequence and $u = (u_{m,n,k})$ be a triple sequence of positive real numbers. Then the inclusion $[M, u, \Delta, c^3_{0\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}] \subset [M, u, \Delta, c^3_{\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}] \subset [M, u, \Delta, \ell^3_{\infty\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}]$ holds.*

Proof 2 *The inclusion*

$$[M, u, \Delta, c^3_{0\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}] \subset [M, u, \Delta, c^3_{\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}]$$

is obvious. We shall prove the inclusion

$$[M, u, \Delta, c^3_{\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}] \subset [M, u, \Delta, \ell^3_{\infty\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}].$$

For this consider a sequence $(x_{m,n,k}) \in [M, u, \Delta, c^3_{\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}]$. Then there exists $\rho_1 > 0$, such that for every $\epsilon > 0$

$$T = \Big\{ (r,s,t) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s,t}} \sum_{m\in I_{r,s,t}} \sum_{n\in I_{r,s,t}} \sum_{k\in I_{r,s,t}} \Big[M \Big\| \frac{u_{m,n,k}\Delta(x_{m,n,k}) - L}{\rho_1},$$

$$z_1, \cdots, z_{n-1} \Big\| \Big]^{p_{m,n,k}} \geq \epsilon, \text{ for some } L \text{ and } \rho > 0 \Big\} \in \mathcal{I}.$$

Since M *is non-decreasing and convex, so for* $\rho = 2\rho_1$, *we have*

$$M\left\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho}, z_1, \cdots, z_{n-1}\right\|$$

$$\leq \quad M\left\|\frac{u_{m,n,k}\Delta(x_{m,n,k}) - L}{\rho_1}, z_1, \cdots, z_{n-1}\right\| + M\left\|\frac{L}{\rho_1}, z_1, \cdots, z_{n-1}\right\|.$$

Suppose that $(r,s,t) \notin T$. *Then by using* (1), *we have*

$$\frac{1}{h_{r,s,t}} \sum_{m \in I_{r,s,t}} \sum_{n \in I_{r,s,t}} \sum_{k \in I_{r,s,t}} \left[M\left\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho}, z_1, \cdots, z_{n-1}\right\|\right]^{p_{m,n,k}}$$

$$\leq K\Bigg\{\frac{1}{h_{r,s,t}} \sum_{m \in I_{r,s,t}} \sum_{n \in I_{r,s,t}} \sum_{k \in I_{r,s,t}} \left[M\left\|\frac{u_{m,n,k}\Delta(x_{m,n,k}) - L}{\rho_1}, z_1, \cdots, z_{n-1}\right\|\right]^{p_{m,n,k}}$$

$$+ \frac{1}{h_{r,s,t}} \sum_{m \in I_{r,s,t}} \sum_{n \in I_{r,s,t}} \sum_{k \in I_{r,s,t}} \left[M\left\|\frac{L}{\rho_1}, z_1, \cdots, z_{n-1}\right\|\right]^{p_{m,n,k}}\Bigg\}$$

$$< K\Bigg\{\epsilon + \frac{1}{h_{r,s,t}} \sum_{m \in I_{r,s,t}} \sum_{n \in I_{r,s,t}} \sum_{k \in I_{r,s,t}} \left[M\left\|\frac{L}{\rho_1}, z_1, \cdots, z_{n-1}\right\|\right]^{p_{m,n,k}}\Bigg\}.$$

Since $\left[M\left\|\frac{L}{\rho_1}, z_1, \cdots, z_{n-1}\right\|\right]^{p_{m,n,k}} \leq \max\left\{1, \left[M\left\|\frac{L}{\rho_1}, z_1, \cdots, z_{n-1}\right\|\right]^{H}\right\}$.
Thus,

$$\frac{1}{h_{r,s,t}} \sum_{m \in I_{r,s,t}} \sum_{n \in I_{r,s,t}} \sum_{k \in I_{r,s,t}} \left[M\left\|\frac{L}{\rho_1}, z_1, \cdots, z_{n-1}\right\|\right]^{p_{m,n,k}} < \infty.$$

Let us take

$$D = K\Bigg\{\epsilon + \frac{1}{h_{r,s,t}} \sum_{m \in I_{r,s,t}} \sum_{n \in I_{r,s,t}} \sum_{k \in I_{r,s,t}} \left[M\left\|\frac{L}{\rho_1}, z_1, \cdots, z_{n-1}\right\|\right]^{p_{m,n,k}}\Bigg\}.$$

