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STEADY CONVECTION IN MHD BÉNARD PROBLEM WITH HALL AND ION-SLIP EFFECTS*

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Abstract

In this paper we study the nonlinear Lyapunov stability of the thermodiffusive equilibrium for a viscous thermoelectroconducting partially ionized fluid in a horizontal layer heated from below.
The classical L^2 norm is too weak to evaluate some stabilizing or instabilizing effects of the electroanisotropic currents.
A more fine Lyapunov function is obtained by reformulating the initial perturbation evolution problem in terms of the poloidal and toroidal scalar fields.
In such a way, if instability occurs as stationary convection, we obtain the coincidence of linear and nonlinear stability bounds.

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1 Introduction

Several variants of the *classical energy method* can be found in literature [1]-[13] to investigate the Lyapunov stability for non-stationary equations, with

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the aim of determining the largest stability domain in the parameter space and, sometimes, the coincidence of the linear and nonlinear stability bounds [1]- [13] .

The papers [14], [15]and [16] dealt with an extension of the Joseph's parametric differentiation method applied to study of the stability of the conduction diffusion state for a binary mixture, in the presence of thermoanisotropic Soret and Dufour currents.

In [17], [18] and [19] the stability problem for a binary mixture in a plane layer, with chemical surface reactions, was reformulated to obtain an equivalent one with better symmetry properties. In such a way the coincidence of linear and nonlinear stability bounds was obtained, in the region of stationary convection of the linear instability theory.

In [20] the nonlinear stability of the thermodiffusive equilibrium for the magnetohydrodynamic anisotropic Bénard problem is studied by the energy splitting [11], obtaining conditional nonlinear asymptotical stability results.

In [21], [22], [23] and [24] the perturbation evolution equations are reformulated by splitting their *potential and solenoidal* parts. The resulting equations, where the unknown are the *independent* poloidal and toroidal scalar fields, allows us to preserve the contribution of some physical effects, such as the rotation [22]-[23] and the magnetic field [24].

If instability occurs as stationary convection we recover the coincidence of the nonlinear stability bound with the linear one obtained by the classical normal modes technique, for the rotating Bénard problem in the hydrodynamic case [21]- [23], for the classical magnetohydrodynamic Bénard problem [24], for the electroanisotropic magnetohydrodynamic Bénard problem [25], in the presence of Hall effect.

This paper dealts with the nonlinear Lyapunov stability of the thermodiffusive equilibrium of a viscous thermoelectroconducting partially ionized fluid in a plane layer heated from below in the Oberbeck-Boussinesq approximation.

After formulating the mathematical model (Sec. 2), we derive some additional evolution perturbation equations in terms of poloidal and toroidal fields (Sec. 3), we study the Lyapunov stability of the thermodiffusive equilibrium (Sec. 4), obtaining the coincidence of linear and nonlinear stability bounds (Sec. 5), if instability occurs as stationary convection.

2 Mathematical model

Let us consider a Newtonian thermo-electroconducting viscous fluid in a horizontal layer S, bounded by the planes $z = 0$ and $z = 1$ in a Cartesian frame of reference $(O, \mathbf{i}, \mathbf{j}, \mathbf{k})$, with $\mathbf{k}$ vertical upwards unit vector.
The fluid, heated from below, is subject to a vertical temperature gradient, in an external constant magnetic field $\mathbf{H_0} = H_0\mathbf{k}$.
In the Oberbeck-Boussinesq approximation the (dimensionless) mathematical model is the following [1], [26], [27]:

$$\begin{cases} \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla P + M^2\mathbf{H}\cdot\nabla\mathbf{H} - [\mathbf{1} - \mathcal{R}(\mathbf{T}-\mathbf{T_0})]\,\mathbf{k} + \mathbf{\Delta v}, \\ \frac{\partial \mathbf{H}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{H}) + \frac{\mathcal{P}_m}{\mathcal{P}_r}\Delta\mathbf{H} + \beta_H\frac{\mathcal{P}_m}{\mathcal{P}_r}\nabla\times(\mathbf{H}\times\nabla\times\mathbf{H}) + \\ \qquad \beta_I\frac{\mathcal{P}_m}{\mathcal{P}_r}\nabla\times\big(\mathbf{H}\times(\mathbf{H}\times\nabla\times\mathbf{H})\big), \\ \frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{T} = \frac{\mathbf{1}}{\mathcal{P}_\mathbf{r}}\mathbf{\Delta T}, \\ \nabla\cdot\mathbf{v} = 0, \\ \nabla\cdot\mathbf{H} = 0, \end{cases} \tag{1}$$

where $\mathbf{v}$, $\mathbf{H}$, T, P are velocity, magnetic, temperature and pressure fields, respectively. T_0 represents a reference temperature. M^2, $\mathcal{R}^2$, $\mathcal{P}_r$, $\mathcal{P}_m$, β_H, β_I denote dimensionless Hartmann, Rayleigh, Prandtl, magnetic Prandtl, Hall and ion-slip numbers, respectively. To the system (1) we add the boundary conditions [26]:

$$\begin{cases} \mathbf{v}\cdot\mathbf{n} = 0, \quad \mathbf{n}\times\mathbf{D}\cdot\mathbf{n} = \mathbf{0}, \quad \mathbf{H} = \mathbf{H}_0, \quad \mathbf{n}\cdot\nabla\times\mathbf{H} = 0, \qquad z = 0, 1, \\ T = T^0, \qquad z = 0, \\ T = T^1, \qquad z = 1, \end{cases} \tag{2}$$

for stress free, thermal conducting and electrically non conducting planes.
In (2) $\mathbf{D}$ is the strain rate tensor, $\mathbf{n}$ is the outer (unit) normal to the boundary ∂S
Moreover $(1)_{1,3,4}$ are, in the Oberbeck-Boussinesq approximation, the balance equation for momentum, energy and mass, $(1)_2$ follows taking into account the generalized Ohm's law for the Maxwell equations Galileo invariant.
We consider, for a not too large temperature gradient β, the conduction state [28]

$$\left(\overline{\mathbf{v}} = \mathbf{0},\ \overline{\mathbf{H}} = \mathbf{H}_0,\ \overline{T} = T^0 - \beta z,\ \overline{P} = P(z)\right), \tag{3}$$

in the periodicity cell $V = \mathcal{V} \times [0,1]$, where $\mathcal{V} = \left[0, \frac{2\pi}{k_x}\right] \times \left[0, \frac{2\pi}{k_y}\right]$ and $k^2 = k_x^2 + k_y^2$ is the wave number.
For increasing gradient β the fluid has a stationary motion, periodic in the x and y directions, i.e. the thermal horizontal convection, that becomes non stationary, till the turbulence [28].

