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CONFORMABLE FRACTIONAL DIFFERENTIAL EQUATIONS IN b-METRIC SPACES*

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Abstract

In this article we investigate some existence results for functional and neutral conformable fractional differential equations in b-metric spaces. Our results are based on the fixed point theory and the $\alpha - \phi$-Geraghty type contraction. Two illustrate examples are given in the last section.

MSC: 26A33

keywords: Conformable integral of fractional order; Conformable fractional order derivative; neutral differential equation; b-metric space; $\alpha - \phi$-Geraghty contraction, fixed point.

1 Introduction

Fractional differential equations have recently been applied in various areas of engineering, mathematics, physics, and other applied sciences. Considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential equations; see the monographs [2, 3, 4, 23, 25, 26, 29].

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The conformable fractional differential operator has been introduced first in [22]. Next, the conformable fractional differential equations has been rapidly developed; see [6, 7, 10, 11, 17, 18, 20, 21, 27, 28], and the reference therein.

The notion of b-metric was proposed by Czerwik [14, 15]. Following these initial papers, the existence fixed point for the various classes of operators in the setting of b-metric spaces have been investigated extensively; see [12, 13, 16, 24], and related references therein.

Neutral fractional differential equations has been studied by many mathematicians; see [1, 5, 30, 31], and the reference therein.

In this paper, first we discuss the existence of solutions for the following class of initial value problems of conformable fractional differential equations

$$\begin{cases}(T^r_{a^+}u)(t) = f(t, u(t)); \ t \in I := [a, b], \\ u(a^+) = u_a \in \mathbb{R}, \end{cases} \tag{1}$$

where $b > a > 0$, $f : I \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $T^r_{a^+}$ is the conformable fractional derivative of order $r \in (0, 1]$.

Next, we consider the following neutral conformable fractional differential equation

$$\begin{cases}u(t) = \varphi(t); \ t \in [-h, a], \\ T^r_{a^+}[u(t) - z(t, u_t)] = f(t, u_t); \ t \in I, \end{cases} \tag{2}$$

where $h > 0$, $\varphi \in \mathcal{C}, f, z : I \times \mathcal{C} \to \mathbb{R}$ is a given continuous function, and $\mathcal{C} := C([-h, a], \mathbb{R})$ is the space of continuous functions on $[-h, a]$.

For any $t \in I$, we define u_t by

$$u_t(s) = u(t + s); \ \text{for } s \in [-h, a].$$

Next, we investigate the following class of infinite delay neutral conformable fractional differential equation

$$\begin{cases}u(t) = \varphi(t); \ t \in (-\infty, a], \\ T^r_{a^+}[u(t) - z(t, u_t)] = f(t, u_t); \ t \in I, \end{cases} \tag{3}$$

where $\varphi : [-\infty, a] \to \mathbb{R}, f, z : I \times \mathcal{B} \to \mathbb{R}$ are given continuous functions, and $\mathcal{B}$ is called a phase space that will be specified later.

For any $t \in I$, we define $u_t \in \mathcal{B}$ by

$$u_t(s) = u(t+s); \text{ for } s \in (-\infty, a].$$

This paper initiates the study of conformable fractional differential equations on b-metric spaces.

2 Preliminaries

Let $C(I)$ be the Banach space of all real continuous functions on I with the norm

$$\|u\|_\infty = \sup_{t \in I} |u(t)|.$$

By $L^1(I)$ we denote the Banach space of measurable functions $u : I \to \mathbb{R}$ with are Lebesgue integrable, equipped with the norm

$$\|u\|_{L^1} = \int_0^T |u(t)| dt.$$

Definition 1. *(Conformable fractional derivatives) [6, 18, 22] The conformable fractional derivative (CFD) of order $0 < r \leq 1$ starting from a of the function $u : I \to \mathbb{R}$ is defined by:*

$$T^r_{a^+} u(t) = \lim_{h \to 0} \frac{u(t + h(t-a)^{1-r}) - u(t)}{h}.$$

Particularly, if u is differentiable, then

$$T^r_{a^+} u(t) = (t-a)^{1-r} \frac{d}{dt} u(t).$$

Definition 2. *(Conformable fractional integral) [6, 18, 22] The conformable fractional integrals of order $r > 0$ of a function $u : I \to \mathbb{R}$ is defined by:*

$$I^r_{a^+} u(t) = \int_a^t (s-a)^{r-1} u(s) ds, \ t \in I.$$

Example 1. *[6] For $0 < r \leq 1$, and $\lambda \in \mathbb{R}$, we have*

$$T^r_0 \lambda = 0, \ T^r_0 t^\lambda = \lambda t^{\lambda - r}, \ T^r_0 e^{\lambda t} = \lambda t^{1-r} e^{\lambda t}; \ t \in I.$$

