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SKEW PROJECTORS AND GENERALIZED OBSERVABLES IN POLARIZATION OPTICS

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Abstract. *The non-Hermitian operators of the non-orthogonal multilayer optical polarizers represent observables in the sense of the generalized quantum theory of measurement. The intimate spectral structure of these polarizers can be disclosed in the frame of skew-angular vector bases and bi-orthonormal vector systems. We show that each of these polarizers corresponds to a skew projector, its operator is "generated" by a skew projector, in the sense of the spectral theorem of linear operators theory. Thus the common feature of all the polarizers (Hermitian and non-Hermitian) is that their "nuclei" are (orthogonal or skew) projectors — the generating projectors.*

Keywords: non-Hermitian operators, non-orthogonal multilayer optical polarizers, skew-angular vector bases

1. Introduction

In the last decades some extensions of the standard Dirac – von Neumann measurement formalism in quantum mechanics were elaborated [1−5].

In the standard formalism, a measurement corresponds to a Hermitian operator yielding its eigenvalues as measurement results, with probabilities determined by the values of the orthogonal projection of the system state on the operator's eigenvectors. In other words an observable is a "projection-valued measure" (PVM).

In a fundamental paper by E. B. Davies and J. T. Lewis [6], the concept of generalized observable is described, which arise when two standard non-commuting observables $\mathcal{A}$ and $\mathcal{B}$ are measured one after the other. The class of observables is extended to the positive operator-valued measure (POVM). Particularly the POVM becomes a PVM when the two standard observables $\mathcal{A}$ and $\mathcal{B}$ commute.

In the standard theory of quantum measurement the postulate of the repeatability plays a central role: if a physical quantity is measured twice in succession in a system, one gets the same value each time. This hypothesis is equivalent to the fact that the class of observables is restricted to (orthogonal) projectors. In the generalized theory of quantum measurement the postulate of repeatability is abandoned.

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Well known, one of the clearest ways of introducing the conceptual and axiomatic framework of quantum mechanics is by analyzing the interaction of photons with polarizing optical devices in (mental) photon-by-photon experiments. The polarization state space being the simplest quantum state space (a bidimensional one), a good physical and mathematical insight is get when the analysis of the basic quantum concepts is pursued in such a space.

So far, the polarization devices taken into account in this approach to the fundamentals of quantum mechanics (e. g. [7−10]) are exclusively of orthogonal kind (orthogonal eigenvectors; normal, more precisely Hermitian, operators).

All the operators describing the basic, "canonical" polarizers (homogeneous linear, circular and elliptical ideal polarizers) are Hermitian [11, 12]. Thus they describe standard observables.

But the polarization device operators provide the most natural frame for analyzing or for exemplifying the concepts of generalized quantum measurement theory. The widespread multilayer polarization devices are generally of non-orthogonal kind (non-orthogonal eigenvectors, non-normal operators) [12−14]. Moreover the two eigenstates corresponding to an arbitrary direction of propagation in some crystal are generally non-orthogonal [15, 16].

In [17] I have exemplified and analyzed in a quantum mechanical operatorial language some optical polarizers of non-orthogonal kind. They are inhomogeneous (two- and three-layer) polarizers. Each layer of such a polarizer is of orthogonal kind, i.e. its operator is Hermitian, it corresponds to a standard observable. But, because of the non-commutativity of these operators, the multilayer polarizer is of non-orthogonal kind. Its operator is non-Hermitian and, nevertheless, it corresponds to an observable. This is a very convincing, clear and simple (2×2 operator) example for the case considered by Davies and Levis [6] of generalized observables which arise in a series of two or more non-commuting standard observables.

Recently, W. M. de Muynck [18] has given an alternative to the Lüders generalization of von Neumann projection, based on the notions of non-orthogonal projections and biorthonormal systems. This technique will be our departure point in the present paper.

Again the optical polarization devices provide the most natural frame of applying and exemplifying these ideas. For the same reason: the space of polarization states in the simplest, a bidimensional one, which allows reaching a good, even intuitive, insight in these abstract ideas.

The aim of this paper is to analyze some non-orthogonal inhomogeneous (multilayer) polarizers by means of the mathematical tools of non-orthogonal projections and biorthonormal systems. We shall see that this is the natural way of interpreting the eigenexpressions and the physical action of this kind of polarizers. Their operators are not projectors, but each of them can be expressed on the basis of a unique (skew) projector conformly to the spectral theorem. This projector may be denominated the *generating projector* of the polarizer.