3 Perturbation model

Let us denote with $\mathbf{v} = \overline{\mathbf{v}} + \mathbf{u}$, $\mathbf{H} = \overline{\mathbf{H}} + \mathbf{h}$, $P = \overline{P} + p$, $T = \overline{T} + \vartheta$ the perturbed fields around the conduction state (3).
Then dimensionless equations governing the evolution of the perturbation $(\mathbf{u}, \mathbf{h}, \mathbf{p}, \vartheta)$ of (3) are the following:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \mathbf{p} + \mathbf{M^2} \left(\dfrac{\partial \mathbf{h}}{\partial \mathbf{z}} + \mathbf{h} \cdot \nabla \mathbf{h} \right) + \mathcal{R} \vartheta \mathbf{k} + \boldsymbol{\Delta} \mathbf{u}, \\ \dfrac{\partial \mathbf{h}}{\partial t} = \dfrac{\partial \mathbf{u}}{\partial z} + \nabla \times (\mathbf{u} \times \mathbf{h}) + \dfrac{\mathcal{P}_\mathbf{m}}{\mathcal{P}_\mathbf{r}} \boldsymbol{\Delta} \mathbf{h} + \beta_\mathbf{H} \dfrac{\mathcal{P}_\mathbf{m}}{\mathcal{P}_\mathbf{r}} \nabla \times \Big((\mathbf{h} + \mathbf{k}) \times \nabla \times \mathbf{h} \Big) + \\ \qquad \beta_I \frac{\mathcal{P}_m}{\mathcal{P}_r} \nabla \times \Big((\mathbf{h} + \mathbf{k}) \times ((\mathbf{h} + \mathbf{k}) \times \nabla \times \mathbf{h}) \Big) \\ \dfrac{\partial \vartheta}{\partial t} + \mathbf{u} \cdot \nabla \vartheta = \dfrac{\mathbf{1}}{\mathcal{P}_\mathbf{r}} \Big(\boldsymbol{\Delta} \vartheta + \mathcal{R} \mathbf{w} \Big), \\ \nabla \cdot \mathbf{u} = 0, \\ \nabla \cdot \mathbf{h} = 0, \end{cases} \tag{4}$$

with the boundary conditions:

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = 0, \mathbf{h} = \mathbf{0}, \vartheta = 0 \text{ on } (\partial V)_0 \cup (\partial V)_1, \tag{5}$$

where

$$(\partial V)_0 = \left\{ (x,y,z) \in \mathbf{R}^3 \mid 0 \le x \le \frac{2\pi}{k_x}, 0 \le y \le \frac{2\pi}{k_y}, z = 0 \right\},$$

$$(\partial V)_1 = \left\{ (x,y,z) \in \mathbf{R}^3 \mid 0 \le x \le \frac{2\pi}{k_x}, 0 \le y \le \frac{2\pi}{k_y}, z = 1 \right\}.$$

If the mean values of the components of velocity and magnetic fields vanish over $\mathcal{V}$, that is if the conditions [4] [29]

$$\begin{aligned}\int_{\mathcal{V}} u(x,y,z)\,dx\,dy &= \int_{\mathcal{V}} v(x,y,z)\,dx\,dy\\ &= \int_{\mathcal{V}} w(x,y,z)\,dx\,dy = 0, \quad \forall z \in [0,1],\end{aligned} \tag{6}$$

$$\begin{aligned}\int_{\mathcal{V}} h_1(x,y,z)\,dx\,dy &= \int_{\mathcal{V}} h_2(x,y,z)\,dx\,dy\\ &= \int_{\mathcal{V}} h_3(x,y,z)\,dx\,dy = 0, \forall z \in [0,1],\end{aligned} \tag{7}$$

are satisfied, then the velocity $\mathbf{u}$ and the magnetic field $\mathbf{h}$ have the unique decomposition [4], [29]:

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2, \qquad \mathbf{h} = \mathbf{h}_1 + \mathbf{h}_2, \tag{8}$$

with

$$\nabla \cdot \mathbf{u}_1 = \nabla \cdot \mathbf{u}_2 = \mathbf{k} \cdot \nabla \times \mathbf{u}_1 = \mathbf{k} \cdot \mathbf{u}_2 = 0, \tag{9}$$

$$\nabla \cdot \mathbf{h}_1 = \nabla \cdot \mathbf{h}_2 = \mathbf{k} \cdot \nabla \times \mathbf{h}_1 = \mathbf{k} \cdot \mathbf{h}_2 = 0, \tag{10}$$

$$\mathbf{u}_1 = \nabla \frac{\partial \chi}{\partial z} - \mathbf{k}\Delta\chi \equiv \nabla \times \nabla \times (\chi\mathbf{k}), \qquad \mathbf{u}_2 = \mathbf{k} \times \nabla\psi = -\nabla \times (\mathbf{k}\psi), \tag{11}$$

$$\mathbf{h}_1 = \nabla \frac{\partial \chi'}{\partial z} - \mathbf{k}\Delta\chi' \equiv \nabla \times \nabla \times (\chi'\mathbf{k}), \qquad \mathbf{h}_2 = \mathbf{k} \times \nabla\psi' = -\nabla \times (\mathbf{k}\psi'). \tag{12}$$

In (11), (12) the doubly-periodic functions χ, χ' and ψ, ψ', are the *poloidal and toroidal potentials*, satisfying [4], [29]:

$$\Delta_1\chi = -\mathbf{k} \cdot \mathbf{u} = -w, \qquad \Delta_1\psi = \mathbf{k} \cdot \nabla \times \mathbf{u}, \tag{13}$$

$$\Delta_1\chi' = -\mathbf{k} \cdot \mathbf{h} = -h_3, \qquad \Delta_1\psi' = \mathbf{k} \cdot \nabla \times \mathbf{h}, \tag{14}$$

where

$\Delta_1 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$. The boundary conditions (5) written in terms of χ, ψ, χ', ψ' become [4]:

$$\chi = \frac{\partial^2 \chi}{\partial z^2} = \frac{\partial \psi}{\partial z} = 0, \qquad \chi' = \frac{\partial \chi'}{\partial z} = \Delta_1\psi' = 0, \qquad z = 0,1. \tag{15}$$

The third component of $(4)_2$, give us:

$$\frac{\partial h_3}{\partial t}=\frac{\partial w}{\partial z}+\nabla\cdot[(\mathbf{u}\times\mathbf{h})\times\mathbf{k}]+\frac{\mathcal{P}_m}{\mathcal{P}_r}\Delta h_3+\beta_H\frac{\mathcal{P}_m}{\mathcal{P}_r}\nabla\cdot\Big[\Big((\mathbf{h}+\mathbf{k})\times\nabla\times\mathbf{h}\Big)\times\mathbf{k}\Big]+ \tag{16}$$
$$\beta_I\frac{\mathcal{P}_m}{\mathcal{P}_r}\nabla\cdot[\Big((\mathbf{h}+\mathbf{k})\times((\mathbf{h}+\mathbf{k})\times\nabla\times\mathbf{h})\Big)\times\mathbf{k}\Big].$$

If every term of (16) belongs to $W^{2,2}(V)$, taking into account (13), (14) and the imbedding of $W^{2,2}(V)$ in $C(\overline{V})$ [30], [31], the equation (16) can be written as follows:

$$\nabla\cdot\Big[\frac{\partial}{\partial t}\nabla_1\chi'-\nabla_1\frac{\partial\chi}{\partial z}+(\mathbf{u}\times\mathbf{h})\times\mathbf{k}-\frac{\mathcal{P}_m}{\mathcal{P}_r}\Delta\nabla_1\chi'+\beta_H\frac{\mathcal{P}_m}{\mathcal{P}_r}\Big((\mathbf{h}+\mathbf{k})\times\nabla\times\mathbf{h})\Big)\times\mathbf{k}+ \tag{17}$$
$$\beta_I\frac{\mathcal{P}_m}{\mathcal{P}_r}\Big((\mathbf{h}+\mathbf{k})\times((\mathbf{h}+\mathbf{k})\times\nabla\times\mathbf{h})\Big)\times\mathbf{k}\Big]=\mathbf{0},$$

where $\nabla_1\equiv\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)$.
It follows that there exists a vector field $\mathbf{B}$ such that

$$\frac{\partial}{\partial t}\nabla_1\chi'-\nabla_1\frac{\partial\chi}{\partial z}+(\mathbf{u}\times\mathbf{h})\times\mathbf{k}-\frac{\mathcal{P}_m}{\mathcal{P}_r}\Delta\nabla_1\chi'+\beta_H\frac{\mathcal{P}_m}{\mathcal{P}_r}\Big((\mathbf{h}+\mathbf{k})\times\nabla\times\mathbf{h}\Big)\times\mathbf{k}+ \tag{18}$$
$$\beta_I\frac{\mathcal{P}_m}{\mathcal{P}_r}\Big((\mathbf{h}+\mathbf{k})\times((\mathbf{h}+\mathbf{k})\times\nabla\times\mathbf{h})\Big)\times\mathbf{k}=\nabla\times\mathbf{B}.$$