Hence,

$$\Bigg\{(r,s,t) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s,t}} \sum_{m \in I_{r,s,t}} \sum_{n \in I_{r,s,t}} \sum_{k \in I_{r,s,t}} \left[M\left\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho}, z_1, \cdots,\right.\right.$$

$$\left.\left. z_{n-1}\right\|\right]^{p_{m,n,k}} > D\Bigg\} \in \mathcal{I}$$ *which implies* $x \in \left[M, u, \Delta, \ell^3_{\infty\mathcal{I}}, p, \|.,\cdots,.\|, \theta_{r,s,t}\right]$.
Hence the proof.

Theorem 3 *Let M and M' be two Orlicz functions which satisfies Δ_2-condition. Then the following inclusion relations hold:*
(1)

$$\begin{aligned}&[M, u, \Delta, c^3_{0\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}]\\ &\subseteq [M' \circ M, u, \Delta, c^3_{0\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}], [M, u, \Delta, c^3_{\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}]\\ &\subseteq [M' \circ M, u, \Delta, c^3_{\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}]\end{aligned}$$

and

$$[M, u, \Delta, \ell^3_{\infty\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}] \subseteq [M' \circ M, u, \Delta, \ell^3_{\infty\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}].$$

(2) $[M, u, \Delta, c^3_{0\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}] \cap [M', u, \Delta, c^3_{0\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}] \subseteq [M + M', u, \Delta, c^3_{0\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}]$, $[M, u, \Delta, c^3_{\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}] \cap [M', u, \Delta, c^3_{\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}] \subseteq [M + M', u, \Delta, c^3_{\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}]$ *and* $[M, u, \Delta, \ell^3_{\infty\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}] \cap [M', u, \Delta, \ell^3_{\infty\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}] \subseteq [M + M', u, \Delta, \ell^3_{\infty\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}]$.

Proof 3 *We prove the result for third inclusion.*
(1) Let $x = (x_{m,n,k}) \in [M, u, \Delta, \ell^3_{\infty\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}]$, then for $\rho > 0$, there exists $D_1 > 0$, such that

$$T_0 = \left\{ (r,s,t) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s,t}} \sum_{m\in I_{r,s,t}} \sum_{n\in I_{r,s,t}} \sum_{k\in I_{r,s,t}} \left[M \left\| \frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_{m,n,k}} > D_1 \right\} \in \mathcal{I}.$$

Let $\sum_1$ *denotes the sum over* $m \in I_{r,s,t}$ *and* $M\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho}, z_1, \cdots, z_{n-1}\| > \delta$, $\sum_2$ *denotes the sum over* $n \in I_{r,s,t}$ *and* $M\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho}, z_1, \cdots, z_{n-1}\| > \delta$, $\sum_3$ *denotes the sum over* $k \in I_{r,s,t}$ *and* $M\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho}, z_1, \cdots, z_{n-1}\| > \delta$, $\sum_4$ *denotes the sum over* $m \in I_{r,s,t}$ *and* $M\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho}, z_1, \cdots, z_{n-1}\| \le \delta$, $\sum_5$ *denotes the sum over* $n \in I_{r,s,t}$ *and* $M\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho}, z_1, \cdots, z_{n-1}\| \le \delta$, $\sum_6$ *denotes the sum over* $k \in I_{r,s,t}$ *and* $M\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho}, z_1, \cdots, z_{n-1}\| \le \delta$.

Since M' is non-decreasing and convex and also satisfies Δ_2-condition, so for $D \geq 1$, we obtain the inequality

$$\frac{1}{h_{r,s,t}} \sum_1 \sum_2 \sum_3 \left[M'\bigg(M\bigg\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho}, z_1, \cdots, z_{n-1}\bigg\|\bigg)\right]^{p_{m,n,k}} \tag{3}$$

$$\leq \max\left\{1, \left(D\frac{1}{\delta}M'(2)\right)^H\right\} \frac{1}{h_{r,s,t}} \sum_1 \sum_2 \sum_3 [M\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho}, z_1, \cdots, z_{n-1}\|]^{p_{m,n,k}}.$$