2. The operators of some basic polarization devices

Let us consider a linear device operator, normal or not, and an orthogonal basis $\{| S_i >\}$, $(i = 1,2)$ in the polarization state space. Labeling by I the identity operator and making use of the closure relation

$$\sum_i | S_i >< S_i | = \mathrm{I} \ , \tag{1}$$

we may write:

$$\mathsf{D} = \mathrm{I}\mathsf{D}\mathrm{I} = \sum_{i,j=1}^{2} | S_i >< S_i | \mathsf{D} | S_j >< S_j | = \sum_{i,j=1}^{2} < S_i | \mathsf{D} | S_j > | S_i >< S_j | . \tag{2}$$

The device operator may be uniquely expanded in a double series of “basic operators” $| S_i >< S_j |$ associated with any orthogonal basis $\{| S_i >|\}$ of the vector space on which the operator is defined, here the polarization states (SOP) vector space. In this expansion all the elementary operators $| S_i >< S_j |$ are of orthogonal kind: $| S_i >< S_i |$ are *orthogonal projectors,* $| S_i >< S_j | (i \neq j)$ are *orthogonal converters (cross-projectors).* In Willard Gibbs’ language, $| S_i >< S_j |$ are *dyads* and the expansion (2) is a *dyadics.* Eq. (2) is the dyadics expansion of the operator D expressed in Dirac’s formalism.

If $\{| S_i >\}$ is an orthonormal basis constituted by the eigenvectors of the device operator D itself (case in which D must be a normal operator):

$$\mathsf{D} | S_i > = \lambda_i | S_i > \ , \tag{3}$$

$$< S_i | S_j > = \delta_{ij} \ , \tag{4}$$

the expansion (2) reduces to:

$$\mathsf{D} = \sum_{i=1}^{2} \lambda_i | S_i >< S_j | \ , \tag{5}$$

which expresses the *spectral theorem* for normal operators and expands the operator D in terms of its *(orthogonal) eigenprojectors*.

Eq. (5) gives the *eigenexpression* of the operator, whereas Eq. (2) is its *improper* expansion.

The basic polarization devices (homogeneous polarizers and retarders of various kinds) are all orthogonal devices. Their eigenvectors are orthogonal, their operators are normal and can be expressed in function of their (orthogonal, normal, perpendicular), projectors as follows:

$$\mathsf{D} = \lambda_M \,| E_M >< E_M | + \lambda_m \,| E_m >< E_m | \;, \tag{6}$$

where we have labeled by indices *M* and *m* the major and the minor eigenvalues and eigenvectors [11] of the operator.

Some usual orthogonal bases in the polarization state space are: $\{| P_x >, | P_y >\}$ (x, y linear polarized states), $\{| P_\theta >, | P_{\theta+90^0} >\}$(linear polarized states of azimuth θ and $\theta + 90^0$), $\{| R >, | L >\}$ (right and left circularly polarized states).

With this notation, the operator of an ideal *x*-polarizer, in its eigenbasis, is $(\lambda_M = 1, \lambda_m = 0)$ the orthogonal projector:

$$\mathsf{P}_{|P_x>} = | P_x >< P_x | \;, \tag{7}$$

the operator of an ideal linear polarizer of azimuth θ is:

$$\mathsf{P}_{|P_\theta>} = | P_\theta >< P_\theta | \;, \tag{8}$$

that of a right-hand circular polarizer

$$\mathsf{P}_{|R>} = | R >< R | \;, \tag{9}$$

a.s.o.

The orthogonal retarders are devices of class SU(2). Their eigenvalues are situated on the unit circle in the complex plane. In a symmetrical form, $\lambda_{M,m} = e^{\pm i\delta/2}$, where δ is the phase delay introduced by the retarder between its eigenvectors. As an example, the eigenform of the operator of a linear retarder δ whose major eigenvector is $| E_M >= | P_\theta >$ takes the form:

$$\mathsf{R}_{|P_\theta>}(\delta) = e^{i\delta/2} \,| P_\theta >< P_\theta | + e^{-i\delta/2} \,| P_{\theta+90^0} >< P_{\theta+90^0} | \;. \tag{10}$$

All the expressions of the kind given above for the device operators are eigenforms of these operators: each operator is expressed in its proper basis.