Lemma 1. *[6, 18, 22] Let $1 \leq r > 0$, and $u \in C(I)$,then*

$$T^r_{a^+} I^r_{a^+} u(t) = u(t).$$

Further, if u is differentiable on I, then

$$I^r_{a^+} T^r_{a^+} u(t) = u(t) - u(a).$$

From the above Lemma, we have the following one:

Lemma 2. *Let $g \in L^1(I)$. Then the Cauchy problem*

$$\begin{cases} T_a^r u(t) = g(t); \ t \in I := [a,b] \\ u(a) = u_a, \end{cases}$$

has a unique solution given by

$$u(t) = u_a + I_{a+}^r g(t).$$

Definition 3. *[8, 9] Let $c \geq 1$ and M be a set. A distance function $d : M \times M \to \mathbb{R}_+^*$ is called b-metric if for all $\mu, \nu, \xi \in M$, the following are fulfilled:*

- *(bM1) $d(\mu, \nu) = 0$ if and only if $\mu = \nu$;*
- *(bM2) $d(\mu, \nu) = d(\nu, \mu)$;*
- *(bM3) $d(\mu, \xi) \leq c[d(\mu, \nu) + d(\nu, \xi)]$.*

The tripled (M, d, c) is called a b-metric space.

Example 2. *[8, 9] Let $d : C(I) \times C(I) \to \mathbb{R}_+^*$ be defined by*

$$d(u,v) = \|(u-v)^2\|_\infty := \sup_{t \in I} \|u(t) - v(t)\|^2; \ for \ all \ u, v \in C(I).$$

It is clear that d is a b-metric with $c = 2$.

Example 3. *[8, 9] Let $X = [0,1]$ and $d : X \times X \to \mathbb{R}_+^*$ be defined by*

$$d(x,y) = |x-y|^2; \ for \ all \ x, y \in X.$$

It is clear that d is not a metric, but it is easy to see that d is a b-metric space with $r \geq 2$.

Let Φ be the set of all increasing and continuous function $\phi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ satisfying the property: $\phi(c\mu) \leq c\phi(\mu) \leq c\mu$, for $c > 1$ and $\phi(0) = 0$. We denote by $\mathcal{F}$ the family of all nondecreasing functions $\lambda : \mathbb{R}_+^* \to [0, \frac{1}{c^2})$ for some $c \geq 1$.

Definition 4. *[8, 9] For a b-metric space (M, d, c), an operator $T : M \to M$ is called a generalized $\alpha - \phi-$Geraghty contraction type mapping whenever there exists $\alpha : M \times M \to \mathbb{R}_+^*$, and some $L \geq 0$ such that for*

$$D(x, y) = \max\left\{d(x, y), d(x, T(x)), d(y, T(y)), \frac{d(x, T(y)) + d(y, T(x))}{2s}\right\},$$

and

$$N(x, y) = \min\{d(x, y), d(x, T(x)), d(y, T(y))\},$$

we have

$$\alpha(\mu, \nu)\phi(c^3 d(T(\mu), T(\nu)) \leq \lambda(\phi(D(\mu, \nu))\phi(D(\mu, \nu)) + L\psi(N(\mu, \nu); \quad (4)$$

for all $\mu, \nu \in M$, where $\lambda \in \mathcal{F}$, $\phi\,\psi \in \Phi$.

Remark 1. *In the case when $L = 0$ in Definition 4, and the fact that*

$$d(x, y) \leq D(x, y);\ \textit{for all}\ x, y \in M,$$

the inequality (4) becomes

$$\alpha(\mu, \nu)\phi(c^3 d(T(\mu), T(\nu)) \leq \lambda(\phi(d(\mu, \nu))\phi(d(\mu, \nu)). \quad (5)$$

Definition 5. *[8, 9] Let M be a non empty set, $T : M \to M$, and $\alpha : M \times M \to \mathbb{R}_+^*$ be a given mappings. We say that T is $\alpha-$admissible if for all $\mu, \nu \in M$, we have*

$$\alpha(\mu, \nu) \geq 1 \to \alpha(T(\mu), T(\nu)) \geq 1.$$

Definition 6. *[8, 9] Let (M, d) be a b-metric space and let $\alpha : M \times M \to \mathbb{R}_+^*$ be a function. M is said to be $\alpha-$regular if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in M such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to \infty$, there exists a subsequence $\{x_{n(k)}\}_{k\in\mathbb{N}}$ of $\{x_n\}_n$ with $\alpha(x_{n(k)}, x) \geq 1$ for all k.*

The following fixed point theorem plays a key role in the proof of our main results.