From the Weyl decomposition theorem of $L^2(V)$ [8], [30], it follows:

$$-(\mathbf{u}\times\mathbf{h})\times\mathbf{k}=\nabla U_0+\nabla\times\mathbf{A}_0, \tag{19}$$

$$-\Big(\mathbf{h}\times\nabla\times\mathbf{h}\Big)\times\mathbf{k}=\nabla U_1+\nabla\times\mathbf{A}_1, \tag{20}$$

$$-\Big[\mathbf{h}\times\Big((\mathbf{h}+\mathbf{k})\times\nabla\times\mathbf{h}\Big)\Big]\times\mathbf{k}=\nabla U_2+\nabla\times\mathbf{A}_2, \tag{21}$$

where U_i and $\mathbf{A}_i$, $i=0,1,2$ are scalar and vector fields verifying the relations:

$$\frac{\partial}{\partial z}U_i+\mathbf{k}\cdot\nabla\times\mathbf{A}_i=0 \qquad i=0,1,2. \tag{22}$$

The scalar functions $U_i\,, i=0,1,2$ are the solutions of some interior Neumann problems [31] in the periodicity cell V, and the necessary and sufficient conditions for the existence of a solution , are fulfilled [31].

Substituting (19), (20), (21) in (18), using the identities

$$\Big(\mathbf{k}\times\nabla\times\mathbf{h}\Big)\times\mathbf{k} = -\nabla\times(\Delta\chi'\mathbf{k}) - \nabla_1\frac{\partial\psi'}{\partial z}, \tag{23}$$

$$\Big[\Big(\mathbf{k}\times(\mathbf{k}\times\nabla\times\mathbf{h})\Big)\Big]\times\mathbf{k} = -\nabla_1(\Delta\chi') + \nabla\times(\frac{\partial\psi'}{\partial z}\mathbf{k}), \tag{24}$$

$$\Big[\Big(\mathbf{k}\times(\mathbf{h}\times\nabla\times\mathbf{h})\Big)\Big]\times\mathbf{k} = \nabla\times(\mathbf{U_1}\mathbf{k}) + \frac{\partial\mathbf{A_1}}{\partial\mathbf{z}} - \nabla(\mathbf{A_1}\cdot\mathbf{k}), \tag{25}$$

and, assuming $\nabla\cdot\mathbf{A}_1 = 0$,

$$\Big[\Big(\mathbf{k}\times(\mathbf{h}\times\nabla\times\mathbf{h})\Big)\Big]\times\mathbf{k} = \nabla\times(U_1\mathbf{k}) + \nabla\times(\mathbf{A_1}\times\mathbf{k}) - \nabla(\mathbf{A_1}\cdot\mathbf{k}), \tag{26}$$

we obtain:

$$\frac{\partial}{\partial t}\nabla_1\chi' = \nabla_1\frac{\partial\chi}{\partial z} + \nabla U_0 + (1+\beta_I)\frac{\mathcal{P}_m}{\mathcal{P}_r}\nabla_1\Delta\chi' + \beta_H\frac{\mathcal{P}_m}{\mathcal{P}_r}\nabla U_1 + \beta_H\frac{\mathcal{P}_m}{\mathcal{P}_r}\nabla_1\frac{\partial\psi'}{\partial z} + \beta_I\frac{\mathcal{P}_m}{\mathcal{P}_r}\nabla\Big(U_2 + \mathbf{A}_1\cdot\mathbf{k}\Big), \tag{27}$$

taking into account that only the null vector belongs to both subspaces of potential and solenoidal vectors [8] [30].
From (27) it follows that

$$\frac{\partial}{\partial z}\Big(U_o + \beta_H\frac{\mathcal{P}_m}{\mathcal{P}_r}U_1 + \beta_I\frac{\mathcal{P}_m}{\mathcal{P}_r}(U_2 + \mathbf{A}_1\cdot\mathbf{k})\Big) = 0 \tag{28}$$

$$\nabla_1(\frac{\partial}{\partial t}\chi' - \frac{\partial\chi}{\partial z} - U_0 - (1+\beta_I)\frac{\mathcal{P}_m}{\mathcal{P}_r}\Delta\chi' - \beta_H\frac{\mathcal{P}_m}{\mathcal{P}_r}(U_1 + \frac{\partial\psi'}{\partial z}) - \beta_I\frac{\mathcal{P}_m}{\mathcal{P}_r}(U_2 + \mathbf{A}_1\cdot\mathbf{k})) = 0. \tag{29}$$

Hence we denote the function on the right hand side of (29) with $F(z)$:

$$\frac{\partial}{\partial t}\chi' - \frac{\partial\chi}{\partial z} - U_0 - (1+\beta_I)\frac{\mathcal{P}_m}{\mathcal{P}_r}\Delta\chi' - \beta_H\frac{\mathcal{P}_m}{\mathcal{P}_r}(U_1 + \frac{\partial\psi'}{\partial z}) - \beta_I\frac{\mathcal{P}_m}{\mathcal{P}_r}(U_2 + \mathbf{A}_1\cdot\mathbf{k}) = F(z). \tag{30}$$

Differentiating (30) with respect to z and taking into account (28) we obtain the evolution equation for χ':

$$\frac{\partial^2}{\partial t\partial z}\chi' = \frac{\partial^2\chi}{\partial z^2} + (1+\beta_I)\frac{\mathcal{P}_m}{\mathcal{P}_r}\Delta\frac{\partial}{\partial z}\chi' + \beta_H\frac{\mathcal{P}_m}{\mathcal{P}_r}\frac{\partial^2\psi'}{\partial z^2} + \frac{d}{dz}F(z). \tag{31}$$