And by continuity of M', we have

$$\frac{1}{h_{r,s,t}} \sum_4 \sum_5 \sum_6 \left[M'\bigg(M\bigg\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho}, z_1, \cdots, z_{n-1}\bigg\|\bigg)\right]^{p_{m,n,k}} \tag{4}$$

$$\leq \frac{1}{h_{r,s,t}} \sum_4 \sum_5 \sum_6 \epsilon^{p_{m,n,k}} \leq \frac{1}{h_{r,s,t}} \sum_4 \sum_5 \sum_6 \max\left\{\epsilon^h, \epsilon^H\right\}.$$

Suppose that $(r,s,t) \notin T_0$. Then by using (3) *and* (4), *we have*

$$\frac{1}{h_{r,s,t}} \sum_{m \in I_{r,s,t}} \sum_{n \in I_{r,s,t}} \sum_{k \in I_{r,s,t}} \left[M'\bigg(M\bigg\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho}, z_1, \cdots, z_{n-1}\bigg\|\bigg)\right]^{p_{m,n,k}}$$

$$\begin{aligned}
&= \frac{1}{h_{r,s,t}} \sum_1 \sum_2 \sum_3 \left[M'\bigg(M\bigg\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho}, z_1, \cdots, z_{n-1}\bigg\|\bigg)\right]^{p_{m,n,k}} \\
&+ \frac{1}{h_{r,s,t}} \sum_4 \sum_5 \sum_6 \left[M'\bigg(M\bigg\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho}, z_1, \cdots, z_{n-1}\bigg\|\bigg)\right]^{p_{m,n,k}} \\
&\leq \max\left\{1, \left(D\frac{1}{\delta}M'(2)\right)^H\right\} D_1 + \max\{\epsilon^h, \epsilon^H\} \\
&= D_2.
\end{aligned}$$

Hence,

$$(r,s,t) \notin U = \bigg\{(r,s,t) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s,t}} \sum_{m \in I_{r,s,t}} \sum_{n \in I_{r,s,t}} \sum_{k \in I_{r,s,t}} \bigg[M'\bigg($$

$$M\bigg\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho}, z_1, \cdots, z_{n-1}\bigg\|\bigg)\bigg]^{p_{m,n,k}} > D_2\bigg\}$$

and thus $U \subset T_0$. This implies $U \in \mathcal{I}$ and consequently $x \in [M' \circ M, u, \Delta, \ell^3_{\infty\mathcal{I}}, p, \|.,\cdots,.\|, \theta_{r,s,t}]$.
(2) Let $x = (x_{m,n,k}) \in [M, u, \Delta, \ell^3_{\infty\mathcal{I}}, p, \|.,\cdots,.\|, \theta_{r,s,t}] \cap [M', u, \Delta, \ell^3_{\infty\mathcal{I}}, p,$

$\|.,\cdots,.\|,\theta_{r,s,t}]$. Then for $\rho>0$, there exist $D_1>0$ and $D_2>0$ such that

$$T_1=\Big\{(r,s,t)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\frac{1}{h_{r,s,t}}\sum_{m\in I_{r,s,t}}\sum_{n\in I_{r,s,t}}\sum_{k\in I_{r,s,t}}\Big[M\Big\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho},z_1,\cdots,z_{n-1}\Big\|\Big]^{p_{m,n,k}}>D_1\Big\}\in\mathcal{I}$$

and

$$T_2=\Big\{(r,s,t)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\frac{1}{h_{r,s,t}}\sum_{m\in I_{r,s,t}}\sum_{n\in I_{r,s,t}}\sum_{k\in I_{r,s,t}}\Big[M'\Big\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho},z_1,\cdots,z_{n-1}\Big\|\Big]^{p_{m,n,k}}>D_2\Big\}\in\mathcal{I}.$$

Let $(r,s,t)\notin T_1\cup T_2$. Then we have

$$\begin{aligned}
&\frac{1}{h_{r,s,t}}\sum_{m\in I_{r,s,t}}\sum_{n\in I_{r,s,t}}\sum_{k\in I_{r,s,t}}\Big[(M+M')\Big\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho},z_1,\cdots,z_{n-1}\Big\|\Big]^{p_{m,n,k}}\\
\le\ & K\Big\{\frac{1}{h_{r,s,t}}\sum_{m\in I_{r,s,t}}\sum_{n\in I_{r,s,t}}\sum_{k\in I_{r,s,t}}\Big[M\Big\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho},z_1,\cdots,z_{n-1}\Big\|\Big]^{p_{m,n,k}}\\
+\ & \frac{1}{h_{r,s,t}}\sum_{m\in I_{r,s,t}}\sum_{n\in I_{r,s,t}}\sum_{k\in I_{r,s,t}}\Big[M'\Big\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho},z_1,\cdots,z_{n-1}\Big\|\Big]^{p_{m,n,k}}\Big\}\\
<\ & K\{D_1+D_2\}\\
=\ & D.
\end{aligned}$$