Theorem 1. *[8, 9] Let (M, d) be a complete b-metric space and $T : M \to M$ be a generalized $\alpha - \phi-$Geraghty contraction type mapping such that*

- *(i) T is $\alpha-$admissible;*
- *(ii) there exists $\mu_0 \in M$ such that $\alpha(\mu_0, T(\mu_0)) \geq 1$;*
- *(iii) either T is continuous or M is $\alpha-$regular.*

Then T has a fixed point. Moreover, if

- *(iv) for all fixed points μ, ν of T, either $\alpha(\mu, \nu) \geq 1$ or $\alpha(\nu, \mu) \geq 1$,*

then T has a unique fixed point.

3 Functional Conformable Fractional Differential Equations

In this section, we are concerned with the existence results of the problem (1).

Let $(C(I), d, 2)$ be the b-metric space with $c = 2$, such that $d : C(I) \times C(I) \to \mathbb{R}_+^*$ is given by:

$$d(u, v) = \|(u - v)^2\|_\infty := \sup_{t \in I} |u(t) - v(t)|^2.$$

Definition 7. *By a solution of the problem (1) we mean a function $u \in C(I)$ that satisfies*

$$u(t) = u_a + \int_a^t (s - a)^{r-1} f(s, u(s)) ds.$$

The following hypotheses will be used in the sequel.

(H_1) There exist $\phi \in \Phi$, $p : C(I) \times C(I) \to (0, \infty)$ such that for each $u, v \in C(I)$, and $t \in I$

$$|f(t, u) - f(t, v)| \leq p(u, v)|u(t) - v(t)|,$$

with

$$\left\| \int_a^t (s - a)^{r-1} p(u, v) ds \right\|_\infty^2 \leq \phi(\|(u - v)^2\|_\infty).$$

(H_2) There exist $\mu_0 \in C(I)$ and a function $\theta : C(I) \times C(I) \to \mathbb{R}$, such that

$$\theta\left(\mu_0(t), u_a + \int_a^t (s - a)^{r-1} f(s, \mu_0(s)) ds\right) \geq 0.$$

(H_3) For each $t \in I$, and $u, v \in C(I)$, we have:

$$\theta(u(t), v(t)) \geq 0$$

implies

$$\theta\left(u_a + \int_a^t (s - a)^{r-1} f(s, u(s)) ds, u_a + \int_a^t (s - a)^{r-1} f(s, v(s)) ds\right) \geq 0.$$

(H_4) If $\{u_n\}_{n \in \mathbb{N}} \subset C(I)$ with $u_n \to u$ and $\theta(u_n, u_{n+1}) \geq 1$, then $\theta(u_n, u) \geq 1$.

Theorem 2. *Assume that hypotheses $(H_1)-(H_4)$ hold. Then the problem (1) has a least one solution defined on I.*

Proof. Consider the operator $N : C(I) \to C(I)$ defined by

$$(Nu)(t) = u_a + \int_a^t (s-a)^{r-1} f(s, u(s))ds.$$

By using Lemma 2, it is clear that the fixed points of the operator N are solutions of (1).

Let $\alpha : C(I) \times C(I) \to (0,\infty)$ be the function defined by:

$$\begin{cases} \alpha(u,v) = 1; & if\ \theta(u(t), v(t)) \geq 0,\ t \in I, \\ \alpha(u,v) = 0; & elese. \end{cases} \tag{6}$$

First, we prove that N is a generalized α-ϕ-Geraghty operator:
For any $u, v \in C(I)$ and each $t \in I$, we have

$$\begin{aligned} |(Nu)(t) - (Nv)(t)| &\leq \int_a^t (s-a)^{r-1} |f(s,u(s)) - f(s,v(s))| ds \\ &\leq \int_a^t (s-a)^{r-1} p(u,v)|u(s) - v(s)| ds \\ &\leq |u(t) - v(t)|^2)^{\frac{1}{2}} \int_a^t (s-a)^{r-1} p(u,v) ds \\ &\leq \|(u-v)^2\|_\infty^{\frac{1}{2}} \int_a^t (s-a)^{r-1} p(u,v) ds. \end{aligned}$$

Thus

$$\begin{aligned} &\alpha(u,v)|(Nu)(t) - (Nv)(t)|^2 \\ &\leq \|(u-v)^2\|_\infty \alpha(u,v) \left\| \int_a^t (s-a)^{r-1} p(u,v) ds \right\|_\infty^2 \\ &\leq \|(u-v)^2\|_\infty \phi(\|(u-v)^2\|_\infty). \end{aligned}$$

Hence

$$\alpha(u,v)\phi(2^3 d(N(u), N(v)) \leq \lambda(\phi(d(u,v))\phi(d(u,v)),$$

where $\lambda \in F$, $\phi \in \Phi$, with $\lambda(t) = \frac{1}{8}t$, and $\phi(t) = t$.
So, N is generalized α-ϕ-Geraghty operator.