If we apply to $(4)_2$ the operator $\mathbf{I}-\mathbf{k}\oplus\mathbf{k}$, where $\mathbf{I}$ is the identity operator, we have

$$\frac{\partial\mathbf{h}^\perp}{\partial t}=\frac{\partial\mathbf{u}^\perp}{\partial z}+[\nabla\times(\mathbf{u}\times\mathbf{h})]^\perp+\frac{\mathcal{P}_m}{\mathcal{P}_r}\Delta\mathbf{h}^\perp+ \tag{32}$$

$$+\beta_H\frac{\mathcal{P}_m}{\mathcal{P}_r}\Big[\nabla\times\Big((\mathbf{h}+\mathbf{k})\times\nabla\times\mathbf{h}\Big)\Big]^\perp+\beta_I\frac{\mathcal{P}_m}{\mathcal{P}_r}\Big[\nabla\times\Big((\mathbf{h}+\mathbf{k})\times((\mathbf{h}+\mathbf{k})\times\nabla\times\mathbf{h})\Big)\Big]^\perp,$$

where $\mathbf{f}^\perp$ denotes the projection of a vector field $\mathbf{f}$ on the plane normal to $\mathbf{k}$. From the Weyl decomposition theorem of $L^2(V)$ [8], [30], it follows:

$$\Big[\nabla\times\Big(\mathbf{u}\times\mathbf{h})\Big]^\perp=\nabla U_0'+\nabla\times\mathbf{A}_0', \tag{33}$$

$$\Big[\nabla\times\Big(\mathbf{h}\times\nabla\times\mathbf{h}\Big)\Big]^\perp=\nabla U_1'+\nabla\times\mathbf{A}_1', \tag{34}$$

$$\Bigg[\nabla\times\Big[\mathbf{h}\times\Big((\mathbf{h}+\mathbf{k})\times\nabla\times\mathbf{h}\Big)\Big]\Bigg]^\perp=\nabla U_2'+\nabla\times\mathbf{A}_2', \tag{35}$$

where U_i' and $\mathbf{A_i'}$, $i=0,1,2$ are scalar and vector fields.
Using the identities

$$\mathbf{u}^\perp=\nabla_1\frac{\partial\chi}{\partial z}-\nabla\times(\psi\mathbf{k})\qquad \mathbf{h}^\perp=\nabla_1\frac{\partial\chi'}{\partial z}-\nabla\times(\psi'\mathbf{k}), \tag{36}$$

$$[\nabla\times(\mathbf{k}\times\nabla\times\mathbf{h})]^\perp=\nabla\times(\Delta\frac{\partial\chi'}{\partial z}\mathbf{k})+\nabla_1\frac{\partial^2\psi'}{\partial z^2}, \tag{37}$$

$$\Big\{\nabla\times\Big[\mathbf{k}\times(\mathbf{k}\times\nabla\times\mathbf{h})\Big]\Big\}^\perp=\nabla_1(\Delta\frac{\partial\chi'}{\partial z})-\nabla\times(\frac{\partial^2\psi'}{\partial z^2}\mathbf{k}), \tag{38}$$

$$\Big\{\nabla\times\Big[\mathbf{k}\times(\mathbf{h}\times\nabla\times\mathbf{h})\Big]\Big\}^\perp=-\nabla\times(\frac{\partial U_1}{\partial z}\mathbf{k})-\frac{\partial^2\mathbf{A_1}}{\partial z^2}+\nabla(\frac{\partial\mathbf{A}_1}{\partial z}\mathbf{k}), \tag{39}$$

and, assuming $\nabla\cdot\mathbf{A}_1=0$, it follows:

$$\Big\{\nabla\times\Big[\mathbf{k}\times(\mathbf{h}\times\nabla\times\mathbf{h})\Big]\Big\}^\perp=-\nabla\times(\frac{\partial U_1}{\partial z}\mathbf{k})-\nabla\times(\frac{\partial\mathbf{A}_1}{\partial z}\times\mathbf{k})+\nabla(\frac{\partial\mathbf{A_1}}{\partial z}\cdot\mathbf{k}). \tag{40}$$

The solenoidal part of (32) give us :

$$-\frac{\partial\nabla\times(\psi'\mathbf{k})}{\partial t}=-\frac{\partial\nabla\times(\psi\mathbf{k})}{\partial z}+\nabla\times\mathbf{A}'_0-\frac{\mathcal{P}_m}{\mathcal{P}_r}\nabla\times(\Delta\psi'\mathbf{k})+\beta_H\frac{\mathcal{P}_m}{\mathcal{P}_r}\Big[\nabla\times\mathbf{A_1'}+ \tag{41}$$

$$\nabla\times(\Delta\frac{\partial\chi'}{\partial z}\mathbf{k})\Big]+\beta_I\frac{\mathcal{P}_m}{\mathcal{P}_r}\Big[\nabla\times\mathbf{A_2'}-\nabla\times(\frac{\partial^2\psi'}{\partial z^2}\mathbf{k})-\nabla\times(\frac{\partial U_1}{\partial z}\mathbf{k})-\nabla\times(\frac{\partial\mathbf{A}_1}{\partial z}\times\mathbf{k})\Big].$$

It follows that

$$-\frac{\partial \psi' \mathbf{k}}{\partial t} = -\frac{\partial \psi \mathbf{k}}{\partial z} + \mathbf{A}'_0 - \frac{\mathcal{P}_m}{\mathcal{P}_r}\Delta\psi'\mathbf{k} + \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r}\Big[\mathbf{A}'_\mathbf{1} + \Delta\frac{\partial \chi'}{\partial z}\mathbf{k}\Big] + \tag{42}$$

$$\beta_I \frac{\mathcal{P}_m}{\mathcal{P}_r}\Big[\mathbf{A}'_\mathbf{2} - \frac{\partial^2 \psi'}{\partial z^2}\mathbf{k} - \frac{\partial U_1}{\partial z}\mathbf{k} - \frac{\partial \mathbf{A}_1}{\partial z} \times \mathbf{k}\Big] + \nabla F_1.$$

Where F_1 is an arbitrary scalar field.
The third component of the previous equation give us:

$$-\frac{\partial \psi'}{\partial t} = -\frac{\partial \psi}{\partial z} + \mathbf{k}\cdot\mathbf{A}'_0 - \frac{\mathcal{P}_m}{\mathcal{P}_r}\Delta\psi' + \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r}\Big[\mathbf{k}\cdot\mathbf{A}'_\mathbf{1} + \Delta\frac{\partial \chi'}{\partial z}\Big] + \tag{43}$$

$$\beta_I \frac{\mathcal{P}_m}{\mathcal{P}_r}\Big[\mathbf{k}\cdot\mathbf{A}'_\mathbf{2} - \frac{\partial^2 \psi'}{\partial z^2} - \frac{\partial U_1}{\partial z}\Big] + \mathbf{k}\cdot\nabla F_1.$$

If we choose the scalar field F_1 such that:

$$\mathbf{k}\cdot\mathbf{A}'_0 + \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r}\mathbf{k}\cdot\mathbf{A}'_1 + \beta_I \frac{\mathcal{P}_m}{\mathcal{P}_r}\Big(\mathbf{k}\cdot\mathbf{A}'_2 - \frac{\partial U_1}{\partial z}\Big) + \frac{\partial F_1}{\partial z} = 0, \tag{44}$$

we obtain the following equation for ψ':

$$\frac{\partial \psi'}{\partial t} = \frac{\partial \psi}{\partial z} + \frac{\mathcal{P}_m}{\mathcal{P}_r}\Delta\psi' - \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r}\Delta\frac{\partial \chi'}{\partial z} + \beta_I \frac{\mathcal{P}_m}{\mathcal{P}_r}\frac{\partial^2 \psi'}{\partial z^2}. \tag{45}$$

In a similar way, starting from $(4)_1$ we obtain the following evolution equation for ψ

$$\frac{\partial \psi}{\partial t} = M^2 \frac{\partial \psi'}{\partial z} + \Delta\psi. \tag{46}$$