Thus, $(r,s,t)\notin U=\Big\{(r,s,t)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\frac{1}{h_{r,s,t}}\sum_{m\in I_{r,s,t}}\sum_{n\in I_{r,s,t}}\sum_{k\in I_{r,s,t}}\Big[(M+M')\Big\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho},z_1,\cdots,z_{n-1}\Big\|\Big]^{p_{m,n,k}}>D\Big\}$.
Hence, we have $T_1\cup T_2\in\mathcal{I}$ and $U\subset T_1\cup T_2$. This implies $U\in\mathcal{I}$. Therefore, $x\in[M+M',u,\Delta,\ell^3_{\infty\mathcal{I}},p,\|.,\cdots,.\|,\theta_{r,s,t}]$. Other inclusions can be proved similarly.

Corollary 1 *Let M be an Orlicz function which satisfies Δ_2 condition, $p=(p_{m,n,k})$ be a bounded triple sequence of positive real numbers, $\theta_{r,s,t}=\{(m_r,n_s,k_t)\}$ be a triple lacunary sequence and $u=(u_{m,n,k})$ be a triple sequence of positive real numbers. Then the inclusion $[u,\Delta,c^3_{0\mathcal{I}},p,\|.,\cdots,.\|,\theta_{r,s,t}]\subset[M,u,\Delta,c^3_{0\mathcal{I}},p,\|.,\cdots,.\|,\theta_{r,s,t}]$, $[u,\Delta,c^3_{\mathcal{I}},p,\|.,\cdots,.\|,\theta_{r,s,t}]\subset[M,u,\Delta,c^3_{\mathcal{I}},p,\|.,\cdots,.\|,\theta_{r,s,t}]$ and $[u,\Delta,\ell^3_{\infty\mathcal{I}},p,\|.,\cdots,.\|,\theta_{r,s,t}]\subset\big[M,u,\Delta,\ell^3_{\infty\mathcal{I}},p,\|.,\cdots,.\|,\theta_{r,s,t}]$.*

Proof 4 *One can easily proof by using* $M(x) = x$ *and* $M'(x) = M(x), \forall\, x \in [0, \infty)$ *in the first part of the Theorem* 3

Theorem 4 *Let* $p_{m,n,k}$ *and* $q_{m,n,k}$ *be two bounded triple sequences of positive real numbers such that* $0 < p_{m,n,k} \leq q_{m,n,k} < \infty$ *and* $\left(\frac{q_{m,n,k}}{p_{m,n,k}}\right)$ *be bounded. Then the following inclusions hold:*
(i) $[M, u, \Delta, c^3_{0\mathcal{I}}, q, \|., \cdots, .\|, \theta_{r,s,t}] \subset [M, u, \Delta, c^3_{0\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}]$.
(ii) $[M, u, \Delta, c^3_{\mathcal{I}}, q, \|., \cdots, .\|, \theta_{r,s,t}] \subset [M, u, \Delta, c^3_{\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}]$.

Proof 5 *Suppose* $x = (x_{m,n,k}) \in [M, u, \Delta, c^3_{0\mathcal{I}}, q, \|., \cdots, .\|, \theta_{r,s,t}]$. *Write*

$$d_{m,n,k} = \left[M\left\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho}, z_1, \cdots, z_{n-1}\right\|\right]^{q_{m,n,k}}$$

and $\xi_{m,n,k} = \frac{p_{m,n,k}}{q_{m,n,k}}$, *so that* $0 < \xi \leq \xi_{m,n,k} \leq 1$. *By using Hölder inequality, we obtain*

$$\frac{1}{h_{r,s,t}} \sum_{m \in I_{r,s,t}} \sum_{n \in I_{r,s,t}} \sum_{k \in I_{r,s,t}} (d_{m,n,k})^{\xi_{m,n,k}}$$