Let $u, v \in C(I)$ such that

$$\alpha(u,v) \geq 1.$$

Thus, for each $t \in I$, we have

$$\theta(u(t), v(t)) \geq 0.$$

This implies from (H_3) that

$$\theta(Nu(t), Nv(t)) \geq 0,$$

which gives

$$\alpha(N(u), N(v)) \geq 1.$$

Hence, N is a α-admissible.

Now, from (H_2), there exists $\mu_0 \in C(I)$ such that

$$\alpha(\mu_0, N(\mu_0)) \geq 1.$$

Finally, from (H_4), If $\mu_{n n\in N} \subset M$ with $\mu_n \to \mu$ and $\alpha(\mu_n, \mu_{n+1}) \geq 1$, then

$$\alpha(\mu_n, \mu) \geq 1.$$

From an application of Theorem 1, we deduce that N has a fixed point u which is a solution of problem (1).

4 Neutral Conformable Fractional Differential Equations

Now, we are concerned with the existence results of the problems (2) and (3). Consider the Banach space

$$C = \{u : (-h, b] \to \mathbb{R},\ u|_{(-h,a]} \in \mathcal{C},\ u|_I \in C(I)\},$$

with the norm

$$\|u\|_C = \max\{\|\varphi\|_{[-h,a]}, \|u\|_\infty\}.$$

Let $(C, d, 2)$ be the b-metric space with $c = 2$, such that $d : C \times C \to \mathbb{R}_+^*$ is given by:

$$d(u, v) = \|(u - v)^2\|_C := \max\{\|\varphi\|_{[-h,a]}, \|(u - v)^2\|_\infty\}.$$

Definition 8. *By a solution of the problem (2) we mean a function $u \in C$ that satisfies*

$$u(t) = \begin{cases} \varphi(t);\ t \in [-h, a], \\ \varphi(a) - z(a, u_a) + z(t, u(t)) + \int_a^t (s - a)^{r-1} f(s, u_s) ds;\ t \in I. \end{cases}$$

Consider the following hypotheses:

(H_{01}) There exist $\psi \in \Phi$, and $p, q : \mathcal{C}\times\mathcal{C} \to (0,\infty)$ such that for each $u, v \in \mathcal{C}$, and $t \in I$

$$|f(t,u) - f(t,v)| \le p(u,v)\|u - v\|_{[-h,a]},$$

and

$$|z(t,u) - z(t,v)| \le q(u,v)\|u - v\|_{[-h,a]},$$

with

$$\left\| q(u,v) + \int_a^t (s-a)^{r-1} p(u,v)ds \right\|_C^2 \le \psi(\|(u-v)^2\|_C).$$

(H_{02}) There exist $\nu_0 \in C(I)$ and a function $\iota : C(I) \times C(I) \to \mathbb{R}$, such that

$$\iota\left(\nu_0(t), \varphi(a) - g(a,u_a) + g(t,\nu_{0t}) + \int_a^t (s-a)^{r-1} f(s,\nu_{0s})ds\right) \ge 0.$$

(H_{03}) For each $t \in I$, and $u, v \in C(I)$, we have:

$$\iota(u(t), v(t)) \ge 0$$

implies $\iota\Big(\varphi(a) - g(a,u_a) + g(t,u_t) + \int_a^t (s-a)^{r-1} f(s,u_s)ds,$

$$\varphi(a) - g(a,v_a) + g(t,v_t) + \int_a^t (s-a)^{r-1} f(s,v_s)ds\Big) \ge 0.$$

(H_{04}) If $\{u_n\}_{n\in\mathbb{N}} \subset C(I)$ with $u_n \to u$ and $\iota(u_n, u_{n+1}) \ge 1$, then $\iota(u_n, u) \ge 1$.