4 Lyapunov stability

If we multiply $(4)_1$ by $\mathbf{u}$, $(4)_2$ by $M^2\mathbf{h}$ and $(4)_3$ by $b\vartheta$, with b a scalar parameter, adding the resulting equations, integrating over V and taking into account the boundary conditions (5), we obtain the energy relation

$$\frac{dE}{dt} = \mathcal{I} - \mathcal{D}, \tag{47}$$

where

$$E(t) = \frac{1}{2}\left(\|u\|^2 + M^2\|h\|^2 + b\|\vartheta\|^2\right), \tag{48}$$

$$\mathcal{I} = \mathcal{R}\left(1 + \frac{b}{\mathcal{P}_r}\right)(\vartheta, w), \tag{49}$$

$$\mathcal{D}=\|\nabla\mathbf{u}\|^2+M^2\frac{\mathcal{P}_m}{\mathcal{P}_r}\|\nabla\mathbf{h}\|^2+\frac{b}{\mathcal{P}_r}\|\nabla\vartheta\|^2+\beta_I\frac{\mathcal{P}_m}{\mathcal{P}_r}\|(\mathbf{h}+\mathbf{k})\times\nabla\times\mathbf{h}\|^2, \quad (50)$$

and $\|\cdot\|$ and $(\cdot,\cdot)$ are, respectively, the norm and the scalar product in $L^2(V)$. In (47) the Hall current disappears, the ion slip current has a stabilizing effect.
To estimate the effects of the anisotropic currents on the stability of the conduction state we define the energy $E^*(t)$ given by

$$E^*(t)=E(t)+d\frac{M^2}{2}\|\nabla_1\frac{\partial\chi'}{\partial z}\|^2+d_1\frac{M^2}{2}\|\nabla_1\frac{\partial\psi'}{\partial z}\|^2+d_2\frac{1}{2}\|\frac{\partial\psi}{\partial z}\|^2 \quad (51)$$

Where d, d_1 e d_2 are some positive parameters we determine later.
If we consider the scalar product of $\Delta_1\chi'$ with the derivative of (31) respect to z, owing the boundary conditions (15) we obtain:

$$\frac{1}{2}\frac{d}{dt}\|\nabla_1\frac{\partial\chi'}{\partial z}\|^2=\left(\Delta_1\chi',\frac{\partial^3\chi}{\partial z^3}\right)+(1+\beta_I)\frac{\mathcal{P}_m}{\mathcal{P}_r}\left(\Delta\frac{\partial^2\chi'}{\partial z^2},\Delta_1\chi'\right)+ \quad (52)$$
$$\beta_H\frac{\mathcal{P}_m}{\mathcal{P}_r}\left(\frac{\partial^3\psi'}{\partial z^3},\Delta_1\chi'\right),$$

because, by using $(14)_1$ and $(7)_3$,

$$\left(\frac{d^2F}{dz^2},\Delta_1\chi'\right)=\int_0^1\frac{d^2F}{dz^2}\,dz\int_{\mathcal{V}}\Delta_1\chi'(x,y,z)\,d\mathcal{V}=0. \quad (53)$$

The scalar product of $\Delta_1\psi'$ with (45), owing the boundary conditions (15) gives us

$$\frac{1}{2}\frac{d}{dt}\|\nabla_1\psi'\|^2=-\left(\Delta_1\psi',\frac{\partial\psi}{\partial z}\right)-\frac{\mathcal{P}_m}{\mathcal{P}_r}\left(\Delta\psi',\Delta_1\psi'\right)+\beta_H\frac{\mathcal{P}_m}{\mathcal{P}_r}\left(\Delta_1\psi',\Delta\frac{\partial\chi'}{\partial z}\right)- \quad (54)$$
$$\beta_I\frac{\mathcal{P}_m}{\mathcal{P}_r}\left(\Delta_1\psi',\frac{\partial^2\psi'}{\partial z^2}\right),$$

From now on, to simplify notations, we will use the subscript notation for the partial differentiation, that is $f_z\equiv\frac{\partial f}{\partial z}$ for a function f depending on z.
From (47), (49), (50), (51), (52), (54) and (46) we have:

$$\frac{dE^*}{dt}<\mathcal{I}^*-\mathcal{D}^*=-\mathcal{D}^*\left(1-\frac{\mathcal{I}^*}{\mathcal{D}^*}\right), \quad (55)$$

where, in terms of poloidal and toroidal fields,

$$\mathcal{I}^* = -\mathcal{R}(1+\frac{b}{\mathcal{P}_r})(\vartheta, \Delta_1\chi)+d\,M^2(\chi_{zzz}, \Delta_1\chi')+M^2\frac{\mathcal{P}_m}{\mathcal{P}_r}d(1+\beta_I)(\Delta_1\chi', \Delta\chi'_{zz}) \tag{56}$$
$$+\beta_H\frac{\mathcal{P}_m}{\mathcal{P}_r}dM^2({\psi'}_{zzz}, \Delta_1\chi') - d_1M^2(\psi_z, \Delta_1\psi') + \beta_H\frac{\mathcal{P}_m}{\mathcal{P}_r}d_1M^2(\Delta\chi'_z, \Delta_1\psi')$$
$$-d_1M^2\frac{\mathcal{P}_m}{\mathcal{P}_r}(\Delta\psi', \Delta_1\psi')-\beta_I\frac{\mathcal{P}_m}{\mathcal{P}_r}d_1M^2(\Delta_1\psi', {\psi'}_{zz})+d_2(\Delta\psi, \psi)+d_2M^2({\psi'}_z, \psi),$$

$$\mathcal{D}^* = \|\nabla\chi_{xz}\|^2 + \|\nabla\chi_{yz}\|^2 + \|\nabla\Delta_1\chi\|^2 + \|\nabla\psi_x\|^2 + \|\nabla\psi_y\|^2 + \tfrac{b}{\mathcal{P}_r}\|\nabla\vartheta\|^2+$$
$$+\, M^2\tfrac{\mathcal{P}_m}{\mathcal{P}_r}\left\{\|\nabla\chi'_{xz}\|^2 + \|\nabla\chi'_{yz}\|^2 + \|\nabla\Delta_1\chi'\|^2 + \|\nabla\psi'_x\|^2 + \|\nabla\psi'_y\|^2\right\}. \tag{57}$$

Taking into account that χ' and ψ' are doubly-periodic functions, the boundedness of the functional $\frac{\mathcal{I}^*}{\mathcal{D}^*}$ can be proved [32]. Using identities:

$$(\Delta_1\chi', \Delta\chi'_{zz}) = -\left[\|\nabla\chi'_{xz}\|^2 + \|\nabla\chi'_{yz}\|^2\right],$$
$$(\Delta_1\psi', \Delta\psi') = \|\nabla\psi'_x\|^2 + \|\nabla\psi'_y\|^2, \tag{58}$$
$$(\Delta_1\psi', {\psi'}_{zz}) = \|\nabla_1\psi'_z\|^2,$$

the energy inequality (55) we can written as follows:

$$\frac{dE^*}{dt} < \mathcal{I}^{**} - \mathcal{D}^{**} = -\mathcal{D}^{**}\left(1 - \frac{\mathcal{I}^{**}}{\mathcal{D}^{**}}\right), \tag{59}$$

with

$$\mathcal{I}^{**} = -\mathcal{R}(1+\frac{b}{\mathcal{P}_r})(\vartheta, \Delta_1\chi)+d\,M^2(\chi_{zzz}, \Delta_1\chi')+M^2\frac{\mathcal{P}_m}{\mathcal{P}_r}d(1+\beta_I)\alpha(\Delta_1\chi', \Delta\chi'_{zz}) \tag{60}$$
$$+\beta_H\frac{\mathcal{P}_m}{\mathcal{P}_r}dM^2({\psi'}_{zzz}, \Delta_1\chi') - d_1M^2(\psi_z, \Delta_1\psi') + \beta_H\frac{\mathcal{P}_m}{\mathcal{P}_r}d_1M^2(\Delta\chi'_z, \Delta_1\psi')$$
$$-d_1M^2\beta\frac{\mathcal{P}_m}{\mathcal{P}_r}(\Delta\psi', \Delta_1\psi')-\beta_I\frac{\mathcal{P}_m}{\mathcal{P}_r}d_1M^2\beta(\Delta_1\psi', {\psi'}_{zz})+d_2\gamma(\Delta\psi, \psi)+d_2M^2({\psi'}_z, \psi).$$