$$= \frac{1}{h_{r,s,t}} \sum_{\substack{m \in I_{r,s,t},\\ d_{m,n,k} \geq 1}} \sum_{\substack{n \in I_{r,s,t},\\ d_{m,n,k} \geq 1}} \sum_{\substack{k \in I_{r,s,t},\\ d_{m,n,k} \geq 1}} (d_{m,n,k})^{\xi_{m,n,k}} + \frac{1}{h_{r,s,t}} \sum_{\substack{m \in I_{r,s,t},\\ d_{m,n,k} < 1}} \sum_{\substack{n \in I_{r,s,t},\\ d_{m,n,k} < 1}} \sum_{\substack{k \in I_{r,s,t},\\ d_{m,n,k} < 1}} (d_{m,n,k})^{\xi_{m,n,k}}$$

$$\leq \frac{1}{h_{r,s,t}} \sum_{\substack{m \in I_{r,s,t},\\ d_{m,n,k} \geq 1}} \sum_{\substack{n \in I_{r,s,t},\\ d_{m,n,k} \geq 1}} \sum_{\substack{k \in I_{r,s,t},\\ d_{m,n,k} \geq 1}} d_{m,n,k} + \frac{1}{h_{r,s,t}} \sum_{\substack{m \in I_{r,s,t},\\ d_{m,n,k} < 1}} \sum_{\substack{n \in I_{r,s,t},\\ d_{m,n,k} < 1}} \sum_{\substack{k \in I_{r,s,t},\\ d_{m,n,k} < 1}} (d_{m,n,k})^{\xi}$$

$$= \frac{1}{h_{r,s,t}} \sum_{\substack{m \in I_{r,s,t},\\ d_{m,n,k} \geq 1}} \sum_{\substack{n \in I_{r,s,t},\\ d_{m,n,k} \geq 1}} \sum_{\substack{k \in I_{r,s,t},\\ d_{m,n,k} \geq 1}} d_{m,n,k}$$

$$+ \sum_{\substack{m \in I_{r,s,t},\\ d_{m,n,k} < 1}} \sum_{\substack{n \in I_{r,s,t},\\ d_{m,n,k} < 1}} \sum_{\substack{k \in I_{r,s,t},\\ d_{m,n,k} < 1}} \left(\frac{1}{h_{r,s,t}} d_{m,n,k}\right)^{\xi} \left(\frac{1}{h_{r,s,t}}\right)^{1-\xi}$$

$$\leq \frac{1}{h_{r,s,t}} \sum_{\substack{m \in I_{r,s,t},\\ d_{m,n,k} \geq 1}} \sum_{\substack{n \in I_{r,s,t},\\ d_{m,n,k} \geq 1}} \sum_{\substack{k \in I_{r,s,t},\\ d_{m,n,k} \geq 1}} d_{m,n,k} + \Bigg(\sum_{\substack{m \in I_{r,s,t},\\ d_{m,n,k} < 1}} \sum_{\substack{n \in I_{r,s,t},\\ d_{m,n,k} < 1}} \sum_{\substack{k \in I_{r,s,t},\\ d_{m,n,k} < 1}}$$

$$\left[\left(\frac{1}{h_{r,s,t}}d_{m,n,k}\right)^{\xi}\right]^{\frac{1}{\xi}}\Bigg)^{\xi}\Bigg(\sum_{\substack{m\in I_{r,s,t},\\ d_{m,n,k}<1}}\sum_{\substack{n\in I_{r,s,t},\\ d_{m,n,k}<1}}\sum_{\substack{k\in I_{r,s,t},\\ d_{m,n,k}<1}}\left[\left(\frac{1}{h_{r,s,t}}\right)^{1-\xi}\right]^{\frac{1}{1-\xi}}\Bigg)^{1-\xi}$$

$$\leq \frac{1}{h_{r,s,t}}\sum_{\substack{m\in I_{r,s,t},\\ d_{m,n,k}\geq 1}}\sum_{\substack{n\in I_{r,s,t},\\ d_{m,n,k}\geq 1}}\sum_{\substack{k\in I_{r,s,t},\\ d_{m,n,k}\geq 1}} d_{m,n,k} + \left(\frac{1}{h_{r,s,t}}\sum_{\substack{m\in I_{r,s,t},\\ d_{m,n,k}<1}}\sum_{\substack{n\in I_{r,s,t},\\ d_{m,n,k}<1}}\sum_{\substack{k\in I_{r,s,t},\\ d_{m,n,k}<1}} d_{m,n,k}\right)^{\xi}.$$