Theorem 3. *Assume that hypotheses $(H_{01})-(H_{04})$ hold. Then the problem (2) has a least one solution defined on $[-h,b]$.*

Proof. Consider the operator $G : C \to C$ defined by

$$(Gu)(t) = \begin{cases} \varphi(t); \ t \in [-h,a], \\ \varphi(a) - g(a,u_a) + g(t,u_t) + \int_a^t (s-a)^{r-1} f(s,u_s)ds; \ t \in I. \end{cases} \tag{7}$$

It is clear that the fixed points of the operator G are solutions of (2).

Let $\alpha : C(I) \times C(I) \to (0,\infty)$ be the function defined in (6).

We start by proving that G is a generalized α-ψ-Geraghty operator: Let $u, v \in C$. For each $t \in [-h,a]$, we have

$$|(Gu)(t) - (Gv)(t)| = 0,$$

and for each $t \in I$, we have

$$\begin{aligned}|(Gu)(t)-(Gv)(t)| &\le |z(t,u_t)-z(t,v_t)| + \int_a^t (s-a)^{r-1}|f(s,u_s)-f(s,v_s)|ds \\ &\le q(u,v)|u_t-v_t| + \int_a^t (s-a)^{r-1}p(u,v)|u_s-v_s|ds \\ &\le |u_t-v_t|^2)^{\frac{1}{2}} q(u,v) + |u_t-v_t|^2)^{\frac{1}{2}} \int_a^t (s-a)^{r-1}p(u,v)ds \\ &\le \|(u-v)^2\|_C^{\frac{1}{2}} \left(q(u,v) + \int_a^t (s-a)^{r-1}p(u,v)ds \right).\end{aligned}$$

Thus, we get

$$\begin{aligned}&\alpha(u,v)|(Gu)(t)-(Gv)(t)|^2 \\ &\le \|(u-v)^2\|_C \alpha(u,v) \left\| q(u,v) + \int_a^t (s-a)^{r-1}p(u,v)ds \right\|_C^2 \\ &\le \|(u-v)^2\|_C \psi(\|(u-v)^2\|_C).\end{aligned}$$

Hence

$$\alpha(u,v)\psi(2^3 d(G(u),G(v)) \le \lambda(\psi(d(u,v))\psi(d(u,v)),$$

where $\lambda \in \digamma$, $\psi \in \Phi$, with $\lambda(t) = \frac{1}{8}t$, and $\psi(t) = t$.
So, G is generalized α-ψ-Geraghty operator.

Let $u, v \in C(I)$ such that

$$\alpha(u,v) \ge 1.$$

Thus, for each $t \in I$, we have

$$\iota(u(t), v(t)) \ge 0.$$

This implies from (H_{03}) that

$$\iota((Gu)(t), (Gv)(t)) \ge 0,$$

which gives

$$\alpha(G(u), G(v)) \ge 1.$$

Hence, N is a α-admissible.

Now, from (H_{02}), there exists $\nu_0 \in C(I)$ such that

$$\alpha(\nu_0, G(\nu_0)) \ge 1.$$

Finally, from (H_{04}), If $\mu_{n n \in N} \subset M$ with $\mu_n \to \mu$ and $\alpha(\mu_n, \mu_{n+1}) \ge 1$, then

$$\alpha(\mu_n, \mu) \ge 1.$$

From an application of Theorem 1, we deduce that G has a fixed point u which is a solution of problem (2).

5 Neutral Conformable Fractional Differential Equations with Infinite Delay

In this section, we establish some existence results for problem (3). Let the space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a seminormed linear space of functions mapping $(-\infty, a]$ into $\mathbb{R}$, and satisfying the following fundamental axioms which were adapted from those introduced by Hale and Kato [19] for ordinary differential functional equations:

(A_1) If $u : (-\infty, b] \to \mathbb{R}$, and $u_a \in \mathcal{B}$, then there are constants $L, M, H > 0$, such that for any $t \in I$ the following conditions hold:

- (i) u_t *is in* $\mathcal{B}$,
- (ii) $\|u_t\|_{\mathcal{B}} \leq K\|u_1\|_{\mathcal{B}} + M \sup_{s\in[a,t]} |u(s)|$,
- (iii) $\|u(t)\| \leq H\|u_t\|_{\mathcal{B}}$.

(A_2) For the function $u(\cdot)$ *in* $(A1)$, u_t is a $\mathcal{B}-$ valued continuous function on I.

(A_3) The space $\mathcal{B}$ is complete.