In order to build up the operator of a composite polarization device ("sandwich") we have to develop the calculus coherently in a unique and adequate basis, i.e. we have to transpose the expression of all the operators of the various layers of the "sandwich" into the same, generally improper, basis.

This transposition can be done by introducing the eigenexpression of the operator in Eq. (2) where $|S_i>$ will be taken the vectors of the new chosen basis. In this algorithm, scalar products between various SOP vectors will appear; these scalar products make the connection between the new basis and the old one. We give here the values of some such scalar products useful for the next calculi:

$$<P_x|P_\theta>=<P_\theta|P_x>=\cos\theta\ , \tag{11}$$

$$<P_y|P_\theta>=<P_\theta|P_y>=\sin\theta\ , \tag{12}$$

$$<L|P_{45^0}>=<P_{45^0}|L>^*=\frac{1}{2}(1+i)=\frac{1}{\sqrt{2}}\ e^{i\pi/4}\ , \tag{13}$$

$$<R|P_{-45^0}>=<P_{-45^0}|R>^*=\frac{1}{2}(1+i)=\frac{1}{\sqrt{2}}\ e^{i\pi/4}\ . \tag{14}$$

By introducing successively the eigenforms (8) and (10) in (2) and making use of (11) – (12), one obtains the improper Cartesian $\{|P_x>,|P_y>\}$ forms of the ideal polarizer and of the δ-retarder of azimuth θ, respectively:

$$\mathsf{P}_{|P_\theta>}=\cos^2\theta|P_x><P_x|+\sin\theta\cos\theta\big[|P_x><P_y|+|P_y><P_x|\big]+\sin^2\theta\,|P_y><P_y| \tag{15}$$

$$\begin{aligned}\mathsf{R}_{|P_\theta>}(\delta)&=\left[\cos^2\theta e^{i\delta/2}+\sin^2\theta e^{-i\delta/2}\right]|P_x><P_x|\\&+2i\sin\theta\cos\theta\sin\frac{\delta}{2}\big[|P_x><P_y|+|P_y><P_x|\big]\\&+\left[\sin^2\theta e^{i\delta/2}+\cos^2\theta e^{-i\delta/2}\right]|P_y><P_y|\\&=\left(\cos\frac{\delta}{2}+i\cos 2\theta\sin\frac{\delta}{2}\right)|P_x><P_x|\\&+i\sin 2\theta\sin\frac{\delta}{2}\big[|P_x><P_y|+|P_y><P_x|\big]\\&+\left(\cos\frac{\delta}{2}-i\cos 2\theta\sin\frac{\delta}{2}\right)|P_y><P_y|\ .\end{aligned} \tag{16}$$

In these improper forms it is also evident that these operators are normal: all the constitutive projectors and converters are of perpendicular kind.

3. Orthonormal and biorthonormal bases in the polarization state space

Any device operator D, normal or non-normal, may be expanded in an orthonormal basis of polarization states $\{|S_i>\}$ in the form (2). The equations

(15) and (16) give two examples of such decomposition: the improper operators of an ideal polarizer and of a δ-retarder, respectively, both of major axis azimuth θ, in the Cartesian SOP basis $\{|P_x>, P_y>\}$. Similar decompositions can be obtained for example in the orthonormal basis of circular polarized states $\{R>, |L>\}$.

The characteristic of such an improper expansion is that the operator is expressed generally by means of two orthogonal projectors and two orthogonal converters (e. g. $\mathsf{Q}_x = |P_x><P_x|$, $\mathsf{Q}_y = |P_y><P_y|$, $\mathsf{C}_{yx} = |P_x><P_y|$, $\mathsf{C}_{xy} = |P_y><P_x|$).

If the operator is normal, its expansion can be reduced to only one or two orthogonal (normal) projectors (e. g. Q_x and Q_y, if the operator is of Cartesian x-y kind). This is its eigenexpression, (5), and we have given above some examples of such a proper expansion, for the operators of some canonical (hence orthogonal) polarization devices (7) − (10).

This reduction of the improper expansion of the operator to its eigenexpression is not only a question of mathematical economy and of adequacy to the physical symmetry of the polarization device, but also one of the deeper understanding of the device structure: it reveals its eigenvectors, eigenvalues, eigenprojectors (the converters are eliminated), shortly all its intimate operatorial properties. Evidently, this is equivalent to the diagonalization of its matrix.