Hence for every $\epsilon > 0$, we have

$$\Bigg\{(r,s,t)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}: \tfrac{1}{h_{r,s,t}}\sum_{m\in I_{r,s,t}}\sum_{n\in I_{r,s,t}}\sum_{k\in I_{r,s,t}}[M\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho}, z_1,\cdots,z_{n-1}\|]^{p_{m,n,k}}$$

$$\geq \epsilon\Bigg\} \subset \Bigg\{(r,s,t)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}: \tfrac{1}{h_{r,s,t}}\sum_{\substack{m\in I_{r,s,t},\\ d_{m,n,k}\geq 1}}\sum_{\substack{n\in I_{r,s,t},\\ d_{m,n,k}\geq 1}}\sum_{\substack{k\in I_{r,s,t},\\ d_{m,n,k}\geq 1}}[M\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho}, z_1,$$

$$\cdots,z_{n-1}\|]^{q_{m,n,k}} \geq \tfrac{\epsilon}{2}\Bigg\}\cup\Bigg\{(r,s,t)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}: \tfrac{1}{h_{r,s,t}}\sum_{\substack{m\in I_{r,s,t},\\ d_{m,n,k}<1}}\sum_{\substack{n\in I_{r,s,t},\\ d_{m,n,k}<1}}\sum_{\substack{k\in I_{r,s,t},\\ d_{m,n,k}<1}}[M$$

$$\|\tfrac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho}, z_1,\cdots,z_{n-1}\|]^{q_{m,n,k}} \geq \left(\tfrac{\epsilon}{2}\right)^{\frac{1}{\xi}}\Bigg\}.$$ *Hence,* $$\Bigg\{(r,s,t)\in\mathbb{N}\times\mathbb{N}\times$$

$$\mathbb{N}: \frac{1}{h_{r,s,t}}\sum_{m\in I_{r,s,t}}\sum_{n\in I_{r,s,t}}\sum_{k\in I_{r,s,t}}\left[M\left\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho}, z_1,\cdots,z_{n-1}\right\|\right]^{p_{m,n,k}} \geq \epsilon\Bigg\}\in\mathcal{I}.$$

Therefore, $x \in [M, u, \Delta, c^3_{0\mathcal{I}}, p, \|.,\cdots,.\|, \theta_{r,s,t}]$.

Theorem 5 *If $\lim_{m,n,k\to\infty} p_{m,n,k} > 0$ and $x_{m,n,k} \to L\big([M, u, \Delta, c^3_{\mathcal{I}}, p, \|.,\cdots,.\|, \theta_{r,s,t}]\big)$, then L is unique.*

Proof 6 *Let $\lim_{m,n,k\to\infty} p_{m,n,k} = p' > 0, x_{m,n,k} \to L\big([M, u, \Delta, c^3_{\mathcal{I}}, p, \|.,\cdots,.\|, \theta_{r,s,t}]\big)$ and $x \to L'\big([M, u, \Delta, c^3_{\mathcal{I}}, p, \|.,\cdots,.\|, \theta_{r,s,t}]\big)$. Then for $L \neq L'$, there exist $\rho_1, \rho_2 > 0$ such that*

$$\Bigg\{(r,s,t)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}: \tfrac{1}{h_{r,s,t}}\sum_{m\in I_{r,s,t}}\sum_{n\in I_{r,s,t}}\sum_{k\in I_{r,s,t}}\left[M\left\|\frac{u_{m,n,k}\Delta(x_{m,n,k})-L}{\rho_1}, z_1,\cdots,\right.\right.$$

$$\left.\left.z_{n-1}\right\|\right]^{p_{m,n,k}} \geq \frac{\epsilon}{2K}\Bigg\}\in\mathcal{I}$$

and

$$\Big\{(r,s,t)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\tfrac{1}{h_{r,s,t}}\sum_{m\in I_{r,s,t}}\sum_{n\in I_{r,s,t}}\sum_{k\in I_{r,s,t}}\left[M\left\|\frac{u_{m,n,k}\Delta(x_{m,n,k})-L'}{\rho_2},z_1,\cdots,z_{n-1}\right\|\right]^{p_{m,n,k}}\geq\frac{\epsilon}{2K}\Big\}\in\mathcal{I}.$$