$$\mathcal{D}^{**} = \|\nabla\chi_{xz}\|^2 + \|\nabla\chi_{yz}\|^2 + \|\nabla\Delta_1\chi\|^2 + \|\nabla\psi_x\|^2 + \|\nabla\psi_y\|^2 + \tfrac{b}{\mathcal{P}_r}\|\nabla\vartheta\|^2+$$
$$+\, M^2\tfrac{\mathcal{P}_m}{\mathcal{P}_r}\Big\{[1 + d(1-\alpha)(1+\beta_I)]\left[\|\nabla\chi'_{xz}\|^2 + \|\nabla\chi'_{yz}\|^2\right] + \|\nabla\Delta_1\chi'\|^2+$$
$$[1 + d_1(1-\beta)]\left[\|\nabla\psi'_x\|^2 + \|\nabla\psi'_y\|^2\right] + d_1(1-\beta)\beta_I\|\nabla_1\psi'_z\|^2\Big\}$$
$$+d_2(1-\gamma)\|\nabla\psi\|^2. \tag{61}$$

Where the constants α, β and γ, we determine later, must satisfy the inequalities

$$1 + d(1-\alpha)(1+\beta_I) > 0, \quad 1-\beta > 0, \quad 1-\gamma > 0, \tag{62}$$

in order to D^{**} be positive definite. Let us define

$$\frac{1}{\sqrt{\mathcal{R}_a^*}} = \max_{\mathcal{K}} \frac{\mathcal{I}^{**}}{\mathcal{D}^{**}} \tag{63}$$

in the class $\mathcal{K}$ of the kinematically admissible functions.
From (55), if $E^*(t)$ is positive definite, it follows that the inequality

$$\sqrt{\mathcal{R}_a^*} \geq 1 \tag{64}$$

is a sufficient condition of the linear and nonlinear asymptotic Lyapunov stability of the conduction state.
In the next section we shall determine the region of the parameters space where the inequality (64) is satisfied.

5 The nonlinear stability bound

We study now the variational problem (63) in terms of the independent fields $(\mathcal{X}, \mathcal{X}', \psi, \psi', \vartheta)$ verifying the boundary conditions (5), (15).
The Euler equations associated to the maximum problem (63) are:

$$\left\{\begin{array}{c}
-\mathcal{R}\left(1+\frac{b}{\mathcal{P}_r}\right)\Delta_1\vartheta - M^2 d\Delta_1\mathcal{X}'_{zzz} + \frac{2}{\sqrt{\mathcal{R}_a^*}}\Delta\Delta\Delta_1\mathcal{X} = 0, \\
-\mathcal{R}\left(1+\frac{b}{\mathcal{P}_r}\right)\Delta_1\mathcal{X} + \frac{2}{\sqrt{\mathcal{R}_a^*}}\frac{b}{\mathcal{P}_r}\Delta\vartheta = 0, \\
d\,\Delta_1\mathcal{X}_{zzz} + 2\frac{\mathcal{P}_m}{\mathcal{P}_r}(1+\beta_I)d\,\alpha\Delta\Delta_1\mathcal{X}'_{zz} + \beta_H\frac{\mathcal{P}_m}{\mathcal{P}_r}\left(d\Delta_1{\psi'}_{zzz} - d_1\Delta\Delta_1{\psi'}_z\right) + \\
+\frac{2}{\sqrt{\mathcal{R}_a^*}}\frac{\mathcal{P}_m}{\mathcal{P}_r}\left[\Delta\Delta\Delta_1\mathcal{X}' + d(1-\alpha)(1+\beta_I)\Delta\Delta_1\mathcal{X}'_{zz}\right] = 0, \\
d_1 M^2\Delta_1{\psi'}_z + d_2 M^2{\psi'}_z + 2\Big[d_2[\gamma + \frac{1}{\sqrt{\mathcal{R}_a^*}}(1-\gamma)]\Delta\psi - \frac{1}{\sqrt{\mathcal{R}_a^*}}\Delta\Delta_1\psi\Big] = 0, \\
-d_1\Delta_1\psi_z + \beta_H\frac{\mathcal{P}_m}{\mathcal{P}_r}\left(d_1\Delta\Delta_1\mathcal{X}'_z - d\Delta_1\mathcal{X}'_{zzz}\right) - 2d_1\beta\frac{\mathcal{P}_m}{\mathcal{P}_r}\left(\Delta\Delta_1\psi' + \beta_I\Delta_1{\psi'}_{zz}\right) - \\
d_2\psi_z - \frac{2}{\sqrt{\mathcal{R}_a^*}}\frac{\mathcal{P}_m}{\mathcal{P}_r}\Big[\Delta\Delta_1\psi' + d_1(1-\beta)[\Delta\Delta_1\psi' + \beta_I\Delta_1\psi' zz]\Big] = 0.
\end{array}\right. \tag{65}$$

In the class of normal mode perturbations

$$(\chi, \chi', \vartheta, \psi, \psi') = (X(z), K(z), \Theta(z), \Psi(z), \Psi'(z)) \exp[i(k_x x + k_y y) + \sigma t], \tag{66}$$

with $\sigma \in \mathbf{C}$, after assuming

$$X(z) = \sum_{n=0}^{\infty} X_n \sin(n\pi z), \tag{67}$$

where $X_n \equiv (X, \sin(n\pi z))$ are the Fourier coefficients, from the Euler equations (65) we obtain:

$$-\left[\mathcal{R}\left(1+\frac{b}{\mathcal{P}_r}\right)\right]^2 \frac{\mathcal{P}_r}{b}\frac{\sqrt{\mathcal{R}_a^*}}{2}k^2 + \frac{2}{\sqrt{\mathcal{R}_a^*}}B_n^3 = \tag{68}$$
$$\frac{M^2 \frac{\mathcal{P}_r}{\mathcal{P}_m} d^2 k^2 n^6 \pi^6 B_n \mathcal{N}}{2\Big\{n^2\pi^2k^2(1+\beta_I)dF_\alpha B_n + \frac{1}{\sqrt{\mathcal{R}_a^*}}B_n^2 k^2\Big\}\mathcal{N} + 2\frac{\beta_H^2}{M^2}n^2\pi^2\mathcal{D}}.$$

In (68)

$$\mathcal{N} = (d_2 - d_1k^2)^2 n^2\pi^2 \frac{\mathcal{P}_r}{\mathcal{P}_m} - \frac{4B_n}{M^2}\Big\{d_1F_\beta[k^4 + k^2n^2\pi^2(1+\beta_I)] + \frac{1}{\sqrt{\mathcal{R}_a^*}}B_nk^2\Big\}\cdot \tag{69}$$
$$\Big\{\frac{1}{\sqrt{\mathcal{R}_a^*}}k^2 - d_2F_\gamma\Big\},$$

$$\mathcal{D} = k^4(d_1B_n - dn^2\pi^2)^2 B_n\Big\{\frac{1}{\sqrt{\mathcal{R}_a^*}}k^2 - d_2F_\gamma\Big\}, \tag{70}$$

$$F_\alpha = \alpha + \frac{1}{\sqrt{\mathcal{R}_a^*}}(1-\alpha), \quad F_\beta = \beta + \frac{1}{\sqrt{\mathcal{R}_a^*}}(1-\beta), \quad F_\gamma = -\gamma - \frac{1}{\sqrt{\mathcal{R}_a^*}}(1-\gamma), \tag{71}$$

with $B_n = n^2\pi^2 + k^2$.
In order to obtain the largest stability domain in the parameter space, we must differentiate (68) with respect to the parameters.
Introducing the function

$$F_{12} = \frac{(d_2 - d_1k^2)^2}{B_n(\frac{1}{\sqrt{\mathcal{R}_a^*}}k^2 - d_2F_\gamma)}, \tag{72}$$