For a non-orthogonal device, for its non-normal operator, this simple scheme does not work. As we shall see, eigenforms of these operators exist too, but they cannot be represented by means of some orthogonal projectors. In other words the spectral theorem can no longer be expressed in the form (5).

Let us label by $|E_i>(i=1, 2)$ the non-orthogonal eigenvectors of a (non-normal) device operator D, corresponding to a (non-orthogonal) polarization device. Such a situation can be managed by means of a technique imported from crystallography, that of the *reciprocal vectors* or *bi-orthogonal systems* [19].

To the non-orthogonal system (here a pair) of vectors $\{|E_i>\}$ we can associate another system of vectors $\{|F_j>\}$, the reciprocal vectors, defined by the equation:

$$<F_j|E_i>=\delta_{ij} \ . \tag{17}$$

That means that any vector of one set and any vector of the other set excluding its own conjugate are mutually orthogonal and that the conjugate vectors are “mutually normed” to unity.

Having in view (17), it is straightforward that the operators:

$$|E_i><F_i|\ .\tag{18}$$

are idempotent. They are projectors in a generalized sense, they are *skew*, *oblique projectors*. By means of them, the *spectral theorem* can be regained for non-orthogonal devices, in the form:

$$\mathsf{D}=\lambda_1|E_1><F_1|+\lambda_2|E_2><F_2|\ .\tag{19}$$

4. Two twisted ideal linear polarizers

The simplest non-orthogonal polarizer can be built up by means of two ideal homogeneous linear polarizers at some relative azimuth $\theta\neq 0,\pi/2$. Let us consider, for the sake of simplicity, a horizontal linear polarizer followed by a linear polarizer of azimuth θ.

The device operator of this sandwich is:

$$\mathsf{P}_1=\mathsf{P}_{|P_\theta>}\mathsf{P}_{|P_x>}=|P_\theta><P_\theta|P_x><P_x|=\cos\theta|P_\theta><P_x|\ .\tag{20}$$

It is constituted by a sequence of two noncommuting orthogonal projectors and by consequence it is not a projector: it is straightforward that it is not idempotent.

Nevertheless the operator (20) corresponds to an observable. For any SOP input it gives θ -linearly polarized output. In terms of the generalized theory of quantum measurement, each of the constitutive orthogonal projectors $\mathsf{P}_{|P_\theta>}$ and $\mathsf{P}_{|P_x>}$ corresponds to a PVM, whereas P_1 corresponds to a POVM. In other words P_1 is a generalized observable.

The eigenvectors and the eigenvalues of the linear polarizer (20) are:

$$|E_1>=|P_\theta>\ ,\quad \text{with}\quad \lambda_1=\cos^2\theta\,,\tag{21}$$

$$|E_2>=|P_y>\ ,\quad \text{with}\quad \lambda_2=0\,.\tag{22}$$

The eigenvectors are not orthogonal, the device is of non-orthogonal kind, its operator is non-normal.

Let us determine the biorthonormal conjugates of the eigenvectors (21), (22) of the operator (20). By using (17) we get:

$$<F_1|E_2>=<F_1|P_y>=0\ \rightarrow |F_1>=a|P_x>$$

$$<F_1|E_1>=<F_1|P_\theta>=1\ \rightarrow\ a^*<P_x|P_\theta>=a^*\cos\theta=1,\ \ a=1/\cos\theta$$

$$|F_1>=\frac{1}{\cos\theta}\left|P_x\right>\tag{23}$$

Similarly one obtains:

$$|F_2>=\frac{1}{\cos\theta}\left|P_{\theta+90^0}\right.> \tag{24}$$

With these four vectors, which constitute the biorthonormal system of eigenvectors of $\mathscr{P}_1$ we form the mathematical entities (18):

$$\mathsf{T}_i=|E_i><F_i|\ ,$$

namely:

$$\mathsf{T}_1=|E_1><F_1|=\frac{1}{\cos\theta}\left|P_\theta><P_x\right|\ , \tag{25}$$

$$\mathsf{T}_2=|E_2><F_2|=\frac{1}{\cos\theta}\left|P_y><P_{\theta+90^0}\right|\ , \tag{26}$$

It is straightforward that they are idempotent. Hence, they are projectors. It is also evident that are not Hermitian, i.e. are not orthogonal projectors. But they are *pseudoorthogonal*, *reciprocal exclusive:*

$$\mathsf{T}_i=|E_i><F_i|\ ,$$

$$\mathsf{T}_1\mathsf{T}_2=0\ , \tag{27}$$

or, in this sense, complementary one to the other.