Consider the space

$$\Omega = \{u : (-\infty, b] \to \mathbb{R},\ u|_{(-\infty,a]} \in \mathcal{B},\ u|_I \in C(I)\}.$$

Definition 9. *By a solution of the problem (3) we mean a function* $u \in \Omega$ *that satisfies*

$$u(t) = \begin{cases} \varphi(t);\ t \in (-\infty, a], \\ \varphi(a) - g(a, u_a) + g(t, u_t) + \int_a^t (s-a)^{r-1} f(s, u_s)ds;\ t \in I. \end{cases}$$

Consider the operator $N_1 :\ \Omega \to \Omega$ defined by:

$$(N_1u)(t) = \begin{cases} \varphi(t);\ t \in (-\infty, a], \\ \varphi(a) - z(a, u_a) + z(t, u_t) + \int_a^t (s-a)^{r-1} f(s, u_s)ds;\ t \in I. \end{cases} \tag{8}$$

Let $x(\cdot) : (-\infty, b] \to \mathbb{R}$ be a function defined by

$$x(t) = \begin{cases} \varphi(t); & t \in (-\infty, a], \\ \varphi(a) - z(a, u_a) & t \in I. \end{cases}$$

Then $x_0 = \varphi$. For each $z \in C(I)$, with $z(0) = 0$, we denote by $\overline{z}$ the function defined by

$$\overline{z} = \begin{cases} 0; & t \in t \in (-\infty, 0], \\ z(t), & t \in I. \end{cases}$$

If $u(\cdot)$ satisfies the integral equation

$$u(t) \quad = \quad \varphi(a) - g(a, u_a) + g(t, u_t) + \textstyle\int_a^t (s-a)^{r-1} f(s, u_s) ds.$$

We can decompose $u(\cdot)$ as $u(t) = \overline{z}(t) + x(t)$; for $t \in I$, which implies that $u_t = \overline{z}_t + x_t$ for every $t \in I$, and the function $z(\cdot)$ satisfies

$$z(t) = z(t, \overline{z}_s + x_s) + \int_a^t (s-a)^{r-1} f(s, \overline{z}_s + x_s) ds.$$

Set

$$C_0 = \{z \in C(I);\ z_0 = 0\},$$

and let $\|\cdot\|_T$ be the norm in C_0 defined by

$$\|z\|_b = \|z_0\|_{\mathcal{B}} + \sup_{t \in I} |z(t)| = \sup_{t \in I} |z(t)|;\ z \in C_0.$$

C_0 is a Banach space with norm $\|\cdot\|_b$. Define the operator $P : C_0 \to C_0$; by

$$(Pz)(t) = z(t, \overline{z}_s + x_s) + \int_a^t (s-a)^{r-1} f(s, \overline{z}_s + x_s) ds. \tag{9}$$

Thus, the operator N_1 has a fixed point is equivalent to P has a fixed point. We turn to proving that P has a fixed point.

Let $(C_0, d, 2)$ be the b-metric space with $c = 2$, such that
$d : C_0 \times C_0 \to \mathbb{R}_+^*$ is given by:

$$d(u, v) = \|(u - v)^2\|_b.$$

As in the prove of Theorem 3, we give without prove the following Theorem:

Theorem 4. *Assume that the following hypotheses hold:*

(H_{001}) *There exist* $\psi \in \Phi$, *and* $p, q : \mathcal{B} \times \mathcal{B} \to (0, \infty)$ *such that for each* $u, v \in \mathcal{B}$, *and* $t \in I$

$$|f(t, u) - f(t, v)| \le p(u, v)\|u - v\|_{\mathcal{B}},$$

and

$$|z(t, u) - z(t, v)| \le q(u, v)\|u - v\|_{\mathcal{B}},$$

with

$$\left\| q(u,v) + \int_a^t (s-a)^{r-1} p(u,v) ds \right\|_b^2 \leq \psi(\|(u-v)^2\|_b),$$

(H_{002}) *There exist* $\nu_1 \in C(I)$ *and a function* $\iota : C(I) \times C(I) \to \mathbb{R}$, *such that*

$$\iota\left(\nu_1(t), z(t,\nu_{1t}) + \int_a^t (s-a)^{r-1} f(s,\nu_{1s}) ds\right) \geq 0,$$

(H_{003}) *For each* $t \in I$, *and* $u, v \in C_0$, *we have:*

$$\iota(u_t, v_t) \geq 0$$

implies $\iota\Big(z(t,u_t) + \int_a^t (s-a)^{r-1} f(s,u_s) ds,$

$$z(t,v_t) + \int_a^t (s-a)^{r-1} f(s,v_s) ds\Big) \geq 0,$$

(H_{004}) *If* $\{w_n\}_{n\in\mathbb{N}} \subset C_0$ *with* $w_n \to u$ *and* $\iota(w_n, w_{n+1}) \geq 1$, *then* $\iota(w_n, w) \geq 1$.