(68) becomes:

$$-\left[\mathcal{R}\left(1+\frac{b}{\mathcal{P}_r}\right)\right]^2 \frac{\mathcal{P}_r}{b}\frac{\sqrt{\mathcal{R}_a^*}}{2}k^2 + \frac{2}{\sqrt{\mathcal{R}_a^*}}B_n^3 = \frac{M^2}{2}\frac{\mathcal{P}_r}{\mathcal{P}_m}d^2k^2n^6\pi^6B_n\cdot \tag{73}$$

$$\frac{G_{12}}{\Big\{n^2\pi^2k^2(1+\beta_I)dF_\alpha B_n+\frac{1}{\sqrt{\mathcal{R}_a^*}}B_n^2k^2\Big\}G_{12}+\frac{\beta_H^2}{M^2}n^2\pi^2k^4(d_1B_n-dn^2\pi^2)^2},$$

where:

$$G_{12}=n^2\pi^2\frac{\mathcal{P}_r}{\mathcal{P}_m}F_{12}-\frac{4}{M^2}\Big\{[k^4+k^2n^2\pi^2(1+\beta_I)]d_1F_\beta+\frac{1}{\sqrt{\mathcal{R}_a^*}}B_nk^2\Big\}. \quad (74)$$

The partial derivative of the right hand side (73), with respect to d_2, is zero iff:

$$d_2=k^2\Big(\frac{2}{F_\gamma\sqrt{\mathcal{R}_a^*}}-d_1\Big). \quad (75)$$

Substituting (75) into (73) we have:

$$\left[-\mathcal{R}\left(1+\frac{b}{\mathcal{P}_r}\right)\right]^2\frac{\mathcal{P}_r}{b}\frac{\sqrt{\mathcal{R}_a^*}}{2}k^2+\frac{2}{\sqrt{\mathcal{R}_a^*}}B_n^3=2M^2\frac{\mathcal{P}_r}{\mathcal{P}_m}d^2k^2n^6\pi^6B_n\cdot \quad (76)$$

$$\frac{H_1}{4B_n\Big\{n^2\pi^2k^2(1+\beta_I)dF_\alpha+\frac{1}{\sqrt{\mathcal{R}_a^*}}B_nk^2\Big\}H_1+\beta_H^2n^2\pi^2k^2(d_1B_n-dn^2\pi^2)^2},$$

where H_1 is given by

$$H_1=M^2\frac{\mathcal{P}_r}{\mathcal{P}_m}\frac{n^2\pi^2}{B_nF_\gamma^2}\Big(d_1F_\gamma-\frac{1}{\sqrt{\mathcal{R}_a^*}}\Big)-\Big\{d_1F_\beta\big[k^2+n^2\pi^2(1+\beta_I)\big]+\frac{1}{\sqrt{\mathcal{R}_a^*}}B_n\Big\}. \quad (77)$$

The partial derivative of the right hand side of (76) , with respect to d_1, is zero iff

$$d_1=-d\frac{n^2\pi^2}{B_n}+\frac{2}{\sqrt{\mathcal{R}_a^*}}\frac{X_n}{Y_{n\beta\gamma}} \quad (78)$$

where

$$X_n=B_n+M^2\frac{\mathcal{P}_r}{\mathcal{P}_m}\frac{n^2\pi^2}{B_nF_\gamma^2} \quad (79)$$

$$Y_{n\beta\gamma}=M^2\frac{\mathcal{P}_r}{\mathcal{P}_m}\frac{n^2\pi^2}{B_nF_\gamma}-F_\beta\big[k^2+n^2\pi^2(1+\beta_I)\big]. \quad (80)$$

Introducing (78) in (76) we have:

$$-\left[\mathcal{R}\left(1+\frac{b}{\mathcal{P}_r}\right)\right]^2\frac{\mathcal{P}_r}{b}\frac{\sqrt{\mathcal{R}_a^*}}{2}k^2+\frac{2}{\sqrt{\mathcal{R}_a^*}}B_n^3=\frac{M^2}{2}d^2\frac{\mathcal{P}_r}{\mathcal{P}_m}n^6\pi^6B_n\cdot \quad (81)$$

$$\frac{Y_{n\beta\gamma}}{\Big\{n^2\pi^2(1+\beta_I)dF_\alpha+\frac{1}{\sqrt{\mathcal{R}_a^*}}B_n\Big\}B_nY_{n\beta\gamma}+\beta_H^2n^2\pi^2B_n\Big(\frac{B_n}{\sqrt{\mathcal{R}_a^*}}\frac{X_n}{Y_{n\beta\gamma}}-dn^2\pi^2\Big)}.$$

The partial derivative of the right hand side of (81), with respect to d, is zero iff

$$d = -\frac{2}{\sqrt{\mathcal{R}_a^*}} \frac{B_n}{Y_{n\beta\gamma}} \frac{Y_{n\beta\gamma}^2 + \beta_H^2 n^2\pi^2 X_n}{n^2\pi^2(1+\beta_I)F_\alpha Y_{n\beta\gamma} - \beta_H^2 n^4\pi^4}. \tag{82}$$

If d is given by (82), (81) becomes:

$$-\mathcal{R}\left(1+\frac{b}{\mathcal{P}_r}\right)^2 \frac{\mathcal{P}_r}{b}\frac{\sqrt{\mathcal{R}_a^*}}{2}k^2 + \frac{2}{\sqrt{\mathcal{R}_a^*}}B_n^3 = -\frac{2}{\sqrt{\mathcal{R}_a^*}}M^2\frac{\mathcal{P}_r}{\mathcal{P}_m}n^6\pi^6 B_n \cdot \tag{83}$$

$$\frac{Y_{n\beta\gamma}^2 + \beta_H^2 n^2\pi^2 X_n}{\left\{n^2\pi^2(1+\beta_I)F_\alpha Y_{n\beta\gamma} - \beta_H^2 n^4\pi^4\right\}^2}.$$

The partial derivative of the right hand side of (83) with respect to $Y_{n\beta\gamma}$ is zero iff

$$Y_{n\beta\gamma} = -F_\alpha(1+\beta_I)X_n. \tag{84}$$

In this case from (83) we obtain:

$$-\left[\mathcal{R}\left(1+\frac{b}{\mathcal{P}_r}\right)\right]^2 \frac{\mathcal{P}_r}{b}\frac{\sqrt{\mathcal{R}_a^*}}{2}k^2 + \frac{2}{\sqrt{\mathcal{R}_a^*}}B_n^3 = -\frac{2}{\sqrt{\mathcal{R}_a^*}}M^2\frac{\mathcal{P}_r}{\mathcal{P}_m}n^6\pi^6 B_n \cdot \tag{85}$$

$$\frac{B_n + M^2\frac{\mathcal{P}_r}{\mathcal{P}_m}\frac{n^2\pi^2}{B_n F_\gamma^2}}{n^4\pi^4(1+\beta_I)\left[B_n + M^2\frac{\mathcal{P}_r}{\mathcal{P}_m}\frac{n^2\pi^2}{B_n F_\gamma^2}\right] + \beta_H^2 n^6\pi^6}.$$