They verify also the completness equation:

$$\begin{aligned}\mathsf{T}_1+\mathsf{T}_2&=\frac{1}{\cos\theta}\left\{|P_\theta><P_x|+|P_y><P_{\theta+90^0}|\right\}=\\&=\frac{1}{\cos\theta}\left\{\left[\cos\theta|P_x>+\sin\theta|P_y>\right]<P_x|\right.\\&\left.+|P_y>\left[-\sin\theta|P_x>+\cos\theta|P_y>\right]\right\}\\&=|P_x><P_x|+|P_y><P_y|=\mathsf{Q}_1+\mathsf{Q}_2=1\ .\end{aligned} \tag{28}$$

where we have labeled by Q_i the perpendicular projectors on *x* and *y*. By consequence they provide a (non-orthogonal) decomposition of the unity.

Let us analyze the physical signification of these two non-orthogonal projectors.

We shall see that the projector T_1 projects any state vector $|S>$ on $|E_1>\equiv|P_\theta>$ along $|E_2>\equiv|P_y>$ and the projector T_2 gives the projection on $|E_2>\equiv|P_y>$ along $|E_1>\equiv|P_\theta>$.

Let us take the projection $\mathcal{T}_1$, (25), of $|S>$:

$$\mathsf{T}_1 |S> = \frac{1}{\cos\theta} |P_\theta><P_x|S> = \frac{<P_x|S>}{\cos\theta} |P_\theta> \ . \tag{29}$$

Here $<P_x|S>$ is the orthogonal component of $|S>$ on $|P_x>$, so that $<P_x|S>/\cos\theta$ is the component of $|S>$ on $|P_\theta>$ along $|P_y>$. Thus $\mathsf{T}_1|S>$, (29), gives the projection of $|S>$ on $|P_\theta>$ along $|P_y>$.

Similarly, one can see that the projector T_2 projects any state vector $|S>$ on $|E_2>\equiv|P_y>$ along $|E_1>\equiv|P_\theta>$:

$$\mathsf{T}_2 |S> = \frac{1}{\cos\theta} |P_y><P_{\theta+90^0}|S> = \frac{<P_{\theta+90^0}|S>}{\cos\theta} |P_y> \ . \tag{30}$$

In (30), $<P_{\theta+90^0}|S>$ is the perpendicular projection of $|S>$ on $|P_{\theta+90^0}>$ and, hence $<P_{\theta+90^0}|S>/\cos\theta$ is the (skew, non-orthogonal) projection of $|S>$ on $|E_2>\equiv|P_y>$ along $|E_1>\equiv|P_\theta>$.

Thus, the two projectors T_1 and T_2 decompose any state vector $|S>$ along the eigenvectors (eigenstates) $(21)-(22)$ of the device (20), exactly like the eigenprojectors of an orthogonal device project any state vector along the orthogonal eigenvectors of the device; T_1 and T_2 are the *eigenprojectors of the non-orthogonal device* (polarizer) (20). They are *skew projectors.*

The eigenvalue of P_1 corresponding to T_2 being $\lambda_2 = 0$, (22), the *spectral theorem for the polarizer* (20) takes the form:

$$\mathsf{P}_1 = \lambda_1 \mathsf{T}_1 \ , \tag{31}$$

which is easy to verify with (20), (21) and (25).

We have to note that while an orthogonal ideal polarizer (e.g. $|P_x><P_x|$) is itself a projector, a non-orthogonal polarizer (e.g. (20)), is not a projector. That happens because its principal eigenvalue, (21), is not the unity. Therefore the operator of a non-orthogonal polarizer is only proportional (not equal) with its principal (non-orthogonal) projector, as shows the spectral theorem equation (31). While an orthogonal polarizer is a (perpendicular) projector, a non-orthogonal polarizer only corresponds to a (non-orthogonal) projector. But physically each of them gives rise to projections (orthogonal and non-orthogonal, respectively) of any incident state on its principal eigenstate. In this sense *they represent both observables (standard* and *generalized*, respectively).