Then the problem (3) has a least one solution defined on $(-\infty, b]$.

6 Examples

Example 1. Let $(C([0,1]), d, 2)$ be the complete b-metric space, such that $d : C([0,1]) \times C([0,1]) \to \mathbb{R}_+^*$ is given by:

$$d(u,v) = \|(u-v)^2\|_C.$$

Consider the following conformable fractional differential problem

$$\begin{cases} (T_{0^+}^r u)(t) = f(t,u(t)); \ t \in [0,1], \\ u(0) = 1, \end{cases} \tag{10}$$

where

$$f(t,u(t)) = \frac{1+\sin(|u(t)|)}{4(1+|u(t)|)}; \ t \in [0,1].$$

Let $t \in (0,1]$, and $u \in C([0,1])$. If $|u(t)| \le |v(t)|$, then

$$
\begin{aligned}
&|f(t,u(t))) - f(t,v(t))| = \left|\frac{1+\sin(|u(t)|)}{4(1+|u(t)|)} - \frac{1+\sin(|v(t)|)}{4(1+|v(t)|)}\right| \\
\le\ & \frac{1}{4}||u(t)|-|v(t)|| + \frac{1}{4}|\sin(|u(t)|) - \sin(|v(t)|)| \\
+\ & \frac{1}{4}\,||u(t)|\sin(|v(t)|) - |v(t)|\sin(|u(t)|)| \\
\le\ & \frac{1}{4}|u(t)-v(t)| + \frac{1}{4}|\sin(|u(t)|) - \sin(|v(t)|)| \\
+\ & \frac{1}{4}\,||v(t)|\sin(|v(t)|) - |v(t)|\sin(|u(t)|)| \\
=\ & \frac{1}{4}|u(t)-v(t)| + \frac{1}{4}(1+|v(t)|)\,|\sin(|u(t)|) - \sin(|v(t)|)| \\
\le\ & \frac{1}{4}|u(t)-v(t)| + \frac{1}{2}(1+|v(t)|) \\
\times\ & \left|\sin\left(\frac{||u(t)|-|v(t)||}{2}\right)\right|\left|\cos\left(\frac{|u(t)|+|v(t)|}{2}\right)\right| \\
\le\ & \frac{1}{4}(2+|v(t)|)|u(t)-v(t)|.
\end{aligned}
$$

The case when $|v(t)| \le |u(t)|$, we get

$$|f(t,u(t)) - f(t,v(t))| \le \frac{1}{4}(2+|u(t)|)|u(t)-v(t)|.$$

So,

$$|f(t,u(t)) - f(t,v(t))| \le \frac{1}{4}\min_{t\in I}\{2+|u(t)|, 2+|v(t)|\}|u(t)-v(t)|.$$

Thus, hypothesis (H_1) is satisfied with

$$p(u,v) = \frac{1}{4}\min_{t\in I}\{2+|u(t)|, 2+|v(t)|\}.$$

Define the functions $\lambda(t) = \frac{1}{8}t$, $\phi(t) = t$, $\alpha : C([0,1]) \times C([0,1]) \to \mathbb{R}_+^*$ with

$$
\begin{cases}
\alpha(u,v) = 1; \ if \ \delta(u(t),v(t)) \ge 0, \ t \in I,\\
\alpha(u,v) = 0; \ else,
\end{cases}
$$

and $\delta : C([0,1]) \times C([0,1]) \to \mathbb{R}$ with $\delta(u,v) = \|u-v\|_C$.
Hypothesis (H_2) is satisfied with $\mu_0(t) = u(0)$. Also, (H_3) holds from the

definition of the function δ. Hence by Theorem 2, problem (10) has at least one solution defined on $[0,1]$.

Example 2. Consider now the following conformable neutral fractional differential problem

$$\begin{cases} u(t)=t;\ t\in[-1,0],\\ T^r_{0^+}\left(u(t)-1-\sin(|u_t|)\right)=\frac{1+\sin(|u_t|)}{1+|u_t|};\ t\in[0,1]. \end{cases} \tag{11}$$