Then for $\rho=\max\{2\rho_1,2\rho_2\}$, *we have*

$$\tfrac{1}{h_{r,s,t}}\sum_{m\in I_{r,s,t}}\sum_{n\in I_{r,s,t}}\sum_{k\in I_{r,s,t}}\left[M\left\|\frac{L-L'}{\rho},z_1,\cdots,z_{n-1}\right\|\right]^{p_{m,n,k}}$$

$$\leq K\Big\{\frac{1}{h_{r,s,t}}\sum_{m\in I_{r,s,t}}\sum_{n\in I_{r,s,t}}\sum_{k\in I_{r,s,t}}\left[M\left\|\frac{u_{m,n,k}\Delta(x_{m,n,k})-L}{\rho_1},z_1,\cdots,z_{n-1}\right\|\right]^{p_{m,n,k}}$$

$$+\frac{1}{h_{r,s,t}}\sum_{m\in I_{r,s,t}}\sum_{n\in I_{r,s,t}}\sum_{k\in I_{r,s,t}}\left[M\left\|\frac{u_{m,n,k}\Delta(x_{m,n,k})-L'}{\rho_2},z_1,\cdots,z_{n-1}\right\|\right]^{p_{m,n,k}}\Big\}.$$

Thus,

$$\Big\{{\scriptstyle (r,s,t)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\frac{1}{h_{r,s,t}}}\sum_{m\in I_{r,s,t}}\sum_{n\in I_{r,s,t}}\sum_{k\in I_{r,s,t}}[M\|\frac{L-L'}{\rho},z_1,\cdots,z_{n-1}\|]^{p_{m,n,k}}\geq\epsilon\Big\}\in\mathcal{I}.$$

Also,

$$\left[M\left\|\frac{L-L'}{\rho},z_1,\cdots,z_{n-1}\right\|\right]^{p_{m,n,k}}\to\left[M\left\|\frac{L-L'}{\rho},z_1,\cdots,z_{n-1}\right\|\right]^{p'}$$

as $m,n,k\to\infty$ *and hence,* $\left[M\left\|\frac{L-L'}{\rho},z_1,\cdots,z_{n-1}\right\|\right]^{p'}=0$, *which implies* $L=L'$. *Hence, the proof.*

Theorem 6 *Let M be an Orlicz function, $p=(p_{m,n,k})$ be a bounded triple sequence of positive real numbers, $\theta_{r,s,t}=\{(m_r,n_s,k_t)\}$ be a triple lacunary sequence and $u=(u_{m,n,k})$ be a triple sequence of positive real numbers. Then the triple sequence $[M,u,\Delta,c^3_{0\mathcal{I}},p,\|.,\cdots,.\|,\theta_{r,s,t}]$ and $[M,u,\Delta,m^3_{0\mathcal{I}},p,\|.,\cdots,.\|,\theta_{r,s,t}]$ are solid.*

Proof 7 *We shall prove the result for $[M,u,\Delta,c^3_{0\mathcal{I}},p,\|.,\cdots,.\|,\theta_{r,s,t}]$, for $[M,u,\Delta,m^3_{0\mathcal{I}},p,\|.,\cdots,.\|,\theta_{r,s,t}]$ it follows in the same manner. Let $(x_{m,n,k})\in[M,u,\Delta,c^3_{0\mathcal{I}},p,\|.,\cdots,.\|,\theta_{r,s,t}]$, then*

$$\mathcal{I}-\lim_{r,s,t}\frac{1}{h_{r,s,t}}\sum_{m\in I_{r,s,t}}\sum_{n\in I_{r,s,t}}\sum_{k\in I_{r,s,t}}[M\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho},z_1,\cdots,z_{n-1}\|]^{p_{m,n,k}}=0.$$

Now consider a sequence of scalar $(\alpha_{m,n,k})$ *such that* $|\alpha_{m,n,k}| \leq 1$, *for all* $m,n,k \in \mathbb{N}$. *Then, we have*

$$\mathcal{I}-\lim_{r,s,t} \frac{1}{h_{r,s,t}} \sum_{m\in I_{r,s,t}} \sum_{n\in I_{r,s,t}} \sum_{k\in I_{r,s,t}} \left[M\left\|\frac{u_{m,n,k}\Delta(|\alpha_{m,n,k}x_{m,n,k}|)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right]^{p_{m,n,k}}$$