In terms of generalized theory of quantum measurement the operator (20) corresponds to a POVM. Indeed its nonzero eigenvalue is positive: $\lambda_1 = \cos^2\theta$.

Finally we note that twisted stacks of N linear polarizers have been recently taken into consideration in the frame of Berry's phase analysis [20].

5. Horizontal linear polarizer followed by a half-wave linear retarder of azimuth θ/2.

Such a sandwich is used in the half-shade analyzer of the polarimeters. Again we are faced with a θ -linear polarizer: the $|P_x>$ linear polarized state given by the first layer (the $\mathsf{P}_{|P_x>}$ polarizer) is shifted by the second layer (the half-wave plate) symmetrically with respect to its principal axis, i.e. into the state $|P_\theta>$.

From our present viewpoint it is a non-orthogonal device formed by two non-commuting orthogonal devices. By means of (16) for $\theta/2$ and $\delta=\pi$ and (15) for $\theta=0$, its operator can be expanded in the $\{|P_x>,|P_y>\}$ basis as follows:

$$\begin{aligned}
\mathsf{P}_2 &= \mathsf{R}_{|P_{\theta/2}>}(\pi)\mathsf{P}_{|P_x>} \\
&= \big[i\cos\theta|P_x><P_x| + i\sin\theta|P_x><P_y| \\
&\quad + i\sin\theta|P_y><P_x| - i\cos\theta|P_y><P_y|\big]|P_x><P_x| \\
&= i\cos\theta|P_x><P_x| + i\sin\theta|P_y><P_x| \quad , \qquad (32)
\end{aligned}$$

This improper expansion of the operator does not reveal directly its intimate structure. By noticing that

$$\cos\theta<P_x| + \sin\theta<P_y| = <P_\theta| \quad , \qquad (33)$$

we obtain what can be called the *eigenexpression* of the operator:

$$\mathsf{P}_2 \sim |P_\theta><P_x| \ . \qquad (34)$$

where ~ stands for „it is the same with"; the complex factor i has no relevance in defining the type of device.

Now it is evident that this operator describes a non-orthogonal linear polarizer of azimuth θ . Its eigenvectors and eigenvalues are:

$$|E_1>=|P_\theta> \ , \quad \text{with} \quad \lambda_1 = <P_x|P_\theta> = \cos\theta \ , \qquad (35)$$

$$|E_2>=|P_y> \ , \quad \text{with} \quad \lambda_2 = 0 \ . \qquad (36)$$

It is worthy to note that, unlike the θ-linear polarizer (20), the θ-linear polarizer (34), (having the same eigenvectors but not the same principal eigenvalue) is not constituted by two orthogonal polarizers. While the operator (20) comes out from a series of two non-commuting orthogonal projectors, each of them corresponding to a POV and describing an observable, the operator (34) comes out from a Hermitian projector and a unitary, SU(2), operator.

Let us form now the biorthonormal conjugates of the eigenvectors (35), (36) of the nonorthogonal linear polarizer P_2 (34). With (17) we get:

$$< F_1 | E_2 >=< F_1 | P_y >= 0 \quad \rightarrow \quad | F_1 >= a | P_x >$$

$$< F_1 | E_1 >=< F_1 | P_\theta >= 1 \quad \rightarrow \quad a^* < P_x | P_\theta >= 1 \ , \quad a = 1/\cos\theta$$

$$| F_1 >= \frac{1}{\cos\theta} | P_x > \ , \tag{37}$$

respectively:

$$< F_2 | E_1 > =< F_2 | P_\theta >= 0 \quad \rightarrow \quad | F_2 >= b | P_{\theta+90^0} >$$

$$< F_2 | E_2 >=< F_2 | P_y >= 1 \quad \rightarrow \quad b^* < P_{\theta+90^0} | P_y >= 1 \ , \quad b = 1/\cos\theta$$

$$| F_2 >= \frac{1}{\cos\theta} | P_{\theta+90^0} > \tag{38}$$

The eigenvectors of the two linear polarizers P_1, (20), and P_2, (34), are the same; only their major eigenvalues are different. Consequently their eigenprojectors are the same too:

$$\mathsf{T}_1 =| E_1 >< F_1 |= \frac{1}{\cos\theta} | P_\theta >< P_x | \ , \tag{39}$$

$$\mathsf{T}_2 =| E_2 >< F_2 |= \frac{1}{\cos\theta} | P_y >< P_{\theta+90^0} | \tag{40}$$

With (34), (35) and (39), the spectral theorem is verified.