For each $t\in[0,1]$, we set

$$f(t,u)=\frac{1+\sin(\|u\|_{[-1,0]})}{1+\|u\|_{[-1,0]})},$$

and

$$z(t,u)=1+\sin(\|u\|_{[-1,0]}).$$

Let $t\in(0,1]$, and $u,v\in C([-1,0])$. Then, we get

$$|f(t,u)-f(t,v)|\le\min_{t\in I}\{2+\|u\|_{[-1,0]}),2+\|v\|_{[-1,0]})\}\|u-v\|_{[-1,0]}),$$

and

$$|z(t,u)-z(t,v)|\le\left|\cos\left(\frac{\|u\|_{[-1,0]})+\|v\|_{[-1,0]})}{2}\right)\right|\|u-v\|_{[-1,0]}).$$

Thus, hypothesis (H_{01}) is satisfied with

$$p(u,v)=\frac{1}{4}\min_{t\in I}\{2+\|u\|_{[-1,0]}),2+\|v\|_{[-1,0]})\},$$

and

$$q(u,v)=\left|\cos\left(\frac{\|u\|_{[-1,0]})+\|v\|_{[-1,0]})}{2}\right)\right|.$$

Define the functions $\lambda(t)=\frac{1}{8}t$, $\psi(t)=t$, $\alpha:C([0,1])\times C([0,1])\to\mathbb{R}^*_+$ with

$$\begin{cases}\alpha(u,v)=1;\ if\ \delta(u(t),v(t))\ge 0,\ t\in I,\\ \alpha(u,v)=0;\ else,\end{cases}$$

and $\delta:C([0,1])\times C([0,1])\to\mathbb{R}$ with $\delta(u,v)=\|u-v\|_C$.
Hypothesis (H_{02}) is satisfied with $\nu_0(t)=2$. Also, (H_{03}) is satisfied from the definition of the function δ. Hence by Theorem 3, problem (11) has at least one solution defined on $[-1,1]$.

Example 3. Let γ be a positive real constant and

$$B_\gamma = \{u \in C((-\infty, 1], \mathbb{R},) : \lim_{\theta \to -\infty} e^{\gamma\theta\theta} u(\theta) \ exists \ in \ \mathbb{R}\}. \tag{12}$$

The norm of B_γ is given by

$$\|u\|_\gamma = \sup_{\theta \in (-\infty, 1]} e^{\gamma\theta} |u(\theta)|.$$

Let $u : (-\infty, 1] \to \mathbb{R}$ be such that $u_0 \in B_\gamma$. Then

$$\lim_{\theta \to -\infty} e^{\gamma\theta} u_t(\theta)$$

$$= \lim_{\theta \to -\infty} e^{\gamma\theta} u(t + \theta - 1) = \lim_{\theta \to -\infty} e^{\gamma(\theta - t + 1)} u(\theta)$$

$$= e^{\gamma(-t+1)} \lim_{\theta \to -\infty} e^{\gamma(\theta)} u_1(\theta) < \infty.$$

Hence $u_t \in B_\gamma$. Finally we prove that

$$\|u_t\|_\gamma \leq K \|u_1\|_\gamma + M \sup_{s \in [0,t]} |u(s)|,$$

where $K = M = 1$ and $H = 1$. We have

$$\|u_t(\theta)\| = |u(t + \theta - 1|.$$

If $t + \theta \leq 1$, we get

$$\|u_t(\beta)\| \leq \sup_{s \in (-\infty, 1]} |u(s)|.$$

For $t + \theta \geq 1$, then we have

$$\|u_t(\beta)\| \leq \sup_{s \in [0,t]} |u(s)|.$$

Thus for all $t + \theta \in I$, we get

$$\|u_t(\beta)\| \leq \sup_{s \in (-\infty, 0]} |u(s)| + \sup_{s \in [0,t]} |u(s)|.$$

Then

$$\|u_t\|_\gamma \leq \|u_1\|_\gamma + \sup_{s \in [0,t]} |u(s)|.$$

It is clear that $(B_\gamma, \| \cdot \|)$ is a Banach space. We can conclude that B_γ a phase space.

Consider now the following problem

$$\begin{cases} u(t)=t;\ t\in(-\infty,0], \\ T^r_{0^+}\left(u(t)-1-\sin(\|u\|_{B_\gamma})\right)=\frac{1+\sin(\|u\|_{B_\gamma})}{1+\|u\|_{B_\gamma}};\ t\in[0,1]. \end{cases} \tag{13}$$

For each $t\in[0,1]$, we set

$$f(t,u)=\frac{1+\sin(\|u\|_{B_\gamma})}{1+\|u\|_{B_\gamma}},$$

and

$$z(t,u)=1+\sin(\|u\|_{B_\gamma}).$$

Simple computations show that all conditions of Theorem 4 are satisfied. Hence, problem (13) has at least one solution defined on $(-\infty,1]$.

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