If we choose

$$F_\alpha = -\frac{1}{\sqrt{1+\beta_I}} \qquad F_\gamma = \frac{1}{\sqrt{1+\beta_I B_n\left(M^2\frac{\mathcal{P}_r}{\mathcal{P}_m}\right)^{-1}}}, \tag{86}$$

after performing the partial derivative with respect to b, (83) becomes

$$\mathcal{R}^2\mathcal{R}_a^* k^2 = B_n^3 + M^2\frac{\mathcal{P}_r}{\mathcal{P}_m}n^2\pi^2 B_n$$

$$\cdot\frac{B_n^2 + \beta_I B_n n^2\pi^2 + M^2\frac{\mathcal{P}_r}{\mathcal{P}_m}n^2\pi^2}{\left(B_n^2 + \beta_I B_n n^2\pi^2 + M^2\frac{\mathcal{P}_r}{\mathcal{P}_m}n^2\pi^2\right)(1+\beta_I) + \beta_H^2 n^2\pi^2 B_n} \tag{87}$$

The relation (80), taking into account (84) allows us to determine $F_\beta = F_\beta(F_\gamma)$.

From (78) (82), (84), $(86)_1$ we obtain

$$d = \frac{2}{\sqrt{\mathcal{R}_a^*}} \frac{B_n}{n^2\pi^2} \frac{1}{\sqrt{1+\beta_I}}, \quad d_1 = 0. \tag{88}$$

From (89) it follows

$$d_2 = \frac{2}{\sqrt{\mathcal{R}_a^*}} k^2 \sqrt{1 + \beta_I B_n \Big(M^2 \frac{\mathcal{P}_r}{\mathcal{P}_m}\Big)^{-1}}, \tag{89}$$

the energy E^* is then positive definite.
From (71) and (86) we obtain

$$\alpha = -\frac{\sqrt{1+\beta_I} + \sqrt{\mathcal{R}_a^*}}{\sqrt{1+\beta_I}(\sqrt{\mathcal{R}_a^*} - 1)}, \tag{90}$$

and

$$\gamma = \frac{\sqrt{1 + \beta_I B_n \Big(M^2 \frac{\mathcal{P}_r}{\mathcal{P}_m}\Big)^{-1}} + \sqrt{\mathcal{R}_a^*}}{\sqrt{1 + \beta_I B_n \Big(M^2 \frac{\mathcal{P}_r}{\mathcal{P}_m}\Big)^{-1}}(1 - \sqrt{\mathcal{R}_a^*})},$$

whence, for $\sqrt{\mathcal{R}_a^*} > 1$ $\alpha < 0$, $\gamma < 0$, and the inequalities $(62)_{1,3}$ are satisfied. The inequality $(62)_2$ is not relevant because in the energy relation the term $(1-\beta)$ disappears being $d_1 = 0$.
The minimum of (87) with respect to $n \in \mathbf{N}$ is attained for $n = 1$, therefore, as a function of $x = \frac{k^2}{\pi^2}$, $\mathcal{R}_a^*$ is given by:

$$\mathcal{R}^2\mathcal{R}_a^* = \pi^4 \frac{1+x}{x} \cdot \tag{91}$$

$$\left\{(1+x)^2 + \frac{M^2}{\pi^2}\frac{\mathcal{P}_r}{\mathcal{P}_m} \frac{(1+x)^2 + \beta_I(1+x) + \frac{M^2}{\pi^2}\frac{\mathcal{P}_r}{\mathcal{P}_m}}{\Big((1+x)^2 + \beta_I(1+x) + \frac{M^2}{\pi^2}\frac{\mathcal{P}_r}{\mathcal{P}_m}\Big)(1+\beta_I) + \beta_H^2(1+x)}\right\}.$$

If $\beta_I = 0$ from (91) we derive

$$\mathcal{R}^2\mathcal{R}_a^* = \pi^4 \frac{1+x}{x} \cdot \tag{92}$$

$$\left\{(1+x)^2 + \frac{M^2}{\pi^2}\frac{\mathcal{P}_r}{\mathcal{P}_m} \frac{(1+x)^2 + \frac{M^2}{\pi^2}\frac{\mathcal{P}_r}{\mathcal{P}_m}}{(1+x)^2 + \frac{M^2}{\pi^2}\frac{\mathcal{P}_r}{\mathcal{P}_m} + \beta_H^2(1+x)}\right\}$$

If $\beta_I = \beta_H = 0$ from (91) we derive

$$\mathcal{R}^2\mathcal{R}_a^* = \pi^4 \frac{1+x}{x}\left\{(1+x)^2 + \frac{M^2}{\pi^2}\frac{\mathcal{P}_r}{\mathcal{P}_m}\right\} \tag{93}$$

The right hand side of (91) represents exactly the critical function of linear instability in presence of Hall and ion-slip currents, if the principle of exchange of stabilities holds [33].
Obviously we find again, if $\beta_H = \beta_I = 0$, the results in [24] and, if $\beta_I = 0$ $\beta_H \neq 0$, in a simpler way, the stability bound obtained in [25]
We have so proved the following

Theorem 1. *If the principle of exchange of stabilties holds, the inequality*

$$1 \leq \mathcal{R}_a^*, \tag{94}$$

with $\mathcal{R}_a^*$ *given by* (91)*, is a sufficient condition of linear and non linear Lyapunov stability, that is, the linear and non linear stability bounds coincide if instability occurs as stationary convection.*

Indeed

$$\mathcal{R}_a^* \geq 1 \Longleftrightarrow \mathcal{R}^2 \leq \mathcal{R}^*$$

$$\mathcal{R}^* = \pi^4 \frac{1+x}{x}\left\{(1+x)^2 + \frac{M^2}{\pi^2}\frac{\mathcal{P}_r}{\mathcal{P}_m}\frac{(1+x)^2+\beta_I(1+x)+M^2\frac{\mathcal{P}_r}{\mathcal{P}_m}}{\left((1+x)^2+\beta_I(1+x)+\frac{M^2}{\pi^2}\frac{\mathcal{P}_r}{\mathcal{P}_m}\right)(1+\beta_I)+\beta_H^2(1+x)}\right\}, \tag{95}$$

where $\mathcal{R}^*$ is exactly the Rayleigh function of the linear instability theory, if the principle of exchange of stabilities holds [33].

Conclusions

The classical L^2 norm is , as well known, *too weak* to evaluate some stabilizing or instabilizing effects of varying-sign terms in the perturbation equations.
Iin literature there are many variants aimed to obtain the largest stability domain in the parameter space.
In [27], in the chapter about variants of the energy method, symmetry and optimality condition, referring to the perturbation evolution equations, we highlighted:

... the central idea of the present variant is to change just these differential equations, and not those integral deduced from them. Moreover this semplification must be done in such a way that the energy relation assumes the simplest form.

Following this approach in this paper the problem governing the perturbation evolution is reformulated in terms of some *essential variables*, by splitting the perturbation equations in *the solenoidal and potential parts*, obtaining some of the equations derived in [33] to study linear instability by the normal modes technique.

Differentiating with respect to the parameters involved in the Lyapunov function, we obtained a nonlinear stability bound that coincides with the linear one, in the subspace of the parameter space where instability occurs as stationary convection.

6 References

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