6. Two layer circular polarizer

The common circular polarizers are made by laminating together a linear polarizer and a linear $\pi/2$ retarder, with the transmission direction of the polarizer at 45^0 to the proper axes of the retarder.

If the fast axis azimuth of the retarder is zero and the azimuth of the polarizer is $+45^0$, the operator of this sandwich is:

$$\mathsf{C} = \mathsf{R}_{|P_x>}(\pi/2)\mathsf{P}_{|P_{45^0}>} \ . \tag{41}$$

With (16) for $\theta = 0$ and $\delta = \pi/2$ and (15) for $\theta = 45^0$ one obtains:

$$\mathsf{C} = \left[e^{i\frac{\pi}{4}} \left|P_x \right> \left< P_x\right| + e^{-i\frac{\pi}{4}} \left|P_y \right> \left< P_y\right| \right] \frac{1}{2} \left[1 + \left|P_x \right> \left< P_y\right| + \left|P_y \right> \left< P_x\right| \right]$$

$$= \frac{1}{2} \left[e^{i\frac{\pi}{4}} \left|P_x \right> \left< P_x\right| + e^{-i\frac{\pi}{4}} \left|P_y \right> \left< P_y\right| + e^{i\frac{\pi}{4}} \left|P_x \right> \left< P_y\right| + e^{-i\frac{\pi}{4}} \left|P_y \right> \left< P_x\right| \right]$$

This, unexpressive, improper form of the operator can be readily led to its, expressive, eigenform:

$$\mathsf{C} = \frac{1}{2} e^{i\frac{\pi}{4}} \left|P_x \right> \left[\left< P_x\right| + \left< P_y\right| \right] + \frac{1}{2} e^{-i\frac{\pi}{4}} \left|P_y \right> \left[\left< P_x\right| + \left< P_y\right| \right]$$

$$= \frac{1}{2} e^{i\frac{\pi}{4}} \left[\left|P_x \right> + e^{-i\frac{\pi}{2}} \left|P_y \right> \right] \left[\left< P_x\right| + \left< P_y\right| \right]$$

$$= e^{i\frac{\pi}{4}} \frac{1}{\sqrt{2}} \left[\left|P_x \right> - i \left|P_y \right> \right] \frac{1}{\sqrt{2}} \left[\left< P_x\right| + \left< P_y\right| \right] = e^{i\frac{\pi}{4}} \, |L><P_{45^0}| \ . \tag{42}$$

This is a non-orthogonal left-circular polarizer. Its eigenvectors and eigenvalues are:

$$|E_1> = |L> \quad \text{with} \quad \lambda_1 = e^{i\frac{\pi}{4}} <P_{45^0}|L> = \frac{1}{\sqrt{2}} \tag{43}$$

$$|E_2> = |P_{-45^0}> \quad \text{with} \quad \lambda_2 = 0 \ . \tag{44}$$

The biorthonormal conjugates of this pair of vectors can be determined by using (17), (13) and (14), as follows:

$$<F_1|E_2> = <F_1|P_{-45^0}> = 0 \quad \to \quad |F_1> = a\,|P_{45^0}>$$

$$<F_1|E_1> = <F_1|L> = 1 \quad \to \quad a^* <P_{45^0}|L> = 1 \ , \quad a = \sqrt{2}\, e^{-i\pi/4}$$

$$|F_1> = \sqrt{2}\, e^{-i\pi/4}\, |P_{45^0}> \ , \tag{45}$$

$$<F_2|E_1> = <F_2|L> = 0 \quad \to \quad |F_2> = b\,|R>$$

$$<F_2|E_2> = <F_2|P_{-45^0}> = 1 \ \to \ b^* <R|P_{-45^0}> = 1 \ , \ b = \sqrt{2}\, e^{i\pi/4} ,$$

$$|F_2> = \sqrt{2}\, e^{i\pi/4}\, |R> \tag{46}$$

The eigenprojectors of the non-orthogonal circular polarizer (42) are:

$$\mathsf{T}_1 = |E_1><F_1| = \sqrt{2}\, e^{i\frac{\pi}{4}}\, |L><P_{45^0}| \ , \tag{47}$$

$$\mathsf{T}_2 = |E_2><F_2| = \sqrt{2}\, e^{-i\frac{\pi}{4}}\, |P_{-45^0}><R| \ . \tag{48}$$

Both are skew projectors (idempotent, non-Hermitian). They give a non-orthogonal decomposition of the unity.