$$\leq \mathcal{I}-\lim_{r,s,t} \frac{1}{h_{r,s,t}} \sum_{m\in I_{r,s,t}} \sum_{n\in I_{r,s,t}} \sum_{k\in I_{r,s,t}} \left[M\left(\left\|\frac{u_{m,n,k}\Delta(x_{m,n,k})}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{m,n,k}}$$

$$= 0.$$

Hence,

$$\mathcal{I}-\lim_{r,s,t} \frac{1}{h_{r,s,t}} \sum_{m\in I_{r,s,t}} \sum_{n\in I_{r,s,t}} \sum_{k\in I_{r,s,t}} [M\|\frac{u_{m,n,k}\Delta(|\alpha_{m,n,k}x_{m,n,k}|)}{\rho}, z_1, \cdots, z_{n-1}\|]^{p_{m,n,k}}$$

$$= 0,$$

for all $m,n,k \in \mathbb{N}$. *Thus,*

$$\alpha_{m,n,k}x_{m,n,k} \in [M, u, \Delta, c^3_{0\mathcal{I}}, p, \|.,\cdots,.\|, \theta_{r,s,t}].$$

Thus the sequence space $[M, u, \Delta, c^3_{0\mathcal{I}}, p, \|.,\cdots,.\|, \theta_{r,s,t}]$ *is solid.*

Theorem 7 *Let* M *be an Orlicz function,* $p = (p_{m,n,k})$ *be a bounded triple sequence of positive real numbers,* $\theta_{r,s,t} = \{(m_r, n_s, k_t)\}$ *be a triple lacunary sequence and* $u = (u_{m,n,k})$ *be a triple sequence of positive real numbers. Then the triple sequence spaces* $[M, u, \Delta, c^3_{0\mathcal{I}}, p, \|.,\cdots,.\|, \theta_{r,s,t}]$, $[M, u, \Delta, c^3_{\mathcal{I}}, p, \|.,\cdots,.\|, \theta_{r,s,t}]$, $[M, u, \Delta, \ell^3_{\infty\mathcal{I}}, p, \|.,\cdots,.\|, \theta_{r,s,t}]$, $[M, u, \Delta, m^3_{0\mathcal{I}}, p, \|.,\cdots,.\|, \theta_{r,s,t}]$ *and* $[M, u, \Delta, m^3_{\mathcal{I}}, p, \|.,\cdots,.\|, \theta_{r,s,t}]$ *are sequence algebras.*

Proof 8 *Let* $(x_{m,n,k}), (y_{m,n,k}) \in [M, u, \Delta, c^3_{0\mathcal{I}}, p, \|.,\cdots,.\|, \theta_{r,s,t}]$. *Then,*

$$\mathcal{I}-\lim_{r,s,t} \frac{1}{h_{r,s,t}} \sum_{m\in I_{r,s,t}} \sum_{n\in I_{r,s,t}} \sum_{k\in I_{r,s,t}} [M\|\frac{u_{m,n,k}\Delta(|x_{m,n,k}|)}{\rho_1}, z_1, \cdots, z_{n-1}\|]^{p_{m,n,k}} = 0$$

and

$$\mathcal{I}-\lim_{r,s,t} \frac{1}{h_{r,s,t}} \sum_{m\in I_{r,s,t}} \sum_{n\in I_{r,s,t}} \sum_{k\in I_{r,s,t}} [M\|\frac{u_{m,n,k}\Delta(|y_{m,n,k}|)}{\rho_2}, z_1, \cdots, z_{n-1}\|]^{p_{m,n,k}} = 0.$$

Now, for $\rho = \rho_1\rho_2 > 0$, *we have*

$$\mathcal{I}-\lim_{r,s,t} \frac{1}{h_{r,s,t}} \sum_{m\in I_{r,s,t}} \sum_{n\in I_{r,s,t}} \sum_{k\in I_{r,s,t}} [M\|\frac{u_{m,n,k}\Delta(|x_{m,n,k}y_{m,n,k}|)}{\rho}, z_1, \cdots, z_{n-1}\|]^{p_{m,n,k}}$$

$$= 0.$$

This implies $(x_{m,n,k}y_{m,n,k}) \in [M, u, \Delta, c^3_{0\mathcal{I}}, p, \|., \cdots, .\|, \theta_{r,s,t}]$.
Hence, $[M, u, \Delta, c^3_{0\mathcal{I}}, p,$
$\|., \cdots, .\|, \theta_{r,s,t}]$ *is a sequence algebra.*

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