Each of them projects any SOP vector on one of the eigenvectors of the polarizer along the other eigenvector: T_1 projects on $|E_1>=|L>$ along $|E_2>=|P_{-45^0}>$, whereas T_2 projects on $|P_{-45^0}>$ along $|L>$.

With (47), (43) and (42), we get:

$$\mathsf{C} = \lambda_1 \mathsf{T}_1 \ , \tag{49}$$

i. e., up to a constant factor, the non-orthogonal circular polarizer reduces to a skew-projector. Its essential physical action is, as those of an orthogonal polarizer, a *projection*, but in this case a *non-orthogonal* one.

Conclusions

In polarization optics two kinds of polarizers are encountered: the orthogonal and the non-orthogonal polarizers.

The homogeneous canonical polarizers are orthogonal. The inhomogeneous (multilayer) polarizers may be orthogonal as well as non-orthogonal [17].

The *orthogonal polarizers* are described by Hermitian operators; particularly, the ideal polarizers by one-dimensional orthogonal projectors. From the viewpoint of quantum theory of measurement these operators are *standard* (Dirac − von Neumann) *observables* and correspond to projection-valued measures.

The *non-orthogonal* multilayer *polarizers* provide the simplest and clearest illustration of the class of operators describing the *generalized observables*. Their operators are not orthogonal projectors, and, moreover, they are not idempotent, i.e. they are not projectors.

Nevertheless, the essence of their physical action is the same as that of the orthogonal (e. g. canonical) polarizers: roughly speaking, they project any incident SOP vector on their major eigenvectors.

The skew – axis (biorthonormal system) analysis of the non-orthogonal polarizers we have presented above is the best mathematical tool for pointing out the similarities as well as the differences between the orthogonal and non-orthogonal polarizers.

Concerning the differences between orthogonal and non-orthogonal polarizers, they are best illustrated by the simplest skew polarizer, analysed in section 4. It corresponds exactly to the scheme proposed by Davies and Lewis in their fundamental paper [6] on the generalized quantum measurement: the operator (20) is a generalized observable arising from two non-commuting standard observables. It is a positive operator-valued measure which is not projection-valued unless the two components commute (here they commute only in the trivial cases $\theta = 0, \pi/2$). Unlike a standard measurement of $|P_\theta>$ (with a canonical $\mathsf{P}_{|P_\theta>}$ polarizer), the generalized measurement of $|P_\theta>$ with the non-Hermitian polarizer P_1, (20), is *conditioned* by the (preparative) measurement of $|P_x>$. Evidently, the „instrument" P_1 is the *composition* of $\mathsf{P}_{|P_\theta>}$ following $\mathsf{P}_{|P_x>}$, in the sense of theorem 2 [6]. The probability measure is that of the *joint distribution* [6] of $\mathsf{P}_{|P_\theta>}$ following $\mathsf{P}_{|P_x>}$.

Finally we have to note that while the linear polarizer presented in section 4 corresponds exactly to the Davies-Lewis scheme, the linear polarizer and the circular polarizer presented in sections 5 and 6 do not correspond exactly to this scheme: their operators are not sequences of noncommuting standard observables (Hermitian, orthogonal projectors) but sequences of a Hermitian projector and a SU(2) operator; the last one does not correspond to an observable. Nevetheless the composite polarizers (34) and (42) correspond to observables.

The most striking common feature of all the polarizers (orthogonal or non-orthogonal) is that they *correspond* to projectors, they *come out of* projectors, they *are generated* by projectors in the sense of the spectral theorem: *the spectral structure of each polarizer is reduced to an* (orthogonal or skew) *projector* — *the generator projector* of the given polarizer.

This structural unity between the operators of the orthogonal and non-orthogonal polarizers comes to strenghten the Yuen's assertion [21] that in fact „*generalised observables* should replace selfadjoint operators as the *standard* description of quantum measurements."

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